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A Letter from the Editor

History, illuminated by theoretical, empirical, and experimental studies, has shown that the institutions chosen by or imposed upon a society have a profound impact on its performance. The theory of mechanism and institution design is about how to devise new mechanisms or institutions, or improve existing ones, to better achieve desired economic or social outcomes. The challenge of design lies in the fact that individuals have different preferences about how society allocates its scarce resources, and private information must implicitly or explicitly be revealed to realize society's goals.

The **Journal of Mechanism and Institution Design** seeks to provide an independent and peer-reviewed open-access online journal that will be a natural English-language home for original analyses of mechanism and institution design. There are three compelling reasons for founding this new Journal. First, over the past few decades, mechanism and institution design has been one of the most flourishing and influential research areas, and we believe that it will continue to grow in importance. Second, we believe that mechanism and institution design can serve as a common language to bridge fields ranging from economics, politics and law, to computer science, mathematics, and engineering, improving communication and productivity. Third, we believe that open access journals are the future of scholarly publishing, and that an independent, non-commercial journal such as ours has even more advantages. The Internet has greatly enhanced the free and wide dissemination of knowledge, but it is most effective when access is open.

The Journal aims to publish original articles that deal with the issues of designing, improving, analyzing and testing economic, financial, political, or social mechanisms and institutions. It seeks scientifically important and socially relevant research, whether theoretical or applied, and whether empirical, experimental, historical, or practical. It strives to maintain a high standard for clarity of thought and expression.

The Journal is published and owned by the Society for the Promotion of Mechanism and Institution Design, a not-for-profit, unincorporated association devoted solely to the development of mechanism and institution design and the dissemination of scientific knowledge of the field. The Society tries to be self-supporting and run the Journal at a minimum cost by requiring only one author of each submitted paper to pay a modest membership fee of the Society to cover the cost of running the Journal. We are committed to handling every submitted paper as quickly as possible through a consistent and fair evaluation.

Working together with our dedicated Editorial Board and professional colleagues, we are confident that this journal will find a secure home in the scientific communities that contribute to mechanism and institution design.

Zaifu Yang, York, December 16, 2016

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CONVERGENCE OF PRICE PROCESSES UNDER TWO DYNAMIC DOUBLE AUCTIONS

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ABSTRACT

We study the convergence of two price processes generated by two dynamic double auctions (DA) and provide conditions under which the two price processes converge to a Walrasian equilibrium in the underlying economy. When the conditions are not satisfied, the price processes may result in a bubble or crash.

Keywords: Double auction mechanisms, incremental subgradient methods, network resource allocations.

JEL Classification Numbers: D44, D50.

1. INTRODUCTION

A double auction (DA) mechanism is a market-clearing system by which dispersed private information feeds into the system sequentially through bilateral trading. With little concentrated information about total demand and supply of an asset or good available to all participants in the marketplace, it

Both authors declare there are no conflicts of interest. This paper supersedes the paper "Bubbles, Crashes and Efficiency with Double Auction Mechanisms" (Ma & Li, 2011), which has been distributed and presented in various conferences. We thank Mark Satterthwaite for introducing us to the topic. Any errors are our own.

is natural to ask whether the price process generated by this DA mechanism converges to an equilibrium of the underlying economy or not.

Both A. Smith (1776) and Hayek (1945) raise a similar question how a market mechanism in a laissez-faire economy, where individual participants with little information about total demand and supply act solely in their self-interests, is able to integrate "dispersed bits of [incomplete] information" correctly into prices. A. Smith (1776) uses his famous "invisible hand" metaphor to describe its magnificence of a price mechanism. Hayek (1945, p. 519) has further explored the idea:

> "The peculiar character of the problem of a rational economic order is determined precisely by the fact that the knowledge of the circumstances of which we must make use never exists in concentrated or integrated form, but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess. The economic problem of society is thus not merely a problem of how to allocate "given" resources $[\cdots]$, it is rather a problem of the utilization of knowledge not given to anyone in its totality."

He goes on by saying: "This mechanism would have been acclaimed as one of the greatest triumphs of the human mind" if "It were the result of deliberate human design" (Hayek, 1945, p. 527). It should be noted that DA mechanisms employed in real exchange markets across the world are deliberately designed by humans.

An answer to the question is important for understanding price determination in an exchange market, since DA mechanisms have been widely used in equity, commodity and currency markets, among others. For example, an answer to the question is vital for understanding the efficient markets hypothesis (Fama, 1965) and the excess volatility puzzle (Shiller, 1981). Nonetheless, it is not easy to come up with an answer. Indeed, does a DA mechanism matter for the price determination of an asset? According to the efficient markets hypothesis, the answer should be no since the price of an asset in an exchange market should always follow its fundamental, with no systematic disparity between the two that can be detected with fundamental or technical analysis. On the other hand, excess volatility suggests that the answer may be yes, since the price of an equity can deviate from its fundamental to a great degree and such a deviation has been realized by a DA mechanism through a sequence of trading between buyer and seller pairs. But, if a DA mechanism really matters, how is it possible for an equity with fundamental value of 100 to be traded, say, at 300 or 50?

The main objective of this paper is to investigate if a DA mechanism can generate a sequence of prices that converges to an equilibrium of the underlying economy when individual demands and supplies are only privately known. To achieve this goal, we study a benchmark model given below:

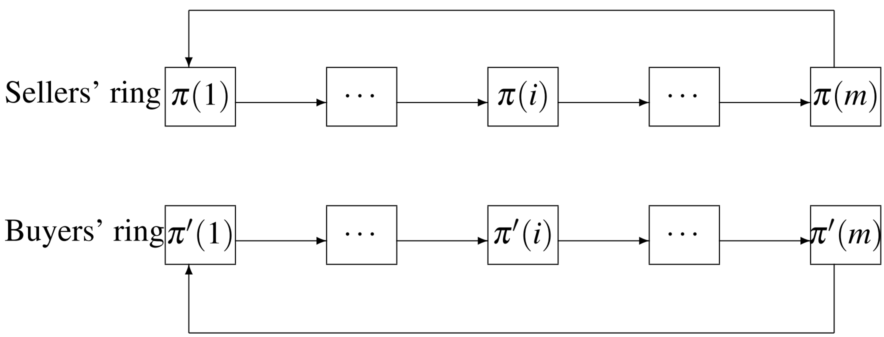
$$\mathscr{P} \qquad \text{minimize } F(y) = \sum_{i=1}^{m} f_i(y) + \sum_{j=1}^{n} g_j(y)$$

subject to $y \in Y$, a nonempty convex subset of R^d_+, where f_i and g_j are real-valued (possibly non-differentiable) convex functions defined on the d-dimensional Euclidean space R^d. A large class of quasilinear economies with m sellers and n buyers can be represented by this form (see Section 2.1). For these economies, the quantity demanded and supplied at prices y for buyer $j = 1, 2, \cdots, n$ and seller $i = 1, 2, \cdots, m$ are just subsets of the subdifferentials $-\partial g_j(y)$ and $\partial f_i(y)$, respectively, using the Fenchel duality (Ma & Nie, 2003). Thus, an equilibrium of the underlying economy studied in this paper is an optimal solution to the problem $\mathscr{P}$.

An Illustrative Example. For simplicity, consider an exchange economy where there is a single object or asset with a finite number of identical copies for sale. In a dynamic double auction, a buyer submits a bid order consisting of a bid price and a bid size, and a seller submits an ask order consisting of an ask price and an ask size, with the bid price at least as high as the ask price. The bid size is the quantity the buyer is willing to buy at the bid price and the ask size is the quantity the seller is willing to sell at the ask price. The price of an object is a weighted average of the bid price and the ask price, with weight $\alpha \in (0, 1)$, as in a static double auction in Chatterjee & Samuelson (1983), Myerson & Satterthwaite (1983), Wilson (1985), and Gresik (1991). Thus, given a sequence of pairs of one buyer and one seller, a sequence of prices is generated by a double auction. The next two questions are, at a given iteration, who will be the buyer and the seller pair and how are bid and ask prices are determined? We provide two specific examples of double auction to address the two questions.

In the first double auction, we assume that the number of buyers equals the number of sellers, and buyers and sellers form two cyclic rings: a buyer ring and a seller ring (Figure 1). This system can be realized if the buyer and seller rings consist of two permutations of agents. A pair of a buyer and a seller is selected according to the two cyclic rings, one pair at a time. The process starts with price X_k at k. Then buyer $\pi'(1)$ and seller $\pi(1)$ are the first pair to submit their bid and ask, respectively, based on the observed X_k. After the iteration of the pair $(\pi'(1), \pi(1))$, the next pair will be buyer $\pi'(2)$ and seller $\pi(2)$. This iteration process ends with the pair $(\pi'(m), \pi(m))$ and the price X_{k+1}.

Figure 1. Iterations under a double auction, where π and π' are two permutations of agents.



We need to determine how a buyer bids and a seller asks. A buyer's bid equals the newly updated price from the previous pair along the two rings plus a price increment that equals the product of the bid step size and the bid size. The bid step size is the price increment for one unit of the object that the buyer is willing to buy. Thus, the more a buyer wants to buy, the higher the bid price increment. An ask price is determined similarly. A seller's ask price equals the newly updated price from the previous pair along the two rings minus a price that equals the product of the ask step size and the ask size. The ask step size is now the price decrement for one unit of the object for sale. Thus, the more a seller wants to sell, the lower the ask price. To be more precise, let $\Phi_{i-1,k}$ be the price at iteration k and designate the next selected pair as $(\pi(i), \pi'(i))$. The ask price $\psi_{\pi(i),k}$ and the bid price $\varphi_{\pi'(i),k}$ are determined by, respectively,

$$\psi_{\pi(i),k} = \Phi_{i-1,k} - a_k \cdot S_{\pi(i)}(\Phi_{i-1,k}),\ \varphi_{\pi'(i),k} = \Phi_{i-1,k} + b_k \cdot D_{\pi'(i)}(\Phi_{i-1,k}), \quad (1)$$

where $S_{\pi(i)}(\Phi_{i-1,k})$ and $D_{\pi'(i)}(\Phi_{i-1,k})$ are the quantity supplied (i.e. the ask size) and demanded (i.e. the bid size) at price $\Phi_{i-1,k}$, respectively. If $S_{\pi(i)}(\Phi_{i-1,k})$ and $D_{\pi'(i)}(\Phi_{i-1,k})$ are set-valued maps, equation (1) should be understood with two selections from the demand and supply. $\{a_k\}$ and $\{b_k\}$ are the ask and bid step sizes, respectively. The price $\Phi_{i,k}$, which is communicated to the next pair, is determined by a weighted average of the bid and ask prices, with weight $\alpha \in (0,1)$:

$$\Phi_{i,k} = \alpha \psi_{\pi(i),k} + (1-\alpha)\varphi_{\pi'(i),k}. \tag{2}$$

Equations (1) and (2) provide the rule on how the price at an iteration evolves from one pair to the other along the two rings. The price process starts at $\Phi_{0,k} = X_k$ and ends with $X_{k+1} = \Phi_{m,k}$ at time k. Then this process repeats with two different permutations of agents. Thus, a sequence of prices X_k, $k = 0,1,2,\cdots$, is generated. Note that we consider the case where m is potentially a large number.

In our second randomized double auction a pair made up of a buyer and a seller is independently selected. Here we do not need the condition that the number of buyers equals the number of sellers because such an auction can be seen as a special case, in which the buyer ring and the seller ring in Figure 1 each consist of a single agent. Thus, equations (1) and (2) provide a sequence of prices $\{X_k\}$ once again.

Results. Assume that the underlying economy has a Walrasian equilibrium and the limit $\lim_{k\to\infty} \frac{b_k}{a_k}$ exists for two diminishing step sizes $\{a_k\}$ and $\{b_k\}$. Suppose there is a positive scalar λ such that (see Assumption 3.2)

$$\sum_{k=0}^{\infty} \left|\frac{b_k}{n} - \lambda \frac{a_k}{m}\right| < +\infty. \tag{3}$$

Then we show that λ must be $\lim_{k\to\infty} \frac{b_k}{a_k}$.[1] Our first main result Theorem 4.4 demonstrates that the price process $\{X_k\}$ must converge to a Walrasian equilibrium price vector of the underlying economy as long as the weight α satisfies the equality $\alpha = \frac{\lambda}{1+\lambda}$. Beyond the existence of Walrasian equilibrium, this convergence result does not depend on privately known demands and supplies. Instead it depends on the two parameters α and λ related to the auction form. If the weight α does not satisfy the equality $\alpha = \frac{\lambda}{1+\lambda}$, then the price process $\{X_k\}$ still converges to a price but it may be higher or lower than the equilibrium price(s). A higher than equilibrium price (i.e. bubble) is obtained when $\alpha < \frac{\lambda}{1+\lambda}$ and a lower than equilibrium price (i.e. crash) is

[1] The converse of this claim is not true. See Example 4.9.

obtained when $\alpha > \frac{\lambda}{1+\lambda}$ by the double auction. For example, α must be right at $\frac{1}{2}$ for $\lambda = 1$ in order for the auction to arrive at a Walrasian equilibrium.

For our randomized double auction, our second major result Theorem 4.7 shows that the above result still holds when λ is defined by $\lambda = \frac{m}{n} \lim_{k\to\infty} \frac{b_k}{a_k}$, where n is the number of buyers (or agents) and m is the number of sellers (or objects), under Assumption 3.3. For example, if $lim_{k\to\infty}\frac{b_k}{a_k} = 1$ and $m = 2n$, then α must be right at $\frac{2}{3}$ for the auction to arrive at a Walrasian equilibrium. Once again, this conclusion does not depend on f_i and g_j, which are unknown to the mechanism designer.

The above two results are proved for a general case where there are multiple heterogeneous objects, with each object having a finite number of identical copies. In the general case, $\{X_k\}$ is a sequence of price vectors rather than a sequence of prices. The condition on λ is also stated for the case where the limit $\lim_{k\to\infty} \frac{b_k}{a_k}$ may not exist.

Noises are identified as a key factor in the formation of bubbles and crashes (Shleifer, 1999). So it is of interest to see how noises may change our results. To examine this issue, we follow Ram et al. (2009) to introduce stochastic noises into buy and sell orders under the two DA mechanisms. Interestingly, our main results still hold for certain noises. This means that not all noises can affect the informational efficiency of a DA mechanism. The relationship between α and λ is still the key for the convergence of the price processes under the two DA mechanisms with stochastic noises.

Literature. The benchmark model is the dual problem of the linear programming relaxation in Bikhchandani & Mamer (1997) that can be reformulated as a convex optimization problem in the price space without constraints by $\mathscr{P}$. An optimal solution to this dual $\mathscr{P}$ (i.e., a minimizer) is a Walrasian equilibrium price vector of the original economy if the zero duality gap condition holds.[2] They also mention that the ascending price auction designed in Kelso & Crawford (1982) for a noted many-to-one job matching market can be used to achieve a Walrasian equilibrium price vector for their economy under the gross substitutes condition. A salient feature of the English auction in Kelso & Crawford (1982) is the constant increase (i.e. step size) of prices for those workers in excess demand at each iteration. Milgrom (2000) studies

[2] This dual approach and its related gradient method in search for an equilibrium in games can be traced back to Arrow & Hurwicz (1957). Beyond its applications in economics and game theory, the dual approach has many applications in other areas (see, e.g., Bertsekas, 2009; Nedić & Ozdaglar, 2009).

an economy similar to that in Bikhchandani & Mamer (1997) and provides an English auction that uses a larger constant price increment in the beginning and a smaller constant price increment near the end for an object in excess demand; the price process generated by such an auction can approach a Walrasian equilibrium at a faster speed when the auction begins, with approximation errors caused by the discrete price increment being reduced near the end of the auction. Xu et al. (2015) provide a subsequent analysis of the two double auctions presented in this paper with constant step sizes and obtain several approximation error bounds, under a more general information communication structure than Figure 1, when α is at random with unknown distributions.

It is often a challenging task to design an efficient auction when the gross substitutes condition is not satisfied and agents have private reservation values over bundles. Sun & Yang (2009, 2014) develop new auction mechanisms that can approach an efficient allocation when goods are substitutes and complements. There are several other auction mechanisms in the literature, see, e.g., Ausubel (2004), Gul & Stacchetti (2000), and references in both Milgrom (2000) and in Sun & Yang (2014).

An English auction is an ascending format of auction in which the price of an item is gradually increased and adjusted in each round according to reported total demand for the given total supply. When the scale of the market is small, this may not cause a problem. However, when it is large, it can be difficult to know the total demand at each round or iteration. A double auction is different, since the process involves pairs of buyers and sellers at each moment in time. A question arises as to whether such an auction process can always achieve a Walrasian equilibrium outcome as is the case with an English auction. The question is a challenge because an equilibrium must be defined with respect to the true demand and supply in totality. But, there is no way to know the total demand and supply in a double auction at any moment in time.

Our double auctions are largely motivated by incremental subgradient methods such as those studied in Kibardin (1980), Nedić & Bertsekas (2001), Ram et al. (2009), and Solodov & Zavriev (1998). In an incremental subgradient method, a single sequence of step sizes is used. They are effective in a unilateral market where one single agent updates her information into the price process. However, they are not as useful in a bilateral market situation using double auctions, where there are two sequences of step sizes, one for the buyers and the other for the sellers. A coordination (or "steering") condition between the two step sizes, such as the one on λ, is required for the convergence of the price process. If the λ condition fails to hold, our convergence results also fail, as shown in Xu et al. (2015, 2016) using numerical simulations. Following the current study, Xu et al. (2014, 2015, 2016) also demonstrate several convergence results for the two double auctions when agents form some Markovian chains in the iteration process. Moreover, Xu et al. (2015) study the convergence of price processes under the two double auctions when the weight α and two step sizes are independently drawn at random, with unknown distributions.

A major issue with double auctions is proving the existence of a Nash equilibrium; see, e.g., Satterthwaite & Williams (1989) and Jackson & Swinkels (2005). However, interdependent reservation values over bundles, assuming unit demand or supply,[3] are not a major concern for a Nash equilibrium. Our two dynamic double auctions are designed for an environment in which such values play an important role, in a model such as the many-to-one job matching in Kelso & Crawford (1982) and the multiple unit demand economy in Bikhchandani & Mamer (1997). These models may also be used for solving resource allocation problems in large scale distributional networks where agents hold dispersed private information; see Subsection 2.2 and e.g., Kelly et al. (1998), Nedić & Ozdaglar (2009), Ram et al. (2009). This analysis has the potential to provide solutions to real world problems such as the business to business trading in a marketplace or in multiagent coordination systems in artificial intelligence (Xia et al., 2005).

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the two DA mechanisms and the main assumptions. Section 4 discusses the main results with DA mechanisms without stochastic noises. Section 5 establishes the main results with stochastic noises. Section 6 concludes.

2. MODEL

We consider the following general problem (see also Bertsekas, 2012):

$$\mathscr{P} \qquad \text{minimize } F(y) \equiv f(y) + g(y)$$

$$\text{subject to } y \in Y,$$

[3] Kojima & Yamashita (2016) offer a latest noted exception along the line of Myerson & Satterthwaite (1983).

where

$$f = \sum_{i=1}^{m} f_i \text{ and } g = \sum_{j=1}^{n} g_j.$$

For all $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, $f_i : R^d \to R$ and $g_j : R^d \to R$ are convex functions and Y is a nonempty convex subset of R^d_+. As Bertsekas (2012) has demonstrated, such a form covers a large class of problems in the literature: a). least squares and related inference problems; b). dual optimization in separable problem; c). problems with many constraints; d). minimization of an expected value - stochastic programming; e). Weber problem in location theory; f). distributed incremental optimization-sensor networks. Here we focus on an application of the problem $\mathscr{P}$ to exchange economies with indivisible assets or goods. Thus we may assume that the price space Y is compact. Since every convex function φ on a compact set Y is regular Lipschitzian, the set of subgradients $\partial\varphi(y)$ for every $y \in Y$ is a nonempty, compact, and convex set, where $\partial\varphi(y)$ is defined by

$$\partial\varphi(y) = \{\eta \mid \varphi(y) + \langle \eta, w - y \rangle \leq \varphi(w), \forall w\};$$

see e.g., Clarke et al. (1988). For any two regular functions φ and ψ at y, the sum $\varphi + \psi$ is regular at y and

$$\partial(\varphi + \psi)(y) = \partial\varphi(y) + \partial\psi(y).$$

We use the following notation

$$F^* = \inf_{y \in Y} F(y), \qquad Y^* = \{y \in Y \mid F(y) = F^*\}, \qquad dist(y, Y^*) = \inf_{y^* \in Y^*} \|y - y^*\|$$

where $\|\cdot\|$ denotes the Euclidean norm.

2.1. Benchmark Economies with Multiple Indivisible Objects

We now show how the problem $\mathscr{P}$ naturally captures a large class of economies or markets typified by the many-to-one job matching model of Kelso & Crawford (1982) and its related economy with indivisible objects in Bikhchandani & Mamer (1997).

Let $M = \{1, 2, \cdots, m\}$ denote the set of objects and $N = \{1, 2, \cdots, n\}$ denote the set of agents. An agent j's reservation value function $u_j : 2^M \to R_+$ is defined over bundles of objects in M such that $u_j(\emptyset) = 0$. A *feasible allocation*

Z is a partition $(Z_0, Z_1, \cdots, Z_n)$ of all objects in M, in which agent j is allocated with the bundle Z_j and Z_0 is the unsold bundle. Let $\bar{Z}$ denote the set of all feasible allocations. A feasible allocation Z^* is Pareto optimal or efficient if

$$V \equiv \sum_{j=1}^{n} u_j(Z_j^*) \geq \sum_{j=1}^{n} u_j(Z_j), \ \ \forall Z \in \bar{Z}.$$

Given a price vetor $p \in R_+^m$, agent j's demand $D_j(p)$ consists of all bundles of M that maximize his surplus, i.e., $D_j(p) = \{S \subset M \mid u_j(S) - \sum_{i \in S} p_i \geq u_j(T) - \sum_{t \in T} p_t, \ \forall T \subset M\}$. A pair (Z, p) of a feasible allocation $Z \in \bar{Z}$ and a price vector $p \in R_+^m$ is a *Walrasian equilibrium* if $p_z = 0$ for all $z \in Z_0$ and $Z_j \in D_j(p)$ for all $j \in N$. It is well-known that a Walrasian equilibrium allocation is efficient. Even though an efficient allocation always exists, a Walrasian equilibrium may not exist, due to the nature of interdependent reservation values. Objects that are complement often cause the problem. Two goods satisfy the gross substitutes (GS) condition if a good that is in demand and whose price is not raised will still be in demand if the price of the other good arises. The GS condition of Kelso & Crawford (1982) is a sufficient condition for existence of a Walrasian equilibrium (Bikhchandani & Mamer, 1997; Gul & Stacchetti, 1999). Extensive studies of this condition can be found in Fujishige & Yang (2003), Hatfield & Kojima (2010), Hatfield & Milgrom (2005) among others. Economies that include complementary goods have been studied by Sun & Yang (2006, 2008, 2014).

Given $p \in R_+^m$, define $\pi_j(p) = u_j(S) - \sum_{i \in S} p_i$ for $S \in D_j(p)$. Note that $\pi_j(p)$ is convex. Then the dual of the linear programming relaxation of the integer programming in Bikhchandani and Mamer can be seen as a convex minimization problem without constraints:

$$V'' = \min_{p \in R_+^m} V(p) \equiv \sum_{i=1}^{m} p_i + \sum_{j=1}^{n} \pi_j(p). \tag{4}$$

Then, we have $V(p) \geq V$ for all $p \in R_+^m$.

Define $e : 2^M \to R^m$ by $e_i(S) = 1$ for $i \in S$ and $e_i(S) = 0$ otherwise. We can write the demand correspondence $\hat{D}_j : Y \to R^m$ by $\hat{D}_j(p) = \{e(S) \mid S \in D_j(p)\}$.

Define $g_j(p) = \pi_j(p)$ for all $j = 1, 2, \cdots, n$ and $f_i : R_+^n \to R$ by $f_i(p) = p_i$ for all $i = 1, 2, \cdots, m$. Thus we obtain the general form $F(p) = f(p) + g(p)$ subject to $p \in Y \subset R_+^m$. Note that the supply of an asset is the interval $[0, 1]$ at zero price. It follows from the Fenchel duality that $\bar{co}\hat{S}_i(p) = \partial f_i(p)$ and

$\partial g_j(p) = -\bar{co}\hat{D}_j(p)$ for all $p \in Y$, where $\bar{co}C$ denotes the closed convex hull of the set C (Ma & Nie, 2003). A vector y is an optimal solution in Y^* if and only if

$$0 \in \sum_{i=1}^{n} \bar{co}\hat{S}_i(y) - \sum_{j=1}^{m} \bar{co}\hat{D}_j(y).$$

Thus, a price vector $p \in Y$ is at an equilibrium only if $0 \in \partial(f+g)(p)$.

Because $\mathscr{P}$ is a dual of the linear programming relaxation of the primal integer programming in Bikhchandani & Mamer (1997) for finding an efficient allocation, a solution y to $\mathscr{P}$ is a Walrasian equilibrium if and only if the duality gap is zero, i.e., $V(y) = V$ (Bikhchandani & Mamer, 1997; Ma & Nie, 2003). Note that the duality gap approaches zero for a large scale economy (as m and n go to infinite) (Bertsekas, 2009).

2.2. A Congestion Control Problem with Production

We introduce a data transmission or congestion control problem on a given network with each link a production function (see, e.g., Kelly et al., 1998). Let $\mathscr{N} = \{1, 2, \cdots, n\}$ denote the set of sources and $\mathscr{L} = \{1, 2, \cdots, L\}$ the set of all undirected links. Each link $l \in \mathscr{L}$ has an increasing and convex cost function $c_l : [0,\infty) \to [0,\infty)$ such that $c_l(0) = 0$, i.e., it costs a link l the amount $c_l(q)$ to produce capacity $q \geq 0$. Let $L(i) \subset \mathscr{L}$ denote the set of links used by source $i \in \mathscr{N}$. The utility function for a source i is defined by $u_i : [0,\infty) \to [0,\infty)$, which is assumed to be increasing and concave. That is, source i gains a utility $u_i(x_i)$ when it sends data at a transmission rate x_i. Let $N(l) = \{i \in \mathscr{N} \mid l \in L(i)\}$ denote the set of sources that use link l. Let $p \in R^L_+$ denote a price vector, i.e., a link charges p_l per unit rate (e.g., packets per second) of data transmission (Kelly et al., 1998). Define $e : 2^L \to R^L$ by $e_l(S) = 1$ if $l \in S$ and $e_l(S) = 0$ otherwise. Define the supply function $S_l : R^L_+ \to [0,\infty)$ by $S_l(p) = \{q \mid p_l q - c_l(q) \geq p_l z - c_l(z), \forall z \geq 0\}$ and the demand function $D_i : R^L_+ \to R^L_+$ by $D_i(p) = \{e(L(i))x_i \mid u_i(x_i) - x_i \sum_{l \in L(i)} p_l \geq u_i(z) - z\sum_{l \in L(i)} p_l, \forall z \geq 0\}$. A triplet $(p; x, q) \in R^L_+ \times R^n_+ \times R^L_+$ is a network equilibrium if a). $e(L(i))x_i \in D_i(p)$ for all $i \in \mathscr{N}$; b). $q_l \in S_l(p)$ for all $l \in \mathscr{L}$; c). $\sum_{i \in N(l)} x_i \leq q_l$ for all $l \in \mathscr{L}$. Define $f_l(p) = p_l q - c_l(q), q \in S_l(p)$ for all $l \in \mathscr{L}$, and $g_i(p) = u_i(x_i) - x_i \sum_{l \in L(i)} p_l, e(L(i))x_i \in D_i(p)$ for all $i \in \mathscr{N}$. Let $f(p) = \sum_{l \in \mathscr{L}} f_l(p)$ and $g(p) = \sum_{i \in \mathscr{N}} g_i(p)$. We obtain the problem $\mathscr{P}$, subject to $p \in R^L_+$. Thus, one can show that $(p; x, q)$ is an equilibrium iff p is an optimal solution to the problem $\mathscr{P}$. Note that Y^* is nonempty. So an equilibrium always exists with transmission rates that are divisible. The problem $\mathscr{P}$ formalized this way is in fact the dual problem of a utility maximization problem on networks. See Kelly et al. (1998), and the examples discussed in Nedić & Ozdaglar (2009) and Ram et al. (2009), where each link is given with a fixed capacity, no production available. A flexible capacity network is often needed in practice. Under some mild assumptions, the first social welfare theorem holds (i.e., the duality gap is zero). Thus, finding an equilibrium under the problem $\mathscr{P}$ is one way to solve the primal utility maximization problem.

3. TWO DOUBLE AUCTIONS

We introduce two new dynamic double auctions. One double auction is designed based on the bilateral cyclic structure presented in Figure 1. The other one is based on a random match between the sellers and the buyers.

3.1. A Cyclic Double Auction (CDA)

Assume that $m = n$, which holds if the initial endowments are owned by the n agents, who act as both sellers and buyers.

CDA Mechanism: Let $\Phi_{0,k} = X_k$. For $i = 1, 2, \cdots, m$, let

$$\begin{aligned}
\psi_{i,k} &= \Phi_{i-1,k} - a_k \bigtriangledown f_i(\Phi_{i-1,k}), && (5)\\
\varphi_{i,k} &= \Phi_{i-1,k} - b_k \bigtriangledown g_i(\Phi_{i-1,k}), && (6)\\
\Phi_{i,k} &= P_Y(\alpha\psi_{i,k} + (1-\alpha)\varphi_{i,k}), \qquad \alpha \in (0,1), && (7)
\end{aligned}$$

where $\bigtriangledown f_i(\Phi_{i-1,k}) \in \partial f_i(\Phi_{i-1,k})$ and $\bigtriangledown g_i(\Phi_{i-1,k}) \in \partial g_i(\Phi_{i-1,k})$. P_Y is the Euclidean projection onto Y because the weighted average of the bid and ask prices may be out of Y in the process. Let

$$X_{k+1} = \Phi_{m,k}. \tag{8}$$

Our explanation of this auction is as follows. X_k is the initial price vector for time k. We want to obtain X_{k+1} with a round of m iterations. To accomplish the task, the m sellers and m buyers form a cyclic seller ring and a cyclic buyer ring in two random orders, i.e., two random permutations of the m agents. Then we rename these agents in the two rings with $i = 1, 2, \cdots, m$. Based on the initial price vector $\Phi_{0,k} = X_k$, seller 1 and buyer 1 are the first pair to submit an ask order and a bid order to determine $\Phi_{1,k}$, which is then communicated to the next pair, seller 2 and buyer 2. This process continues until it reaches the last pair, seller m and buyer m, to obtain $\Phi_{m,k}$. The price vector X_{k+1} is set to be $\Phi_{m,k}$ to end the m iterations in the cycle. After obtaining X_{k+1}, the m agents are reshuffled and renamed with two new random permutations and the process proceeds with X_{k+1} in the same manner as with X_k. The auction starts with a given $X_0 \in Y \subset R^d_{++}$, and generates a sequence of price vectors $\{X_k\}$, $k \geq 0$.

The ask order for seller i at k consists of a vector of ask prices $\psi_{i,k}$ and an ask size $\bigtriangledown f_i(\Phi_{i-1,k})$, where $\bigtriangledown f_i(\Phi_{i-1,k})$ is on the supply curve $\partial f_i(\Phi_{i-1,k})$. The relationship between ask prices and sizes is given by equation (14), where a_k is the ask step size at round k, which is the price decrement for one unit of object for sale. The ask prices are lower than $\Phi_{i-1,k}$, the newly updated prices from the previous pair. The more the seller wants to sell an object, the lower the ask prices.

Similarly, the bid order for buyer i at k consists of a vector of bid prices $\varphi_{i,k}$ and a bid size $\bigtriangledown g_i(\Phi_{i-1,k})$, where $\bigtriangledown g_i(\Phi_{i-1,k})$ is on the demand curve $-\partial g_i(\Phi_{i-1,k})$. The relationship between bid prices and sizes is given by equation (15), where b_k is the bid step size at round k, which is the price increment for one unit of object to buy. The bid prices are higher than $\Phi_{i-1,k}$, the newly updated prices from the previous pair. The more the buyer wants to buy an object, the higher the bid prices. The prices $\Phi_{i,k}$ are a weighted average of the ask and bid prices, with a weight in $(0,1)$, as in Chatterjee & Samuelson (1983).

We provide conditions on the weight α and the two step sizes $\{a_k\}$ and $\{b_k\}$ so that the price process $\{X_k\}$ converges to an optimal solution in Y^*.

It may be useful to compare CDA with the noted cyclic incremental subgradient method (e.g., Nedić & Bertsekas, 2001).

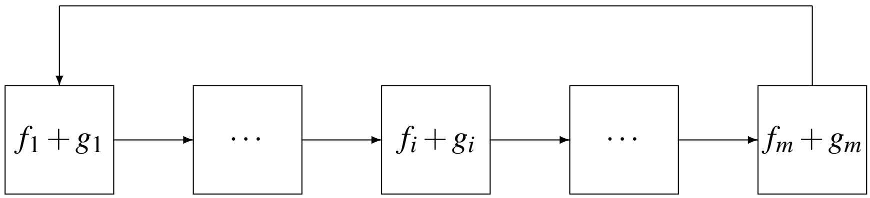
Incremental Subgradient Method: Assume $m = n$ and $X_0 \in Y$. Let $\Phi_{0,k} = X_k$. For $i = 1, 2, \cdots, m$, let

$$\Phi_{i,k} = P_Y(\Phi_{i-1,k} - a_k \bigtriangledown (f_i + g_i)(\Phi_{i-1,k})),$$

where $\bigtriangledown (f_i + g_i)(\Phi_{i-1,k}) \in \partial (f_i + g_i)(\Phi_{i-1,k})$. Let $X_{k+1} = \Phi_{m,k}$. P_Y is the Euclidean projection onto the set Y.

In the incremental subgradient method, m agents form a single ring in an arbitrary order (Figure 2). Prices are iterated along the ring one agent at a time. At each iteration, only one agent i reveals his buy size in $-\partial g_i(\Phi_{i-1,k})$

Figure 2. An Iteration under the Incremental Subgradient Method.



together with his sell size in $\partial f_i(\Phi_{i-1,k})$ at prices $\Phi_{i-1,k}$. A key feature of their algorithm is that adjustment in prices at each iteration depends on the chosen step size a_k and the individual excess supply $\bigtriangledown(f_i + g_i)(\Phi_{i-1,k})$. They show that if the step size $\{a_k\}$ is diminishing, their algorithm generates a sequence of prices that converges to an equilibrium in Y^*. CDA is different from the incremental subgradient method in two ways. First, the buyer and seller cyclic rings are different. We consider f and g as two different sides of the market. Second, there are two sequences of step sizes $\{a_k\}$ and $\{b_k\}$ in CDA. This makes the convergence results for the incremental subgradient method inapplicable to CDA because there is a new λ condition. Even if the λ condition is satisfied, the weight α and the parameter λ must be in a right combination so that the process $\{X_k\}$ can converge to an optimal solution in Y^*. We need to consider two step sizes because the market using double auctions is bilateral, in contrast to the unilateral market under the incremental subgradient method.

3.2. A Randomized Double Auction (RDA)

Let w_k be a random variable taking equiprobable values from the set $\{1, 2, \cdots, m\}$ and w'_k be a random variable taking equiprobable values from the set $\{1, 2, \cdots, n\}$. Let $\bigtriangledown f_{w_k}(X_k) \in \partial f_{w_k}(X_k)$ and $\bigtriangledown g_{w'_k}(X_k) \in \partial g_{w'_k}(X_k)$, where if w_k takes a value i, then the vector $\partial f_{w_k}(X_k)$ is $\partial f_i(X_k)$, similarly for g. Here we do not need to assume that $m = n$.

Our sequence $\{X_k\}$ is generated by RDA mechanism as below.

RDA Mechanism: Given X_k, let

$$\psi_{w_k,k+1} = X_k - a_k \bigtriangledown f_{w_k}(X_k), \tag{9}$$

$$\varphi_{w'_k,k+1} = X_k - b_k \bigtriangledown g_{w'_k}(X_k). \tag{10}$$

Let

$$X_{k+1} = P_Y(\alpha \psi_{w_k,k+1} + (1-\alpha)\varphi_{w'_k,k+1}), \tag{11}$$

where $\alpha \in (0,1)$ and P_Y is the Euclidean projection onto Y.

Our explanation of this auction is as follows. X_k is the price vector at time k. We want to obtain X_{k+1} by incorporating individual demand and supply information from a pair of buyer and seller, with the seller randomly selected from the set of sellers and the buyer randomly selected from the set of buyers. The bid prices and bid size as well as the ask price and ask bid are determined in the same way as in CDA. This is equivalent to the case where a seller randomly matches with a buyer. Again, the auction starts with $X_0 \in Y$. Then (9)-(11) generate a sequence of prices X_k, $k = 1, 2, \cdots$.

We are interested in the convergence of $\{X_k\}$ and the conditions under which $\{X_k\}$ converges to an optimal solution in Y^*. Because f and g are privately known to agents, not to the mechanism designer, our conditions cannot be imposed on f and g. This provides a challenge because information about f and g has never been revealed in totality in the two auction processes. The two auction forms are in spirit close to the decentralized market mechanism as stated in A. Smith (1776) and Hayek (1945), in contrast to a centralized market mechanism.

3.3. Key Assumptions

We assume that the step sizes satisfy the following diminishing conditions in Assumption 3.1, which is standard for many convergence results in the literature. Assumptions 3.2 and 3.3 are the two new assumptions for two double auctions CDA and RDA. To achieve an equilibrium in Y^*, we must have a right combination of α and λ.

Assumption 3.1 (Diminishing step sizes). *Assume that the two sequences $\{a_k\}$ and $\{b_k\}$ of step sizes are such that (i). $a_k > 0$ and $b_k > 0$; (ii). $\sum_{k=0}^{\infty} a_k = +\infty$ and $\sum_{k=0}^{\infty} b_k = +\infty$; (iii). $\sum_{k=0}^{\infty} a_k^2 < +\infty$ and $\sum_{k=0}^{\infty} b_k^2 < +\infty$.*

Assumption 3.2. *Assume that the two sequences $\{a_k\}$ and $\{b_k\}$ of step sizes* *are such that there exists some positive λ to ensure*

$$\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty. \tag{12}$$

Assumption 3.3. *Assume that the two sequences $\{a_k\}$ and $\{b_k\}$ of step-sizes are such that there exists some positive λ to ensure*

$$\sum_{k=0}^{\infty} |\frac{b_k}{n} - \lambda \frac{a_k}{m}| < +\infty. \tag{13}$$

Note the difference between Assumptions 3.2 and 3.3. Assumption 3.2 is for CDA where $m = n$, while Assumption 3.3 is for RDA where m may be different from n. These λ conditions (12) and (13) will be discussed in Subsection 4.4.

4. MAIN RESULTS

In this section we prove our two main results for CDA (5)-(8) and RDA (9)-(11). Lemma 4.1 below is the key for both mechanisms.

4.1. Main Result: CDA

Since Y is compact, there exist scalars $C_1, C_2, \cdots, C_m$ and $D_1, D_2, \cdots, D_m$ such that

$$\|h\| \le C_i, \qquad \forall h \in \partial f_i(X_k) \cup \partial f_i(\Phi_{i-1,k}), i = 1, 2, \cdots, m, k = 0, 1, 2, \cdots$$

and

$$\|\ell\| \le D_i, \qquad \forall \ell \in \partial g_i(X_k) \cup \partial g_i(\Phi_{i-1,k}), i = 1, 2, \cdots, m, k = 0, 1, 2, \cdots.$$

Note that $m = n$.

Lemma 4.1. *Let $\{X_k\}$ be the sequence generated by CDA (5)-(8). Then for all $y \in Y$ and $k \ge 0$, we have*

$$\begin{aligned} \|X_{k+1} - y\|^2 &\le \|X_k - y\|^2 - 2a_k\alpha(f(X_k) - f(y)) - 2b_k(1-\alpha)(g(X_k) - g(y)) \\ &+ (\alpha a_k C + (1-\alpha) b_k D)^2, \end{aligned}$$

where $C = \sum_{i=1}^{m} C_i$ *and* $D = \sum_{i=1}^{m} D_i$.

Remark. This lemma shows that the quadratic distance of the price process $\{X_k\}$ to an equilibrium in Y^* can be bounded and that it is possible for the price process to approach an equilibrium in Y^*. Nonetheless, a naive choice y in Y^* does not work because of this term $[a_k \alpha f(X_k) + b_k(1-\alpha)g(X_k)] - [a_k \alpha f(y) + b_k(1-\alpha)g(y)]$, which is not a sum form $f+g$ as defined in $\mathscr{P}$. For a choice y in Y^*, there is no guarantee that the term $[a_k \alpha f(X_k) + b_k(1-\alpha)g(X_k)] - [a_k \alpha f(y) + b_k(1-\alpha)g(y)]$ is always nonnegative. This is why the convergence results for the incremental subgradient method with diminishing step sizes in Nedić & Bertsekas (2001) does not apply to CDA.

Proof. Denote $h_{i,k} = \bigtriangledown f_i(\Phi_{i-1,k})$ and $\ell_{i,k} = \bigtriangledown g_i(\Phi_{i-1,k})$ for all $i = 1, 2, \cdots, m$. By the non-expansive property of projection, we have

$$\begin{aligned}
\|\Phi_{i,k} - y\|^2 &\leq \|\alpha \psi_{i,k} + (1-\alpha)\varphi_{i,k} - y\|^2 \\
&= \|\alpha(\psi_{i,k} - y) + (1-\alpha)(\varphi_{i,k} - y)\|^2 \\
&= \alpha^2 \|\psi_{i,k} - y\|^2 + (1-\alpha)^2 \|\varphi_{i,k} - y\|^2 \\
&\quad +2\alpha(1-\alpha)\langle(\psi_{i,k} - y), (\varphi_{i,k} - y)\rangle \\
&= \alpha^2 \|\Phi_{i-1,k} - y - a_k h_{i,k}\|^2 + (1-\alpha)^2 \|\Phi_{i-1,k} - y - b_k \ell_{i,k}\|^2 \\
&\quad +2\alpha(1-\alpha)\langle(\Phi_{i-1,k} - y - a_k h_{i,k}), (\Phi_{i-1,k} - y - b_k \ell_{i,k})\rangle \\
&= \alpha^2 \|\Phi_{i-1,k} - y\|^2 - 2a_k \alpha^2 \langle h_{i,k}, (\Phi_{i-1,k} - y)\rangle + \alpha^2 a_k^2 \|h_{i,k}\|^2 \\
&\quad +(1-\alpha)^2 \|\Phi_{i-1,k} - y\|^2 - 2b_k(1-\alpha)^2 \langle \ell_{i,k}, (\Phi_{i-1,k} - y)\rangle \\
&\quad +(1-\alpha)^2 b_k^2 \|\ell_{i,k}\|^2 + 2\alpha(1-\alpha)\|\Phi_{i-1,k} - y\|^2 \\
&\quad -2\alpha(1-\alpha)a_k \langle h_{i,k}, (\Phi_{i-1,k} - y)\rangle \\
&\quad -2\alpha(1-\alpha)b_k \langle \ell_{i,k}, (\Phi_{i-1,k} - y)\rangle \\
&\quad +2\alpha(1-\alpha)a_k b_k \langle h_{i,k}, \ell_{i,k}\rangle \\
&= \|\Phi_{i-1,k} - y\|^2 - 2(1-\alpha)b_k \langle \ell_{i,k}, (\Phi_{i-1,k} - y)\rangle \\
&\quad +\|\alpha a_k h_{i,k} + (1-\alpha)b_k \ell_{i,k}\|^2 - 2\alpha a_k \langle h_{i,k}, (\Phi_{i-1,k} - y)\rangle
\end{aligned}$$

Since we have $\|h_{i,k}\| \leq C_i, \|\ell_{i,k}\| \leq D_i$ for all $k = 0, 1, 2, \cdots$, we obtain

$$\begin{aligned}
\|\Phi_{i,k} - y\|^2 &\leq \|\Phi_{i-1,k} - y\|^2 + (\alpha a_k C_i + (1-\alpha)b_k D_i)^2 \\
&\quad -2\langle(\alpha a_k h_{i,k} + (1-\alpha)b_k \ell_{i,k}), (\Phi_{i-1,k} - y)\rangle
\end{aligned}$$

Summing over $i = 1, 2, \cdots, m$, we get

$$\begin{aligned}\sum_{i=1}^{m} \|\Phi_{i,k} - y\|^2 \ &\leq \ \sum_{i=1}^{m} \|\Phi_{i-1,k} - y\|^2 + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2 \\ &\quad -2 \sum_{i=1}^{m} \langle (\alpha a_k h_{i,k} + (1-\alpha) b_k \ell_{i,k}), (\Phi_{i-1,k} - y) \rangle\end{aligned}$$

So we have

$$\begin{aligned}\|X_{k+1} - y\|^2 \ &\leq \ \|X_k - y\|^2 + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2 \\ &\quad -2 \sum_{i=1}^{m} \langle (\alpha a_k h_{i,k} + (1-\alpha) b_k \ell_{i,k}), (\Phi_{i-1,k} - y) \rangle\end{aligned}$$

By the definition of subgradients $h_{i,k}$ and $\ell_{i,k}$,

$$\langle h_{i,k}, (y - \Phi_{i-1,k}) \rangle \leq f_i(y) - f_i(\Phi_{i-1,k})$$

and

$$\langle \ell_{i,k}, (y - \Phi_{i-1,k}) \rangle \leq g_i(y) - g_i(\Phi_{i-1,k}).$$

Then

$$\begin{aligned}\|X_{k+1} - y\|^2 \ &\leq \ \|X_k - y\|^2 - 2(1-\alpha) b_k \sum_{i=1}^{m} (g_i(\Phi_{i-1,k}) - g_i(y)) \\ &\quad + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2 - 2\alpha a_k \sum_{i=1}^{m} (f_i(\Phi_{i-1,k}) - f_i(y))\end{aligned}$$

So

$$\begin{aligned}\|X_{k+1} - y\|^2 \ &\leq \ \|X_k - y\|^2 - 2\alpha a_k (f(X_k) - f(y)) - 2(1-\alpha) b_k (g(X_k) - g(y)) \\ &\quad -2\alpha a_k \sum_{i=1}^{m} (f_i(\Phi_{i-1,k}) - f_i(X_k)) + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2 \\ &\quad -2(1-\alpha) b_k \sum_{i=1}^{m} (g_i(\Phi_{i-1,k}) - g_i(X_k))\end{aligned}$$

Next we need to estimate $f_i(\Phi_{i-1,k}) - f_i(X_k)$ and $g_i(\Phi_{i-1,k}) - g_i(X_k)$.

Lemma 4.1.1. $\|\Phi_{i-1,k} - X_k\| \leq \sum_{j=1}^{i-1} (\alpha a_k C_j + (1-\alpha) b_k D_j)$.

Proof. We show Lemma 4.1.1 by induction. Note that $\Phi_{0,k} - X_k = 0$. Assume that it holds for $i-1$. Then

$$\begin{aligned}
\|\Phi_{i,k} - X_k\| &= \|(\alpha\psi_{i,k} + (1-\alpha)\varphi_{i,k}) - X_k\| \\
&\leq \alpha\|\psi_{i,k} - X_k\| + (1-\alpha)\|\varphi_{i,k} - X_k\| \\
&\leq \alpha\|\Phi_{i-1,k} - a_k h_{i,k} - X_k\| + (1-\alpha)\|\Phi_{i-1,k} - b_k \ell_{i,k} - X_k\| \\
&\leq \|\Phi_{i-1,k} - X_k\| + \alpha a_k C_i + (1-\alpha) b_k D_i \text{ by induction hypothesis} \\
&\leq \sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha) b_k D_j) + \alpha a_k C_i + (1-\alpha) b_k D_i.
\end{aligned}$$

This completes the proof of Lemma 4.1.1. □

So

$$\|f_i(\Phi_{i-1,k}) - f_i(X_k)\| \leq \sum_{j=1}^{i-1} C_i(\alpha a_k C_j + (1-\alpha) b_k D_j)$$

and

$$\|g_i(\Phi_{i-1,k}) - g_i(X_k)\| \leq \sum_{j=1}^{i-1} D_i(\alpha a_k C_j + (1-\alpha) b_k D_j).$$

Plugging into (14), we have

$$\begin{aligned}
\|X_{k+1} - y\|^2 &\leq \|X_k - y\|^2 - 2\alpha a_k(f(X_k) - f(y)) - 2(1-\alpha) b_k(g(X_k) - g(y)) \\
&\quad + 2\sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha) b_k D_i) \sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha) b_k D_j) \\
&\quad + \sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha) b_k D_i)^2 \\
&= \|X_k - y\|^2 - 2\alpha a_k(f(X_k) - f(y)) - 2(1-\alpha) b_k(g(X_k) - g(y)) \\
&\quad + (\sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha) b_k D_i))^2 \\
&= \|X_k - y\|^2 - 2\alpha a_k(f(X_k) - f(y)) - 2(1-\alpha) b_k(g(X_k) - g(y)) \\
&\quad + (\alpha a_k C + (1-\alpha) b_k D)^2.
\end{aligned}$$

This completes the proof of Lemma 4.1. □

Remark. To apply Lemma 4.1, we must choose a right y for CDA. To do so, we need to define the following $\mathscr{P}(\alpha, \lambda)$:

$$\mathscr{P}(\alpha, \lambda) \qquad \text{minimize}_{y \in Y}\ F(y, \alpha, \lambda) \equiv (\alpha f + \lambda(1-\alpha) g)(y)$$

where $\alpha \in (0,1)$. The parameter λ is some positive scalar. Then we introduce the following notation

$$F^*(\alpha,\lambda) = \inf_{y\in Y} F(y,\alpha,\lambda), \qquad Y^*(\alpha,\lambda) = \{y \in Y \mid F(y,\alpha,\lambda) = F^*(\alpha,\lambda)\},$$

and $\text{dist}(x, Y^*(\alpha,\lambda))$, the Euclidean distance. Note that Y^* and $Y^*(\alpha,\lambda)$ are related but they are not the same, depending on α and λ.

Proposition 4.2. *Let Assumptions 3.1 and 3.2 hold. Let $\{X_k\}$ be the price sequence generated by CDA (5)-(8). Then*

$$\liminf_{k\to\infty} dist(X_k, Y^*(\alpha,\lambda)) = 0.$$

Proof. From Lemma 4.1, we obtain for all $y^* \in Y^*(\alpha,\lambda)$ and $k \geq 0$,

$$\begin{aligned} \|X_{k+1} - y^*\|^2 \ &\leq\ \|X_k - y^*\|^2 - 2(b_k - \lambda a_k)(1-\alpha)(g(X_k) - g(y^*)) \\ &\quad -2a_k[(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y^*)] \\ &\quad +(\alpha a_k C + (1-\alpha) b_k D)^2. \end{aligned}$$

Since Y is compact, g is continuous, image $(g(Y))$ is bounded. That means there exists $M > 0$ such that $|g(y)| \leq M$ for all $y \in Y$. Hence, for any $N = 1, 2, \cdots$,

$$\begin{aligned} 0 \ &\leq\ \|X_{N+1} - y^*\|^2 \leq \|X_0 - y^*\|^2 + \sum_{k=0}^{N} (\alpha a_k C + (1-\alpha) b_k D)^2 \\ &\quad -2\sum_{k=0}^{N} a_k[(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y^*)] \\ &\quad +2(\sum_{k=0}^{N} |b_k - \lambda a_k|) \cdot (1-\alpha) \cdot 2M \\ &= \ I - II + III + IV. \end{aligned}$$

I is a constant. When N goes to infinity, $III < +\infty$ since $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$ and

$$IV \leq 2(\alpha^2 C^2 \sum_{k=0}^{\infty} a_k^2 + (1-\alpha)^2 D^2 \sum_{k=0}^{\infty} b_k^2) < +\infty.$$

Thus, $II < +\infty$. We obtain

$$\liminf_{k\to\infty}[(\alpha f+(1-\alpha)\lambda g)(X_k)-(\alpha f+(1-\alpha)\lambda g)(y^*)]=0.$$

Otherwise, $\exists\delta > 0$ such that for any natural number k

$$(\alpha f+(1-\alpha)\lambda g)(X_k)-(\alpha f+(1-\alpha)\lambda g)(y^*)>\delta.$$

And then $II > \delta\sum_{k=0}^{\infty} a_k = +\infty$, a contradiction.

Now take a subsequence $\{X_{n_k}\}$ of $\{X_k\}$ such that

$$0\le(\alpha f+(1-\alpha)\lambda g)(X_{n_k})-(\alpha f+(1-\alpha)\lambda g)(y^*)<\frac{1}{k}.$$

By the fact that Y is compact, $\{X_{n_k}\}$ has at least one accumulation point y_0, say, and since $\alpha f+(1-\alpha)\lambda g$ is continuous, we have that

$$\lim_{k\to\infty}(\alpha f+(1-\alpha)\lambda g)(X_{n_k})=(\alpha f+(1-\alpha)\lambda g)(y_0).$$

By the definition of $\{X_{n_k}\}$, we know that

$$\lim_{k\to\infty}(\alpha f+(1-\alpha)\lambda g)(X_{n_k})=(\alpha f+(1-\alpha)\lambda g)(y^*),\ y^*\in Y^*(\alpha,\lambda).$$

Hence, $y_0\in Y^*(\alpha,\lambda)$. So

$$\liminf_{k\to\infty} dist(X_k,Y^*(\alpha,\lambda))=0.$$

This completes the proof. □

Proposition 4.3. *Let Assumptions 3.1 and 3.2 hold. Then the sequence* $\{X_k\}$ *in Proposition 4.2 converges to an optimal solution* $y_0\in Y^*(\alpha,\lambda)$.

Proof. Let

$$\begin{aligned}\delta_k &= 2a_k[(\alpha f+(1-\alpha)\lambda g)(X_k)-(\alpha f+(1-\alpha)\lambda g)(y_0)]\\ &\quad +2(|b_k-\lambda a_k|)\cdot(1-\alpha)\cdot 2M+(\alpha a_k C+(1-\alpha)b_k D)^2>0.\end{aligned}$$

Then $\sum_{k=0}^{\infty}\delta_k = II + III + IV < +\infty$ (see the proof of Proposition 4.2). We also have that

$$\begin{aligned}\|X_{k+1}-y_0\|^2 &\leq \|X_k - y_0\|^2 + (\alpha a_k C + (1-\alpha)b_k D)^2 \\ &\quad -2a_k[(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y_0)] \\ &\quad +2(|b_k - \lambda a_k|)\cdot(1-\alpha)\cdot 2M \\ &\leq \|X_k - y_0\|^2 + \delta_k.\end{aligned}$$

Then applying Proposition 1.3 in Correa & Lemaréchal (1993) to the result in Proposition 4.2, we have that

$$\lim_{k\to\infty} X_k = y_0.$$

□

Theorem 4.4. *Let Assumptions 3.1 and 3.2 hold. Assume that* $\alpha = \frac{\lambda}{1+\lambda}$ *and* $\alpha \in (0,1)$. *Then the sequence* $\{X_k\}$ *generated by CDA (5)-(8) converges to an optimal solution in* Y^*.

Proof. It follows from Proposition 4.3 and the definition of $Y^*(\alpha,\lambda)$, which is the same as Y^* when $(1-\alpha)\lambda = \alpha$ holds. This completes the proof. □

Remark. Suppose the limit $\lim_{k\to\infty}\frac{b_k}{a_k}$ exists and Assumptions 3.1 and 3.2 hold. Then we will see in Section 4.4 that $\lambda = \lim_{k\to\infty}\frac{b_k}{a_k}$. Assume that $\lambda = 1$. Then Theorem 4.4 shows that the price process $\{X_k\}$ converges to a Walrasian equilibrium of the underlying economy if $\alpha = \frac{1}{2}$. Once λ changes, we must also change α accordingly in order to achieve a Walrasian equilibrium. Otherwise the price process $\{X_k\}$ still converges but it may not converge to a Walrasian equilibrium of the original economy. In particular, Theorem 4.4 fails if Assumption 3.2 does not hold, as shown in Xu et al. (2015, 2016) by numerical simulations.

4.2. Main Result for RDA

Assumption 4.5. *The sequence* $\{w_k\}(\{w'_k\})$ *is a sequence of independent random variables, each uniformly distributed over the set* $\{1,2,\cdots,m\}$ $(\{1,2,\cdots,n\})$.

Furthermore, the two sequences $\{w_k\}$ and $\{w'_k\}$ are independent of the sequence $\{X_k\}$.

Since Y is compact, f and g are regular, we have that the two sets of subgradients $\{\bigtriangledown f_{w_k}(X_k), k = 0, 1, 2, \cdots\}$ and $\{\bigtriangledown g_{w'_k}(X_k), k = 0, 1, 2, \cdots\}$ are bounded. That is, there exist some positive constants C_0 and D_0 such that, with probability 1, $\| \bigtriangledown f_{w_k}(X_k)\| \leq C_0$ and $\| \bigtriangledown g_{w'_k}(X_k)\| \leq D_0, \forall k \geq 0$.

Proposition 4.6. *Let Assumptions 3.1, 3.3, and 4.5 hold. Then the sequence $\{X_k\}$ generated by RDA (9)-(11) converges to an optimal solution in $Y^*(\alpha,\lambda)$ with probability 1.*

Proof. Since Y is compact and g is continuous, there exists M such that $|g(y)| \leq M$ for all $y \in Y$. We obtain for all k and $y \in Y^*(\alpha,\lambda)$, as in the proof of Proposition 4.2, by applying Lemma 4.1 to the case with $m = 1$:

$$
\begin{aligned}
E\{\|X_{k+1}-y\|^2|\mathscr{F}_k\} \leq\ & \|X_k-y\|^2 - 2(1-\alpha)\frac{b_k}{n}(g(X_k)-g(y)) \\
& +(\alpha a_k C_0 + (1-\alpha)b_k D_0)^2 - 2\alpha\frac{a_k}{m}(f(X_k)-f(y)) \\
\leq\ & \|X_k-y\|^2 + (\alpha a_k C_0 + (1-\alpha)b_k D_0)^2 \\
& -\frac{2a_k}{m}[(\alpha f+\lambda(1-\alpha)g)(X_k) - (\alpha f+\lambda(1-\alpha)g)(y)] \\
& +2|\frac{b_k}{n} - \lambda\frac{a_k}{m}|\cdot|g(X_k)-g(y)| \\
\leq\ & \|X_k-y\|^2 + 4M|\frac{b_k}{n} - \lambda\frac{a_k}{m}| + (\alpha a_k C_0 + (1-\alpha)b_k D_0)^2 \\
& -\frac{2a_k}{m}[(\alpha f+\lambda(1-\alpha)g)(X_k) - (\alpha f+\lambda(1-\alpha)g)(y)]
\end{aligned}
$$

where $\mathscr{F}_k = \{X_0, X_1, \cdots, X_k\}$.

Two definitions are in order. A sample path is a sequence of $\{X_k\}$. For each $y^* \in Y^*(\alpha,\lambda)$, let Ω_{y^*} denote the set containing all sample paths $\{X_k\}$ such that

$$2\sum_{k=0}^{\infty}\frac{a_k}{m}[(\alpha f+\lambda(1-\alpha)g)(X_k) - (\alpha f+\lambda(1-\alpha)g)(y^*)] \leq K < +\infty,$$

and that $\{\|X_k - y^*\|\}$ converges. We need the following.

Supermartingale Convergence Theorem (Theorem 3.1 in Nedić & Bertsekas, 2001). *Let X_k, Z_k and W_k, $k = 0,1,2,\cdots$, be three sequences of random variables and let $\mathscr{F}_k$, $k = 0,1,2,\cdots$, be sets of random variables such that $\mathscr{F}_k \subset \mathscr{F}_{k+1}$ for all k. Suppose that:*

(a) The random variables X_k, Z_k, and W_k are nonnegative, and are functions of the random variables in $\mathscr{F}_k$.

(b) For each k, we have $E\{X_{k+1}|\mathscr{F}_k\} \leq X_k - Z_k + W_k$.

(c) There holds $\sum_{k=0}^{\infty} W_k < \infty$.

Then, we have $\sum_{k=0}^{\infty} Z_k < \infty$, and the sequence X_k converges to a nonnegative random variable X, with probability 1.

By the supermartingale convergence theorem, for each $y^* \in Y^*(\alpha,\lambda)$, we have that Ω_{y^*} is a set of probability 1. Let $\{v_i\}$ be a countable subset of the relative interior $relint(Y^*(\alpha,\lambda))$ that is dense in $Y^*(\alpha,\lambda)$. Define $\Omega = \bigcap_{i=1}^{\infty} \Omega_{v_i}$. Then Ω has probability 1 since

$$Prob(\bigcup_{i}^{\infty} \bar{\Omega}_{v_i}) \leq \sum_{i=1}^{\infty} Prob(\bar{\Omega}_{v_i}) = 0.$$

For each sample path in Ω, the sequence $\|X_k - v_i\|$ converges so that $\{X_k\}$ is bounded. By

$$2\sum_{k=0}^{\infty} \frac{a_k}{m}[(\alpha f + \lambda(1-\alpha)g)(X_k) - (\alpha f + \lambda(1-\alpha)g)(y)] \leq K < +\infty,$$

we have

$$\lim_{k\to\infty}[(\alpha f + \lambda(1-\alpha)g)(X_k) - (\alpha f + \lambda(1-\alpha)g)(y)] = 0.$$

Otherwise, if there exist $\delta > 0$ such that for all k,

$$(\alpha f + \lambda(1-\alpha)g)(X_k) - (\alpha f + \lambda(1-\alpha)g)(y) > \delta,$$

then we have

$$2\sum_{k=0}^{\infty} \frac{a_k}{m}[(\alpha f + \lambda(1-\alpha)g)(X_k) - (\alpha f + \lambda(1-\alpha)g)(y)] > \frac{\delta}{m}\sum_{k=0}^{\infty} a_k = +\infty,$$

which is impossible.

Continuity of $\alpha f + \lambda(1-\alpha)g$ implies that all the limit points of $\{X_k\}$ are belong to $Y^*(\alpha,\lambda)$. Since $\{v_i\}$ is a dense subset of Y^* and $\|X_k - v_i\|$ converges, it follows that $\{X_k\}$ cannot have more than one limit point, so it must converge to some vector $y \in Y^*(\alpha,\lambda)$. This completes the proof of Proposition 4.6. □

Remark. In the proof above, we choose y^* in $Y^*(\alpha,\lambda)$. With such a choice, there is no guarantee that the term

$$2\alpha\frac{a_k}{m}(f(X_k)-f(y^*)) - 2(1-\alpha)\frac{b_k}{n}(g(X_k)-g(y^*))$$

is nonnegative for all k, as required in supermartingale convergence theorem. This is why we need Assumption 3.3 because we need to take the term $4M|\frac{b_k}{n} - \lambda\frac{a_k}{m}|$ out of it.

Theorem 4.7. *Let Assumptions 3.1, 3.3 and 4.5 hold. Assume that* $(1-\alpha)\lambda = \alpha$ *and* $\alpha \in (0,1)$. *Then the sequence* $\{X_k\}$ *generated by RDA (9)-(11) converges to an optimal solution in* Y^* *with probability 1.*

Proof. It follows from Proposition 4.6 and the definition of $Y^*(\alpha,\lambda)$, which coincides with Y^* when the condition $(1-\alpha)\lambda = \alpha$ holds. This completes the proof. □

Remark. Under the assumptions in Theorem 4.7, and also assume that the limit $\lim_{k\to\infty}\frac{b_k}{a_k}$ exists and equals 1. Moreover, assume that $m = 2n$. Then $\lambda = 2$ because $\lambda = \frac{m}{n}\lim_{k\to\infty}\frac{b_k}{a_k}$, by Proposition 4.10 in Section 4.4. Theorem 4.7 shows that the price process $\{X_k\}$ converges to a Walrasian equilibrium of the underlying economy, with probability 1, if $\alpha = \frac{2}{3}$. Once λ, n or m has a change, we must also change α accordingly in order to achieve a Walrasian equilibrium. Otherwise the price process $\{X_k\}$ still converges but it may not converge to a Walrasian equilibrium of the underlying economy. Once again, Assumption 3.3 is the key. If it does not hold, then Theorem 4.7 fails again, as shown in Xu et al. (2015, 2016).

4.3. A Numerical Example

The following example has been also studied by Xu et al. (2014, 2015, 2016). Now we use this example to illustrate why Theorems 4.4 and 4.7 may fail to converge to a Walrasian equilibrium of the underlying economy if the condition $\lambda = \frac{\alpha}{1-\alpha}$ does not hold. There are three sellers, $i = 1,2,3$, with each seller $i = 1,2,3$ an initial endowment of $(i+1)$ units of an identical (divisible) good. There are five buyers $j = 1,2,\cdots,5$, each buyer j's consumer's surplus or profit function $g_j : R_+ \to R$ is obtained from

$$g_j(y) = \max_{q\geq 0} u_j(q) - qy,$$

where $u_j : [0,\infty) \to R_+$ is j's utility function given by $u_j(q) = (j+1) + 2\sqrt{(j+1)q}$. The supply curve for each seller is $S_i(y) = [0, i+1]$ for $y = 0$ and $S_i(y) = i+1$ for $y > 0$, $i = 1,2,3$. The demand curve $D_j(y) = q_j^*$, where $u_i'(q_j^*) = y$ for $y >> 0$, $j = 1,2,\cdots,5$. In this example, we can set $f_i(y) = (i+1)y$ for $i \in I = \{1,2,3\}$ and $g_j(y) = (j+1) + \frac{j+1}{y}$ for $j \in J = \{1,2,\cdots,5\}$ so that $D_j(y) = q_j^* = \frac{j+1}{y^2}$. Thus, the equilibrium price equals $y^* = \sqrt{\frac{20}{9}} = 1.49$.

Note that the three sellers and five buyers have no knowledge where the equilibrium price 1.49 is. Neither do they know the total demand and supply. Each of them just submits their bid and ask based on their own private information. The two double auctions acting as a clearinghouse integrate individually "dispersed and incomplete information" (Hayek, 1945) into prices. While the equilibrium price equals $y^* = 1.49$, the price, to which the price process under RDA converges, is $\sqrt{\frac{\lambda(1-\alpha)}{\alpha}}y^*$. Thus, if $\alpha = 0.1$ and $\lambda = 1$, the price process under RDA converges to the price $3y^*$, 200% higher than the original Walrasian equilibrium price y^*. If $\alpha = 0.5$ while $\lambda = 9$, the price process converges to $3y^*$ as well. A crash price is also possible. For example, with $\alpha = 0.8$ and $\lambda = 1$, the price process converges to $\frac{1}{2}y^*$, 50% lower than the equilibrium price of the original economy. Xu et al. (2015, 2016) provide simulations that are consistent with these theoretical predictions under Theorem 4.7.

For Theorem 4.4, we may assume that there are five sellers to the above example. Now the original Walrasian equilibrium becomes $y^* = 1$. With $\alpha = 0.1$ and $\lambda = 1$, the CDA generates a sequence of prices that converges to 3, 200% higher than the original equilibrium price $y^* = 1$. Note that the initial distribution of endowments among the five agents is not that important.

Assume that $\lim_k \frac{b_k}{a_k}$ exists and equals 1. There are some λ such that the λ conditions in Assumptions 3.2 and 3.3 hold. Then we know that $\lambda = \lim_k \frac{b_k}{a_k} = 1$ for CDA and RDA for five sellers and five buyers economy, the two auctions generate price sequences that converge to the equilibrium price 1 when $\alpha = \frac{1}{2}$. Any other α can result in a price that is either higher or lower than the equilibrium price 1.

Consider the three sellers and five buyers economy with RDA again. Also assume that $\lim_k \frac{b_k}{a_k}$ exists and equals 1 and there are some λ such that Assumption 3.3 holds. Then $\lambda = \frac{3}{5}$. In order for RAD to achieve the equilibrium price 1.49, one must have α equal $\frac{3}{8}$. Any other α will result in a price that is either higher or lower than the equilibrium price 1.49. Therefore, whether DA can achieve an equilibrium depends on a combination of α and λ and the λ condition.

4.4. A Discussion about the λ Condition

We have seen that λ plays a key role. In Assumption 3.2, we need λ to satisfy the condition such that $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$. Is there such a λ for any two sequences $\{a_k\}$ and $\{b_k\}$ that satisfy Assumption 3.1? Unfortunately the answer is not always affirmative. The relative strength between ask and bid step sizes is quite subtle for CDA or RDA mechanism.

Example 4.8. Let

$$a_k = \begin{cases} \frac{1}{k}, & k \text{ is odd} \\ \\ \frac{1}{k^2}, & k \text{ is even} \end{cases}$$

$$b_k = \begin{cases} \frac{1}{k^2}, & k \text{ is odd} \\ \\ \frac{1}{k}, & k \text{ is even} \end{cases}$$

Then

$$\begin{aligned} \sum_{k=0}^{\infty} |a_k - \lambda b_k| &\geq \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} |a_k - \lambda b_k| \\ &= \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} a_k - \lambda \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} b_k \\ &= \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} \frac{1}{k} - \lambda \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} \frac{1}{k^2} = +\infty \end{aligned}$$

for any λ.

The next example shows that even if the limit $\lim_{k\to\infty} \frac{b_k}{a_k}$ exits and equals 1, there may not exist λ that satisfies Assumption 3.2.

Example 4.9. Let $a_k = \frac{1}{k^{\frac{3}{4}}}$ and $b_k = a_k(1 + a_k^{\frac{1}{3}})$. Then

$$\lim_{k\to\infty} \frac{b_k}{a_k} = \lim_{k\to\infty}(1 + \frac{1}{k^{\frac{1}{4}}}) = 1.$$

But

$$\sum_{k=0}^{\infty} |b_k - a_k| = \sum_{k=0}^{\infty} a_k^{\frac{4}{3}} = \sum_{k=0}^{\infty} \frac{1}{k} = +\infty.$$

Note that, by Proposition 4.10 below, if there exists a λ that satisfies Assumption 3.2, it can only be 1. Hence, no λ exists and satisfies Assumption 3.2 here.

The following answers what λ must be.

Proposition 4.10. *Let Assumption 3.1 hold. If* $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$ *for some* λ*, then the following must hold*

$$\liminf_{k\to\infty} \frac{b_k}{a_k} \le \lambda \le \limsup_{k\to\infty} \frac{b_k}{a_k}.$$

Proof. If there exist $\delta > 0$ and k_0 such that $\frac{b_k}{a_k} - \lambda > \delta$ for all $k \ge k_0$, then

$$\sum_{k=k_0}^{\infty} |b_k - \lambda a_k| \ge \delta \sum_{k=k_0}^{\infty} a_k = +\infty, \qquad \text{a contradiction.}$$

Hence, $\liminf_{k\to\infty} \frac{b_k}{a_k} \le \lambda$ is one necessary condition for $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$. Similarly, $\limsup_{k\to\infty} \frac{b_k}{a_k} \ge \lambda$. This completes the proof. □

Proposition 4.11. *Let Assumption 3.1 hold. If there exists a* λ *such that* $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$*, then it must be unique.*

Proof. Suppose, on the contrary, that there are two λ and λ' such that

$$\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty \text{ and } \sum_{k=0}^{\infty} |b_k - \lambda' a_k| < +\infty.$$

Then

$$|\lambda - \lambda'| \sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} |b_k - \lambda a_k| + \sum_{k=0}^{\infty} |b_k - \lambda' a_k| < +\infty.$$

But $\sum_{k=0}^{\infty} a_k = +\infty$, a contradiction. This completes the proof. □

Thus, there exists at most one λ that satisfies Assumption 3.2 for any two given sequences $\{a_k\}$ and $\{b_k\}$ satisfying Assumption 3.1. But for any given λ, there are a family of step sizes $\{a_k\}$ and $\{b_k\}$ that satisfy Assumptions 3.1 and 3.2. Let $a_k = \frac{1}{k}$ and $b_k = 2a_k + ca_k^2$. Then Assumption 3.2 is satisfied with $\lambda = 2$ for any positive finite number c and any $\{a_k\}$ that satisfies Assumption 3.1.

5. DA MECHANISM WITH STOCHASTIC NOISES

Let $f = \sum_{i=1}^{m} f_i$ and $g = \sum_{i=1}^{m} g_i$. Assume X_0 is a random initial vector. Let $\varepsilon_{i,k}$ and $\delta_{i,k}$ denote two independent random noise vectors. (DA) mechanism with stochastic noises is defined as follows.

Let $\Phi_{0,k} = X_k$. For $i = 1, 2, \cdots, m$, let

$$\begin{aligned}
\psi_{i,k} &= \Phi_{i-1,k} - a_k(h_{i,k} + \varepsilon_{i,k}),\ h_{i,k} \in \partial f_i(\Phi_{i-1,k}) && (14)\\
\varphi_{i,k} &= \Phi_{i-1,k} - b_k(\ell_{i,k} + \delta_{i,k}),\ \ell_{i,k} \in \partial g_i(\Phi_{i-1,k}) && (15)\\
\Phi_{i,k} &= P_Y(\alpha\psi_{i,k} + (1-\alpha)\varphi_{i,k}), \qquad \alpha \in [0,1]. && (16)
\end{aligned}$$

Let

$$X_{k+1} = \Phi_{m,k}. \tag{17}$$

P_Y is the Euclidean projection onto Y.

We define $\mathscr{F}_k^i$ to be the σ-algebra generated by the sequence

$$\Phi_{0,0}, \Phi_{1,0}, \cdots, \Phi_{m,0}, \cdots, \Phi_{i,k}.$$

Note that $\mathscr{F}_k^0$ is also denoted as $\mathscr{F}_k$.

Assumption 5.1. There exist deterministic scalar sequences $\{\mu_k\}$, $\{\nu_k\}$, $\{\tau_k\}$ and $\{\sigma_k\}$ that satisfy the following inequalities for all i and k:

$$\begin{aligned} \|E[\varepsilon_{i,k}|\mathscr{F}_k^{i-1}]\| &\leq \mu_k, \qquad \|E[\delta_{i,k}|\mathscr{F}_k^{i-1}]\| \leq \tau_k; \\ E[\|\varepsilon_{i,k}\|^2|\mathscr{F}_k^{i-1}] &\leq \nu_k^2, \qquad E[\|\delta_{i,k}\|^2|\mathscr{F}_k^{i-1}] \leq \sigma_k^2. \end{aligned}$$

Note that $\mu_k \leq \nu_k$ and $\tau_k \leq \sigma_k$ for all $k = 0, 1, \cdots$. The noise terms in the assumption above are similar to those in Ram et al. (2009). Their results, however, do not apply to (DA) mechanism (14)-(17) because there are two sequences of step-sizes that interact together to determine the price iteration process $\{X_k\}$.

The following lemma is the key and its proof is provided in the Appendix.

Lemma 5.2. *Let Assumption 5.1 hold. Then the sequence $\{X_k\}$ generate by (DA) mechanism with stochastic noises (14)-(17) is such that for any step-size rule and any $y \in Y$,*

$$\begin{aligned} E[\|X_{k+1} - y\|^2|\mathscr{F}_{k-1}^m] \leq\ & \|X_k - y\|^2 - 2\alpha a_k(f(X_k) - f(y)) \\ & -2(1-\alpha)b_k(g(X_k) - g(y)) \\ & +2(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k)\sum_{i=1}^{m} E[\|\Phi_{i-1,k} - y\||\mathscr{F}_{k-1}^m] \\ & +(\alpha a_k C + (1-\alpha)b_k D + \alpha m a_k \nu_k + (1-\alpha)m b_k \sigma_k)^2, \end{aligned}$$

Note that $\mathscr{F}_{k-1}^m = \mathscr{F}_k^0$.

5.1. Main Result with Stochastic Noises

Assumption 5.3. The following holds:

$$\sum_{k=0}^{\infty} a_k\mu_k < \infty, \sum_{k=0}^{\infty} b_k\tau_k < \infty, \sum_{k=0}^{\infty} a_k^2\nu_k^2 < \infty, \sum_{k=0}^{\infty} b_k^2\sigma_k^2 < \infty.$$

Proposition 5.4. *Let Assumptions 3.1, 3.2, 5.1 and 5.3 hold. Then the sequence $\{X_k\}$ generated by (DA) mechanism with stochastic noises (14)-(17) converges to an optimal solution $y^* \in Y^*(\alpha, \lambda)$, with probability 1.*

Proof. By Lemma 5.2, for any $y^* \in Y^*(\alpha,\lambda)$, we have that

$$\begin{aligned}
&E[\|X_{k+1}-y^*\|^2|\mathscr{F}^m_{k-1}] \le \|X_k-y^*\|^2+M_k\\
&-2\alpha a_k(f(X_k)-f(y^*))-2(1-\alpha)b_k(g(X_k)-g(y^*))\\
&+2(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k)\sum_{i=1}^m E[\|\Phi_{i-1,k}-y^*\||\mathscr{F}^m_{k-1}],
\end{aligned}$$

where $M_k=(\alpha a_k C+(1-\alpha)b_k D+m\alpha a_k\nu_k+m(1-\alpha)b_k\sigma_k)^2$.

Since

$$\begin{aligned}
E[\|\Phi_{i-1,k}-y^*\||\mathscr{F}^m_{k-1}] &\le E[\|\Phi_{i-1,k}-X_k\||\mathscr{F}^m_{k-1}]+\|X_k-y^*\|\\
&\le \sum_{j=1}^{i-1}(\alpha a_k C_j+(1-\alpha)b_k D_j+\alpha a_k\nu_k\\
&\quad +(1-\alpha)b_k\sigma_k)+\|X_k-y^*\|.
\end{aligned}$$

In the second inequality above we have used Lemma 5.2.1 in the Appendix and Assumption 5.1.

Hence,

$$\begin{aligned}
2(\alpha a_k\mu_k &+ (1-\alpha)b_k\tau_k)\sum_{i=1}^m E[\|\Phi_{i-1,k}-y^*\||\mathscr{F}^m_{k-1}]\\
&\le 2(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k)\sum_{i=1}^m\{\sum_{j=1}^{i-1}(\alpha a_k C_j\\
&\quad +(1-\alpha)b_k D_j+\alpha a_k\nu_k+(1-\alpha)b_k\sigma_k)+\|X_k-y^*\|\}\\
&\le 2(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k)\sum_{i=1}^m\sum_{j=1}^{i-1}\{\alpha a_k C_j\\
&\quad +(1-\alpha)b_k D_j+\alpha a_k\nu_k+(1-\alpha)b_k\sigma_k\}\\
&\quad +m(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k)(\|X_k-y^*\|^2+1).
\end{aligned}$$

In the last inequality above we have used the inequality $a^2+1\ge 2a$.

And then

$$\begin{aligned}
E[\|X_{k+1}-y^*\|^2|\mathscr{F}^m_{k-1}] &\le (1+m(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k))\|X_k-y^*\|^2\\
&\quad +(M_k+N_k)-2\alpha a_k(f(X_k)-f(y^*))\\
&\quad -2(1-\alpha)b_k(g(X_k)-g(y^*)),
\end{aligned}$$

where

$$\begin{aligned} N_k &= 2(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \sum_{i=1}^{m} \sum_{j=1}^{i-1} \{\alpha a_k C_j + (1-\alpha) b_k D_j \\ &\quad + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k\} + m(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k). \end{aligned}$$

We need the following lemma in our proof below.

Lemma 3.2 in Ram et al. (2009): *Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots$ be a sequence of sub σ-fields of $\mathscr{F}$. Let u_k, v_k and w_k, $k = 0, 1, 2, \cdots$, be non-negative $\mathscr{F}_k$-measurable random variables and let $\{q_k\}$ be a deterministic sequence. Assume that $\sum_{k=0}^{\infty} q_k < \infty$, $\sum_{k=0}^{\infty} w_k < \infty$, and*

$$E\{u_{k+1} | \mathscr{F}_k\} \le (1+q_k) u_k - v_k + w_k$$

hold with probability 1. Then, with probability 1, the sequence $\{u_k\}$ converges to a non-negative random variable and $\sum_{k=0}^{\infty} v_k < \infty$.

To apply Lemma 3.2 in Ram et al. (2009), let $q_k = m(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k)$ and $W_k = M_k + N_k$.

Then

$$\begin{aligned} \sum_{k=0}^{\infty} q_k &= m\alpha \sum_{k=0}^{\infty} a_k \mu_k + m(1-\alpha) \sum_{k=0}^{\infty} b_k \tau_k < +\infty \\ \sum_{k=0}^{\infty} W_k &= \sum_{k=0}^{\infty} (M_k + N_k). \end{aligned}$$

Since, using $a^2 + b^2 \ge 2ab$ and Assumption 5.3,

$$\begin{aligned} \sum_{k=0}^{\infty} M_k &= \sum_{k=0}^{\infty} (\alpha a_k C + (1-\alpha) b_k D + m\alpha a_k \nu_k + m(1-\alpha) b_k \sigma_k)^2 \\ &\le 4 \sum_{k=0}^{\infty} [(\alpha a_k C)^2 + ((1-\alpha) b_k D)^2 + (m\alpha a_k \nu_k)^2 + (m(1-\alpha) b_k \sigma_k)^2] \\ &< \infty \end{aligned}$$

and, by $\mu_k \leq \nu_k$ and $\tau_k \leq \sigma_k$ in Assumption 5.1,

$$\begin{aligned}
\sum_{k=0}^{\infty} N_k &= \sum_{k=0}^{\infty} 2(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \sum_{i=1}^{m} \sum_{j=1}^{i-1} \{\alpha a_k C_j \\
&\quad + (1-\alpha) b_k D_j + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k\} + \sum_{k=0}^{\infty} q_k \\
&\leq \sum_{k=0}^{\infty} [\sum_{i=1}^{m} (\alpha a_k \nu_k + (1-\alpha) b_k \sigma_k + \alpha a_k C_i + (1-\alpha) b_k D_i)]^2 + \sum_{k=0}^{\infty} q_k \\
&\leq \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} q_k < \infty,
\end{aligned}$$

we have that $\sum_{k=0}^{\infty} W_k < \infty$.

Therefore, we get, with probability 1, the sequence $\|X_k - y^*\|^2$ converges to some non-negative random variable for every $y^* \in Y(\alpha, \lambda)$. Also with probability 1, we have

$$\sum_{k=0}^{\infty} (\alpha a_k (f(X_k) - f(y^*)) + (1-\alpha) b_k (g(X_k) - g(y^*))) < +\infty,$$

which implies that

$$\begin{aligned}
&\sum_{k=0}^{\infty} a_k [(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y^*)] \\
&\leq \sum_{k=0}^{\infty} (\alpha a_k (f(X_k) - f(y^*)) + (1-\alpha) b_k (g(X_k) - g(y^*))) \\
&+ \sum_{k=0}^{\infty} (1-\alpha) |b_k - \lambda a_k| |g(X_k) - g(y^*)| \\
&< +\infty.
\end{aligned}$$

Since Y is compact and g is continuous, the image of g is bounded. Assume $\exists M > 0$ such that $|g(y)| \leq M$ for all $y \in Y$. Then

$$\sum_{k=0}^{\infty} (1-\alpha) |b_k - \lambda a_k| |g(X_k) - g(y^*)| \leq 2M(1-\alpha) \sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty.$$

Since $\sum_{k=0}^{\infty} a_k = +\infty$, then

$$\liminf_{k \to \infty} (\alpha f + (1-\alpha)\lambda g)(X_k) = (\alpha f + (1-\alpha)\lambda g)(y^*),$$

with probability 1.

By considering a sample path for which

$$\liminf_{k\to\infty}(\alpha f+(1-\alpha)\lambda g)(X_k)=(\alpha f+(1-\alpha)\lambda g)(y^*)$$

and $\|X_k-y^*\|^2$ converges for any y^*, we conclude that the sample sequence must converge to some y^* in view of continuity of f. Hence, the sequence $\{X_k\}$ converges to some optimal solution in $Y^*(\alpha,\lambda)$ with probability 1. This completes the proof of Proposition 5.4x . □

Remark. Strictly speaking, the term $2\alpha a_k(f(X_k)-f(y^*))+2(1-\alpha)b_k[g(X_k)-g(y^*)]$ may not be nonnegative so that we cannot directly apply Lemma 3.2. But this will not cause a problem as long as Assumptions 3.1 and 3.2 are satisfied because we can always follow the proof of Proposition 4.6 to remove the term $4M|\frac{b_k}{n}-\lambda\frac{a_k}{m}|$ from it.

Immediately we obtain from Proposition 5.4 the following.

Theorem 5.5. *Let Assumptions 3.1, 3.2, 5.1 and 5.3 hold. Assume that* $\lambda=\frac{\alpha}{1-\alpha}$, $\alpha\in(0,1)$. *Then the sequence* $\{X_k\}$ *generated by the DA mechanism with stochastic noises (14)-(17) converges to an optimal solution in* Y^*, *with probability 1.*

5.2. RDA Mechanism with Stochastic Noises

Recall that w_k is a random variable taking equiprobable values from the set $\{1,2,\cdots,m\}$ and w_k' is a random variable taking equiprobable values from the set $\{1,2,\cdots,n\}$. Also recall that $h_{w_k}(X_k)\in\partial f_{w_k}(X_k)$ and $\ell_{w_k'}(X_k)\in\partial g_{w_k'}(X_k)$, where if w_k takes a value j, then the vector $\partial f_{w_k}(X_k)$ is $\partial f_j(X_k)$, similarly for g.

Our sequence $\{X_k\}$ is generated by (RDA) mechanism with stochastic noises as below.

Given X_k, let

$$\psi_{k+1}=X_k-a_k(h_{w_k}(X_k)+\varepsilon_{w_k,k}),\qquad h_{w_k}(X_k)\in\partial f_{w_k}(X_k) \tag{18}$$

and

$$\varphi_{k+1}=X_k-b_k(\ell_{w_k'}(X_k)+\delta_{w_k',k}),\qquad \ell_{w_k'}(X_k)\in\partial g_{w_k'}(X_k). \tag{19}$$

And set

$$X_{k+1} = P_Y(\alpha\psi_{k+1} + (1-\alpha)\varphi_{k+1}), \qquad \alpha \in [0,1]. \tag{20}$$

P_Y is the Euclidean projection onto Y.

We define $\mathscr{F}_k$ to be the σ-field generated by $X_0, X_1, \cdots, X_k$.

Assumption 5.8. The sequence $\{w_k\}(\{w'_k\})$ is a sequence of independent random variables, each uniformly distributed over the set $\{1,2,\cdots,m\}(\{1,2,\cdots,n\})$. Furthermore, the two sequences $\{w_k\}$ and $\{w'_k\}$ are independent of the sequence $\{X_k\}$.

Proposition 5.9. *Let Assumptions 3.1, 3.3, 5.1, 5.3, and 5.8 hold. Then the sequence $\{X_k\}$ generated by (RDA) with randomization and stochastic noises (18)-(20) converges to an optimal solution in $Y^*(\alpha,\lambda)$, with probability 1.*

Proof. The proof is similar to those of Proposition 4.6 and Proposition 5.4 and thus omitted. □

The following result follows from Proposition 5.9 and the definition of $Y^*(\alpha,\lambda)$, which is the same as Y^* when the equality $\lambda = \frac{\alpha}{1-\alpha}$ holds.

Theorem 5.10. *Let Assumptions 3.1, 3.3, 5.1, 5.3, and 5.8 hold. Assume that $\lambda = \frac{\alpha}{1-\alpha}$, $\alpha \in (0,1)$. Then the sequence $\{X_k\}$ generated by (RDA) mechanism with stochastic noises (18)-(20) converges to an optimal solution in Y^*, with probability 1.*

6. CONCLUDING REMARKS

This paper studies two dynamic double auctions and examines the question of whether the price processes they generate can converge to a Walrasian equilibrium of the underlying economy. We show that the weight α and the λ condition are important for the convergence of these price processes. With the right combinations of α and λ, the price processes generated by the two auctions converge to a Walrasian equilibrium of the underlying economy. If the combination is not right, the price processes may generate a bubble or crash. Numerical examples show that such a bubble or crash can reach an enormous level, as shown in Xu et al. (2015, 2016), which provide extensions of the convergence results presented in Ma & Li (2011) to more complicated environments. Our results and those in the literature imply that the form of double auctions does matter very much for the price determination of an asset or a good traded in an exchange market. Because human emotion such as fear and greed may affect the two parameters, our results also shed some important light on how human emotion may impact the price of an asset in an exchange market that uses double auctions as clearinghouses.

In a laissez-faire economy, private information can be successively incorporated into the price of a good through individual decisions of what to buy or sell. Without knowing what may be the price at equilibrium for a good, the market via an invisible hand can reach an equilibrium. Such a view is the foundation for economic analyses based on equilibrium. The incremental subgradient method in Nedić & Bertsekas (2001) can be used to show how this may be done in theory for a quasilinear economy, with some intervention from a central authority by setting the step size rules properly. Such an approach is especially important for market mechanisms since every individual has a piece of private information while the market equilibrium prices must reflect all relevant private information. Chen et al. (2016) provide a different approach for a totally uncoordinated and decentralized market, in which every firm and every worker can form a matching pair randomly and seek opportunities to improve their individual positions. They prove that without any clearinghouse or coordination, starting with any matching and any salary scheme system, stable or not, a natural decentralized random matching process converges to a Walrasian equilibrium with probability one in finite time. It remains open to question if there is a good way to integrate the two approaches.

The study of the competitive efficiency of a DA mechanism started with experiments for an identical good in V. L. Smith (1962, 1965) where an artificial market was created with competitive equilibrium unknown to the buyers and sellers. In these experiments the DA mechanism converged quickly to a neighbor of the competitive equilibrium, even with a few participants. A great number of experiments have been conducted since then and a similar result has been obtained (Friedman & Rust, 1993). In recent years the competitive efficiency of a DA mechanism has been retested in experiments with more complicated environments, which are deliberately designed to be a proxy of an exchange market. V. L. Smith et al. (1988) show that both bubbles and crashes can be generated by a DA mechanism under these environments. Therefore, the allocative efficiency of a DA mechanism is a complicated issue. We show in this paper that the efficiency of our two DA mechanisms depends on the relative strength of the two step-sizes of the bid and the ask through a parameter λ and the weight how these bids and asks enter the price process. The two parameters may be considered as two "steering" factors because they act just like a steering in a vehicle; a different combination of the two directs the price process to different places.

Our study of the two DA mechanisms is applicable to a market where there are potentially a large number of agents and a large number of assets. The primary task of our paper is to provide an explanation of the price determination of a good. Our results are closely related to those obtained with the incremental subgradient method in Nedić & Bertsekas (2001) and Ram et al. (2009). Because the problem $\mathscr{P}$ has so many other applications (Bertsekas, 2009, 2012), our DA mechanisms provide an alternative explanation of how an optimal solution can be approached for those environments (e.g., distributed and neural networks).

7. APPENDIX

This appendix contains the proof of Lemma 5.2 which is divided into several steps.

Proof of Lemma 5.2. By non-expansive property of projection,

$$
\begin{aligned}
\|\Phi_{i,k}-y\|^2 &\leq \|\alpha\psi_{i,k}+(1-\alpha)\varphi_{i,k}-y\|^2 \\
&= \|\alpha(\psi_{i,k}-y)+(1-\alpha)(\varphi_{i,k}-y)\|^2 \\
&= \alpha^2\|\psi_{i,k}-y\|^2+(1-\alpha)^2\|\varphi_{i,k}-y\|^2+2\alpha(1-\alpha)\langle\psi_{i,k}-y,\varphi_{i,k}-y\rangle \\
&= \alpha^2\|\Phi_{i-1,k}-y-a_kh_{i,k}-a_k\varepsilon_{i,k}\|^2 \\
&\quad +(1-\alpha)^2\|\Phi_{i-1,k}-y-b_k\ell_{i,k}-b_k\delta_{i,k}\|^2 \\
&\quad +2\alpha(1-\alpha)\langle\Phi_{i-1,k}-y-a_kh_{i,k}-a_k\varepsilon_{i,k},\Phi_{i-1,k}-y-b_k\ell_{i,k}-b_k\delta_{i,k}\rangle \\
&= \alpha^2\|\Phi_{i-1,k}-y-a_kh_{i,k}\|^2+\alpha^2a_k^2\|\varepsilon_{i,k}\|^2 \\
&\quad -2\alpha^2\langle\Phi_{i-1,k}-y-a_kh_{i,k},a_k\varepsilon_{i,k}\rangle+2\alpha(1-\alpha)a_kb_k\langle\varepsilon_{i,k},\delta_{i,k}\rangle \\
&\quad +(1-\alpha)^2\|\Phi_{i-1,k}-y-b_k\ell_{i,k}\|^2+(1-\alpha)^2b_k^2\|\delta_{i,k}\|^2 \\
&\quad -2(1-\alpha)^2\langle\Phi_{i-1,k}-y-b_k\ell_{i,k},b_k\delta_{i,k}\rangle \\
&\quad +2\alpha(1-\alpha)\langle\Phi_{i-1,k}-y-a_kh_{i,k},\Phi_{i-1,k}-y-b_k\ell_{i,k}\rangle \\
&\quad -2\alpha(1-\alpha)\langle\Phi_{i-1,k}-y-a_kh_{i,k},b_k\delta_{i,k}\rangle \\
&\quad -2\alpha(1-\alpha)\langle\Phi_{i-1,k}-y-b_k\ell_{i,k},a_k\varepsilon_{i,k}\rangle \\
&= \|\alpha(\Phi_{i-1,k}-y-a_kh_{i,k})+(1-\alpha)(\Phi_{i-1,k}-y-b_k\ell_{i,k})\|^2 \\
&\quad +\|\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}\|^2 \\
&\quad -2\alpha\langle\Phi_{i-1,k}-y-a_kh_{i,k},\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}\rangle \\
&\quad -2(1-\alpha)\langle\Phi_{i-1,k}-y-b_k\ell_{i,k},\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}\rangle \\
&= \|\Phi_{i-1,k}-y\|^2-2\alpha a_k\langle h_{i,k},(\Phi_{i-1,k}-y)\rangle \\
&\quad -2(1-\alpha)b_k\langle\ell_{i,k},(\Phi_{i-1,k}-y)\rangle \\
&\quad +\|\alpha a_kh_{i,k}+(1-\alpha)b_k\ell_{i,k}\|^2+\|\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}\|^2 \\
&\quad -2\alpha\langle\Phi_{i-1,k}-y-a_kh_{i,k},\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}\rangle \\
&\quad -2(1-\alpha)\langle\Phi_{i-1,k}-y-b_k\ell_{i,k},\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}\rangle
\end{aligned}
$$

Taking conditional expectations with respect to the σ-field $\mathscr{F}_k^{i-1}$ leads to

$$\begin{aligned}
E[\|\Phi_{i,k}-y\|^2|\mathscr{F}_k^{i-1}] &\leq \{\|\Phi_{i-1,k}-y\|^2-2\alpha a_k\langle h_{i,k},(\Phi_{i-1,k}-y)\rangle \\
&\quad -2(1-\alpha)b_k\langle \ell_{i,k},(\Phi_{i-1,k}-y)\rangle+\|\alpha a_k h_{i,k}+(1-\alpha)b_k\ell_{i,k}\|^2\} \\
&\quad +\{E[\|\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}\|^2|\mathscr{F}_k^{i-1}] \\
&\quad -2\alpha\langle\Phi_{i-1,k}-y-a_kh_{i,k},E[\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}|\mathscr{F}_k^{i-1}]\rangle \\
&\quad -2(1-\alpha)\langle\Phi_{i-1,k}-y-b_k\ell_{i,k}, \\
&\quad E[\alpha a_k\varepsilon_{i,k}+(1-\alpha)b_k\delta_{i,k}|\mathscr{F}_k^{i-1}]\rangle\} \\
&= I+II.
\end{aligned}$$

Consider II first. We have that, by Assumption 5.1,

$$\begin{aligned}
II &\leq (\alpha a_k\nu_k+(1-\alpha)b_k\sigma_k)^2+2\alpha(\|\Phi_{i-1,k}-y\|+a_k\|h_{i,k}\|)(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k) \\
&\quad +2(1-\alpha)(\|\Phi_{i-1,k}-y\|+b_k\|\ell_{i,k}\|)(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k) \\
&= (\alpha a_k\nu_k+(1-\alpha)b_k\sigma_k)^2+2\|\Phi_{i-1,k}-y\|(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k) \\
&\quad +2\alpha a_kC_i(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k)+2(1-\alpha)b_kD_i(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k).
\end{aligned}$$

Now consider I. Since $h_{i,k}\in\partial f_i(\Phi_{i-1,k})$ and $\ell_{i,k}\in\partial g_i(\Phi_{i-1,k})$ so that

$$\langle h_{i,k},(y-\Phi_{i-1,k})\rangle\leq f_i(y)-f_i(\Phi_{i-1,k})$$

and

$$\langle \ell_{i,k},(y-\Phi_{i-1,k})\rangle\leq g_i(y)-g_i(\Phi_{i-1,k}),$$

we have that

$$\begin{aligned}
I &\leq \|\Phi_{i-1,k}-y\|^2-2\alpha a_k(f_i(\Phi_{i-1,k})-f_i(y))-2(1-\alpha)b_k(g_i(\Phi_{i-1,k})-g_i(y)) \\
&\quad +\|\alpha a_kC_i+(1-\alpha)b_kD_i\|^2.
\end{aligned}$$

Taking the expectations conditional on $\mathscr{F}_{k-1}^m=\mathscr{F}_k^0$, we obtain from $I+II$ that

$$\begin{aligned}
E[\|\Phi_{i,k}-y\|^2|\mathscr{F}_{k-1}^m] &\leq E[\|\Phi_{i-1,k}-y\|^2|\mathscr{F}_{k-1}^m]-2\alpha a_k(f_i(X_k)-f_i(y)) \\
&\quad -2(1-\alpha)b_k(g_i(X_k)-g_i(y)) \\
&\quad +2E[\|\Phi_{i-1,k}-y\||\mathscr{F}_{k-1}^m](\alpha a_k\mu_k+(1-\alpha)b_k\tau_k)+M_{i,k},
\end{aligned}$$

where

$$\begin{aligned}
M_{i,k} &= (\alpha a_kC_i+(1-\alpha)b_kD_i)^2+(\alpha a_k\nu_k+(1-\alpha)b_k\sigma_k)^2 \\
&\quad +2\alpha a_kC_i(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k)+2(1-\alpha)b_kD_i(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k) \\
&\quad +2\alpha a_kE[\|f_i(\Phi_{i-1,k})-f_i(X_k)\||\mathscr{F}_{k-1}^m] \\
&\quad +2(1-\alpha)b_kE[\|g_i(\Phi_{i-1,k})-g_i(X_k)\||\mathscr{F}_{k-1}^m].
\end{aligned}$$

Note that $\Phi_{0,k} = X_k$ and $\Phi_{m,k} = X_{k+1}$. Taking sum over $i = 1, 2, \cdots, m$, we have that

$$\begin{aligned} E[\|X_{k+1} - y\|^2 | \mathscr{F}^m_{k-1}] \quad \leq \quad & \|X_k - y\|^2 - 2\alpha a_k (f(X_k) - f(y)) \\ & -2(1-\alpha) b_k (g(X_k) - g(y)) + \sum_{i=1}^m M_{i,k} \\ & +2(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \sum_{i=1}^m E[\|\Phi_{i-1,k} - y\| | \mathscr{F}^m_{k-1}]. \end{aligned}$$

Next we consider $\sum_{i=1}^m M_{i,k}$.

Lemma 5.2.1. *We claim that*

$$\|\Phi_{i-1,k} - X_k\| \leq \sum_{j=1}^{i-1} [\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \|\varepsilon_{j,k}\| + (1-\alpha) b_k \|\delta_{j,k}\|].$$

Proof of Lemma 5.2.1. We prove by induction.

$$\begin{aligned} \|\Phi_{i,k} - X_k\| \quad \leq \quad & \|(\alpha \psi_{i,k} + (1-\alpha)\varphi_{i,k}) - X_k\| \\ \leq \quad & \alpha \|\psi_{i,k} - X_k\| + (1-\alpha)\|\varphi_{i,k} - X_k\| \\ = \quad & \alpha \|\Phi_{i-1,k} - a_k h_{i,k} - a_k \varepsilon_{i,k} - X_k\| \\ & +(1-\alpha)\|\Phi_{i-1,k} - b_k \ell_{i,k} - b_k \delta_{i,k} - X_k\| \\ \leq \quad & \|\Phi_{i-1,k} - X_k\| + \alpha a_k \|h_{i,k}\| + (1-\alpha) b_k \|l_{i,k}\| \\ & +\alpha a_k \|\varepsilon_{i,k}\| + (1-\alpha) b_k \|\delta_{i,k}\|. \end{aligned}$$

By induction, we get that

$$\|\Phi_{i,k} - X_k\| \leq \sum_{j=1}^{i} [\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \|\varepsilon_{j,k}\| + (1-\alpha) b_k \|\delta_{j,k}\|].$$

This completes the proof of Lemma 5.2.1. □

We now continue the proof of Lemma 5.2 and have

$$\begin{aligned} E[\|f_i(\Phi_{i-1,k}) - f_i(X_k)\| | \mathscr{F}^m_{k-1}] \quad \leq \quad & E[C_i \sum_{j=1}^{i-1} (\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \|\varepsilon_{j,k}\| \\ & +(1-\alpha) b_k \|\delta_{j,k}\|) | \mathscr{F}^m_{k-1}] \\ \leq \quad & C_i \sum_{j=1}^{i-1} \{\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \nu_k \\ & +(1-\alpha) b_k \sigma_k\}; \end{aligned}$$

and

$$\begin{aligned}
E[\|g_i(\Phi_{i-1,k}) - g_i(X_k)\| | \mathscr{F}^m_{k-1}] &\leq E[D_i \sum_{j=1}^{i-1} \{\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \|\varepsilon_{j,k}\| \\
&\quad +(1-\alpha) b_k \|\delta_{j,k}\|\} | \mathscr{F}^m_{k-1}] \\
&\leq D_i \sum_{j=1}^{i-1} \{\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \nu_k \\
&\quad +(1-\alpha) b_k \sigma_k\}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=1}^m M_{i,k} &\leq \sum_{i=1}^m (\alpha a_k C_i + (1-\alpha) b_k D_i)^2 + m(\alpha a_k \nu_k + (1-\alpha) b_k \sigma_k)^2 \\
&\quad +2 \sum_{i=1}^m (\alpha a_k C_i + (1-\alpha) b_k D_i)(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \\
&\quad +2 \sum_{i=1}^m (\alpha a_k C_i + (1-\alpha) b_k D_i) \sum_{j=1}^{i-1} \{\alpha a_k C_j + (1-\alpha) b_k D_j \\
&\quad +\alpha a_k \nu_k + (1-\alpha) b_k \sigma_k\} \text{ (since } \mu_k \leq \nu_k \text{ and } \tau_k \leq \sigma_k) \\
&\leq \sum_{i=1}^m (\alpha a_k C_i + (1-\alpha) b_k D_i + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k)^2 \\
&\quad +2 \sum_{i=1}^m (\alpha a_k C_i + (1-\alpha) b_k D_i + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k) \times \\
&\quad \sum_{j=1}^{i-1} (\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k) \\
&= (\sum_{i=1}^m (\alpha a_k C_i + (1-\alpha) b_k D_i + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k))^2 \\
&= (\alpha a_k C + (1-\alpha) b_k D + \alpha m a_k \nu_k + (1-\alpha) m b_k \sigma_k)^2.
\end{aligned}$$

This completes the proof of Lemma 5.2. □

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SCHOOL CHOICE UNDER COMPLETE INFORMATION: AN EXPERIMENTAL STUDY

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ABSTRACT

We present an experimental study of three school choice mechanisms under complete information, using the designed environment in Chen & Sönmez (2006). We find that the top trading cycles (TTC) mechanism outperforms both the Gale-Shapley deferred acceptance (DA) and the Boston immediate acceptance (BOS) mechanism in terms of truth-telling and efficiency, whereas DA is more stable than either TTC or BOS. Compared to the incomplete information setting in Chen & Sönmez (2006), the performance of both TTC and BOS improves with more information, whereas that of DA does not.

Keywords: School choice, experiment, mechanism design.

JEL Classification Numbers: C78, C92, D82.

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1. INTRODUCTION

School choice mechanisms affect the educational experiences and outcomes of many students around the world. The past two decades have witnessed major reforms in this domain. In designing practical markets, institutions have relied on economic theory, computation, and controlled laboratory experiments (Roth, 2002). For example, shortly after the publication of Abdulkadiroğlu & Sönmez (2003), New York City high schools replaced their previous mechanism with a variant of the student-proposing deferred-acceptance (DA) mechanism (Gale & Shapley, 1962; Abdulkadiroğlu, Pathak, & Roth, 2005). In the school choice reforms in Boston, matching theorists directly influenced the adoption of the DA mechanism (Abdulkadiroğlu, Pathak, Roth, & Sönmez, 2005). In this case, experimental data helped persuade the Boston public school authorities to switch from the Boston immediate acceptance mechanism (BOS) to DA (Chen & Sönmez, 2006). In parallel with these reforms, policy makers in Chicago and in New England independently abandoned the Boston mechanism and adopted versions of the DA (Pathak & Sönmez, 2013). Laboratory experiments provide the first data for institutional redesigns when field data is not yet available. Even when field data is available, lab experiments compare the performance of different mechanisms at a level of detail that cannot be obtained from field data.

In an incomplete information setting, Chen & Sönmez (2006) present the first experimental study of three well-known school choice mechanisms, BOS, DA and the top trading cycles (TTC). They find that DA outperforms TTC in truthful preference revelation, despite the strategy-proofness of both mechanisms. Furthermore, they show that TTC does not outperform DA in efficiency, although theoretically it is efficient whereas DA is not. Among the three mechanisms, BOS performs the worst in terms of truthful preference revelation and efficiency. While a stability comparison is not presented in Chen & Sönmez (2006), using the same experimental setting, Calsamiglia et al. (2010) find that DA is more stable than TTC, which in turn is weakly more stable than BOS.

In this paper, we ask two questions. First, how do the three school choice mechanisms perform in a complete information setting? Second, how does information provision affect the performance of each mechanism?

To answer these questions, we run an experiment under the complete information setting, using the same set of parameters as in the designed environment in Chen & Sönmez (2006), which enables us to compare our results with their earlier study. Our results show that, in the complete information setting, TTC outperforms DA, which in turn outperforms BOS in truth-telling. Consistent with theory, TTC outperforms both DA and BOS in efficiency, whereas DA and BOS generate similar efficiency levels. In terms of stability, DA outperforms TTC and BOS by a large margin, whereas BOS and TTC achieve the same level of stability. In summary, information affects the performance of the three mechanisms in different ways. More information improves the performance of both TTC and BOS, but does not change either efficiency or stability under DA.

Our findings have clear policy implications. A market designer who values efficiency over stability should adopt TTC and encourage information provision. In comparison, a market designer who values stability over efficiency should adopt DA.

The rest of the paper is organized as follows. Section 2 reviews the experimental school choice literature. Section 3 introduces the school choice problem and summarizes the theoretical properties of the three mechanisms. Section 4 presents the experimental design. Section 5 summarizes the main results. Section 6 concludes the paper. Instructions for experiment are given in the Supplement before references.

2. LITERATURE REVIEW

With the development of matching theory in the school choice domain (Abdulkadiroğlu & Sönmez, 2003), a growing number of laboratory experiments have tested the performance of school choice mechanisms as well as participant behavior under different incentives.

In the first experimental study of school choice mechanisms, Chen & Sönmez (2006) deploy a setting with 36 students and 7 schools per match. Schools differ in their capacity, location, quality and strength. Student preferences are induced in two different environments. One is the designed environment, where student preferences are generated based on their distance to the school and their interests. To provide a robustness check, a random environment is used where student preferences are randomly generated. They use an incomplete information setting, where students only know their own preference, their district schools, and school capacities. They find that DA outperforms TTC in truthful preference revelation, whereas TTC does not outperform DA in efficiency. Of the three mechanisms, BOS performs the worst in truthful preference revelation and efficiency. We use the same set of parameters as in their designed environment but with complete information.

Closely related to our study, Pais & Pintér (2008) investigate the impact of different information conditions on the performance of the same three school choice mechanisms, but with a relatively small group size (five students and three schools per match). They find that under complete information, TTC outperforms both BOS and DA in terms of truthful preference revelation and efficiency, while the three mechanisms perform similarly in stability. Overall, they find that TTC is less sensitive to the amount of information provided to the participants.

Subsequent experimental studies have examined the impact of a limit on the number of schools in the rank order list (Calsamiglia et al., 2010), participant risk attitude and preference intensities (Klijn et al., 2013), peer information sharing in networks (Ding & Schotter, 2016) and intergenerational advice (Ding & Schotter, 2015) on participant behavior.

Two other studies investigate the effects of information conditions on individual behavior and mechanism performance. In an interim Bayesian setting, Featherstone & Niederle (2013) observe that BOS achieves higher efficiency than DA when school priorities involve ties which are broken randomly. More recently, Chen & He (2016) study endogenous information acquisition under BOS and DA in the school choice setting. They find that information provision of students' own and other's preferences improves efficiency.

3. THREE SCHOOL CHOICE MECHANISMS

In this section, we introduce the school choice problem and the three mechanisms. In a *school choice problem*, there are a number of students, and a number of schools. Each student has strict preference over all schools, whereas each school has a maximum capacity and a strict priority ordering of all students. School priorities are imposed by the school district based on state and local laws (Abdulkadiroğlu & Sönmez, 2003).

The outcome of a school choice problem is referred to as a *matching*, which is an assignment of school seats to students such that each student is assigned one seat and no school assigns more seats than its capacity. A matching is *Pareto efficient* if there is no matching which assigns each student a weakly better school and at least one student a strictly better school. A *blocking pair* consists of a student-school pair (i, s) such that: (1) student i prefers school s to her assignment under μ and (2) student i has higher priority at school s than some other student who is assigned a seat at school s under μ. A matching μ eliminates *justified envy* if there is no blocking pair.

Finally, a *student assignment mechanism* is a systematic procedure that selects a matching for each school choice problem. A mechanism is Pareto efficient if it always selects a Pareto efficient matching; it is *stable* if it always selects a matching which eliminates justified envy; it is *strategy-proof* if no student can possibly benefit by unilaterally misrepresenting her preferences.

In the school choice literature, three mechanisms have been studied extensively: the Boston immediate acceptance mechanism, the Gale-Shapley deferred acceptance mechanism and the top trading cycle mechanism. We now briefly describe each mechanism and summarize their theoretical properties.

The *Boston immediate acceptance mechanism (BOS)* asks students to submit rank order lists (ROL) of schools. Together with the pre-announced capacity of each school, BOS uses pre-defined rules to determine the school priority ranking over students and consists of the following rounds:

Round 1. Each school considers all students who rank it first and assigns its seats in order of their priority at that school until either there is no seat left at that school or no such student left.

Generally, in:

Round $(k > 1)$. The kth choice of the students who have not yet been assigned is considered. Each school that still has available seats assigns the remaining seats to students who rank it as their kth choice in order of their priority at that school until either there is no seat left at that school or no such student left.

The process terminates after any round k when either every student is assigned a seat at some school, or the only students who remain unassigned have listed no more than k choices.

Ergin & Sönmez (2006) characterize the Nash equilibria of the BOS mechanism, showing that its equilibria are either equal to or Pareto inferior to the dominant strategy outcome of DA. The BOS mechanism was adopted for student assignment to public schools in Boston from 1999 to 2005, when it was replaced by DA (Abdulkadiroğlu & Sönmez, 2003; Abdulkadiroğlu et al., 2006). It has been widely used in other regions as well, including Cambridge, Denver, Minneapolis, Seattle, St. Petersburg-Tampa (Chen & Sönmez, 2006) and Beijing (He, 2014). Despite its popularity, BOS is neither strategy-proof nor stable. BOS also tends to favor strategically sophisticated students (Pathak & Sönmez, 2008). Because of these shortcomings, Abdulkadiroğlu & Sönmez (2003) propose two strategy-proof alternatives, DA and TTC.

The *Gale-Shapley deferred acceptance mechanism (DA)* is proposed by Gale & Shapley (1962) in the college admission context. It is then analyzed in the school choice context by Abdulkadiroğlu & Sönmez (2003). Specifically, the mechanism collects school capacities and student ROLs for schools. With strict rankings of schools over students that are determined by pre-specified rules, it proceeds as follows:

Round 1. Every student applies to her first choice. Each school rejects the least ranked students in excess of its capacity and temporarily holds the others.

Generally, in:

Round $(k > 1)$. Every student who is rejected in Round $(k-1)$ applies to the k^{th} choice on her list. Each school pools together new applicants and those on hold from Round $(k-1)$. It then rejects the least ranked students in excess of its capacity. Those who are not rejected are temporarily held.

The process terminates after any Round k when no rejections are issued. Each school is then matched with those students whom it is currently holding.

DA is strategy-proof and stable. Though it is not efficient, it Pareto dominates any other mechanism that eliminates justified envy (Dubins & Freedman, 1981; Roth, 1982). As mentioned in the introduction, various versions of the DA has been adopted to replace BOS in Boston and New York City for school choice, throughout England for public school assignment (Pathak & Sönmez, 2013), and by many provinces in Chinese college admissions (Chen & Kesten, 2017).

While DA preserves stability at the cost of Pareto efficiency, the *top trading cycles mechanism (TTC)* preserves efficiency at the cost of stability (Abdulkadiroğlu & Sönmez, 2003). In what follows, we adopt the description of the TTC from Chen & Sönmez (2006). In our experimental setting, each student has a high priority at her district school and low priority at other schools. At each school, the priority among high priority students, as well as the priority among low priority students, is determined with a single tie-breaking lottery. In this case, the TTC mechanism works as follows:

1. For each school, a priority ordering of students is determined.

2. Each student submits a preference ranking of the schools.

3. Based on the submitted preferences and priorities, the student assignment is determined as follows:

   (a) Each participant is tentatively assigned a seat at her district school.

   (b) All participants are lined up in an initial queue based on the tie-breaking lottery.

   (c) An application to the highest-ranked school is made on behalf of the participant at the top of the queue.

   - If the application is made to her district school, then her tentative assignment is finalized. The participant and her assignment are removed from the system. The process continues with the next participant in line.
   - If the application is made to another school, say school s, then the first participant in the queue who tentatively holds a seat at school s is moved to the top of the queue directly in front of the requester.

   (d) Whenever the queue is modified, the process continues similarly: an application to the highest ranked school with still available seats is made on behalf of the participant at the top of the queue.

   - If the application is made to her district school, then her tentative assignment is finalized. The participant and her assignment are removed from the system. The process continues with the next participant in line.
   - If the application is made to another school, say school s, then the first participant in the queue who tentatively holds a seat at school s is moved to the top of the queue directly in front of the requester.

   (e) A mutually-beneficial exchange is obtained when a cycle of applications is made in sequence (e.g., I apply to John's district school, John applies to your district school and you apply to my district school). In this case, the Pareto-improving exchange is carried out and the participants, as well as their assignments, are removed from the system.

   (f) The process continues until all participants are assigned a school seat.

TTC has been adopted by the city of New Orleans in its school choice program implemented since 2012 (Vanacore, 2012). In theory, the TTC mechanism has an efficiency advantage over both BOS and DA. The DA mechanism, on the other hand, is stable, whereas neither BOS or TTC is. In terms of preference manipulation, we expect a high (low) proportion of truth-telling under both the DA and TTC (BOS), based on the strategy-proofness of DA and TTC. We will summarize the theoretical properties of each mechanism as hypotheses in Section 5.

4. EXPERIMENTAL DESIGN

We use the same set of parameters as the designed environment in Chen & Sönmez (2006). There are 36 students and 7 schools per match. Schools have different capacities. Schools A and B are small and high quality schools, with 3 seats each, whereas school C to G are lower quality schools with 6 seats each. Each student has a district school where she has the highest priority.

The induced student preferences over schools is generated by a utility function, which depends on the school quality, proximity, and a random factor. The utility function of each student has three components, $u^i(S) = u^i_p(S) + u^i_q(S) + u^i_r(S)$. Here $u^i_p(S)$ represents the proximity utility for student i at school S. This utility is 10 if student i lives within the walk zone of school S and zero otherwise. The second component, $u^i_q(S)$, represents the quality utility for student i at school S. For odd-labelled students (i.e., for students who are gifted in sciences), $u^i_q(A) = 20$, $u^i_q(B) = 40$, and $u^i_q(S) = 10$ for $S \in \{C,D,E,F,G\}$. For even-labelled students (i.e., for students who are gifted in arts), $u^i_q(A) = 40$, $u^i_q(B) = 20$, and $u^i_q(S) = 10$ for $S \in \{C,D,E,F,G\}$. Lastly, the third component, $u^i_r(S)$, represents a random utility (uniform in the range 0-40) which captures diversity in tastes.

Table 1 presents the monetary payoff of each student for being matched with each school. Boldfaced numbers indicate that the student lives within the school district of that school. For example, participant with ID #1 lives in the school district of school A. She will get $13 dollars if she gets admitted to school A, $16 dollars if she gets admitted to school B, and so on.

Our design differs from Chen & Sönmez (2006) in the amount of information provided to our participants. While a student knows only her own row of the payoff table, her own district school and school capacities in Chen & Sönmez (2006), we implement a complete information setting, where students

Table 1: Payoff Table

	Schools						
Student ID	A	B	C	D	E	F	G
1	**13**	16	9	2	5	11	7
2	**16**	13	11	7	2	5	9
3	**11**	13	7	16	2	9	5
4	16	**13**	11	5	2	7	9
5	11	**16**	2	5	13	7	9
6	16	**13**	7	9	11	2	5
7	13	16	**9**	5	11	7	2
8	16	9	**11**	2	13	7	5
9	16	13	**2**	5	9	7	11
10	16	7	**9**	5	2	11	13
11	7	16	**11**	9	5	2	13
12	13	16	**9**	11	2	7	5
13	9	16	2	**13**	11	5	7
14	16	5	2	**9**	7	13	11
15	13	16	9	**11**	2	7	5
16	16	13	11	**5**	9	7	2
17	13	16	5	**7**	2	9	11
18	16	13	5	**9**	7	11	2
19	11	16	7	5	**13**	9	2
20	16	13	7	9	**5**	2	11
21	13	16	2	7	**9**	11	5
22	16	11	7	2	**9**	5	13
23	16	13	7	2	**5**	11	9
24	16	13	11	5	**9**	2	7
25	13	16	2	5	11	**9**	7
26	16	13	5	9	7	**2**	11
27	7	11	5	2	13	**9**	16
28	16	13	7	2	11	**5**	9
29	7	11	16	13	2	**9**	5
30	16	9	7	2	5	**11**	13
31	11	16	7	2	5	9	**13**
32	13	9	16	2	5	7	**11**
33	13	16	11	9	7	5	**2**
34	16	11	2	7	5	13	**9**
35	7	16	2	5	11	13	**9**
36	16	13	5	7	9	2	**11**

know every student's preference, i.e., the entire payoff table, school priority and capacity for every school, and every student's position in the randomly generated tie-breaker before making a decision.

In the experiment, each subject is randomly assigned an ID number and is seated in a chair in a classroom. All sessions are conducted by hand. At the beginning of the first session, participants are asked to volunteer to generate the random tie-breaker. One volunteer is asked to come to the front of the classroom and draw ping pong balls (labelled 1-36), one at a time, from an urn. The volunteer announces the number on each ball, which is recorded on a transparency in public by the experimenter. This tie-breaker is then used in all subsequent sessions. The random tie-breaker used in our experiment is as follows:

$$[6,19,15,10,33,24,1,28,30,32,4,25,21,14,23,16,5,35,29,8,22,2,13,12,\\ 18,9,20,17,7,34,11,31,3,36,27,26].$$

This means that participant with ID #6 has the highest priority, participant with ID #19 has the second highest priority, and so on.

At the beginning of each session, after subjects are seated in a classroom, the experimenter reads the instructions aloud. Subjects are then given fifteen minutes to read the instructions at their own pace and to make their decisions. At the end of fifteen minutes, the experimenter collects the decision sheets. The session ends after the decision sheets are collected. The experimenter then puts the subject decisions and the lottery into a computer program to generate the allocations, announces the allocations by email and pays the subjects after the email announcement.

Table 2: Features of experimental sessions

Mechanism	Subjects per session	# of sessions	Total # of subjects
BOS_c	36	2	72
DA_c	36	2	72
TTC_c	36	2	72

Table 2 summarizes features of experimental sessions. We conduct two independent sessions for each mechanism in spring 2004 at the University of Michigan. Subjects are undergraduate students from the University of

Michigan. Each sessions has 36 subjects. This gives us 72 subjects for each mechanism and 216 subjects in total. Each session consists of one round only. The session lasts between 45-60 minutes, with the first 20-25 minutes being used for instructions. The average payment (including a \$3 participation fee) is \$14.73. No subject participates in more than one session. In addition, when we compare subject behavior under complete information to that under incomplete information, we re-use the data from the 216 subjects in the designed environment in Chen & Sönmez (2006). Thus, we use data from a total of 432 subjects in our analysis. Experimental instructions are included in the Appendix. Data are available from the authors upon request.

5. RESULTS

Several questions are important in evaluating the mechanisms. The first is whether individuals report their preferences truthfully. The second is the rankings of mechanisms in terms of efficiency or stability. The third is whether the experimental results are robust to changes in the information condition.

In presenting the results, we introduce some shorthand notations. Let $x > y$ denote that a measure under mechanism x is greater than the corresponding measure under mechanism y at the 5% significance level. Let $x \geq y$ denote that a measure under mechanism x is greater than the corresponding measure under mechanism y at the 10% significance level. Let $x \sim y$ denote that a measure under mechanism x is not significantly different from the corresponding measure under mechanism y at the 10% significance level.

We first examine whether individuals reveal their preferences truthfully, and if not, how they manipulate their preferences under each of the three mechanisms. We then report how information affects truth-telling. Based on their theoretical properties, we formulate the following hypothesis:

Hypothesis 1 (Truth-telling)**.** *(a) Under DA or TTC, participants will be more likely to reveal their preferences truthfully than under BOS. (b) Participants will be equally likely to reveal their preferences truthfully under either DA or TTC. (c) The likelihood of truth-telling under either DA or TTC remains the same when more information is provided.*

Note that in Hypothesis 1(c), we are silent about whether the proportion of truth-telling under BOS might change when more information is provided. This is due to the fact that the extent to which information might influence school choice strategies depends on the environment (Ergin & Sönmez, 2006; Abdulkadiroğlu et al., 2011). Following the experimental school choice literature (Chen & Sönmez, 2006; Pais & Pintér, 2008; Calsamiglia et al., 2010; Pais et al., 2011), we separate participant strategies into three categories: truth-telling, district school bias (DSB), and other strategies. Formally, *district school bias* is defined as putting one's district school into a higher position than that in one's true preference order. Under TTC and DA, the ranking of schools below one's district school does not matter. Therefore, under these two mechanisms, we code a ROL as truthful as long as the list from one's first choice to one's district school is truthful. In comparison, under BOS, we use the complete ROL to measure truth-telling.

Table 3: Proportion of truth-telling and misrepresentations

Complete Information				Incomplete Information (CS 2006)			
Mechanism	Truth-telling	DSB	Other	Mechanism	Truth-telling	DSB	Other
BOS_c	0.194	0.611	0.194	BOS_i	0.111	0.750	0.139
DA_c	0.542	0.153	0.306	DA_i	0.722	0.083	0.194
TTC_c	0.708	0.083	0.208	TTC_i	0.500	0.292	0.208

Table 3 reports the proportion of each strategy category. The summary of statistics in the incomplete information setting is generated from the data in the designed environment in Chen & Sönmez (2006) (shortened as CS 2006). Compared to the earlier study, we combine the small school bias and similar preference bias into the "other" category. Table 4 reports results from proportion of t-tests for each pair of comparison across mechanisms as well as information conditions. We summarize the results below.

Table 4: Comparing proportion of truth-telling across mechanisms and information conditions

Complete Information			Incomplete Information			Complete vs. Incomplete		
(1) Hypotheses	(2) z-stat	(3) p-value	(4) Hypotheses	(5) z-stat	(6) p-value	(7) Hypotheses	(8) z-stat	(9) p-value
$DA_c > BOS_c$	4.320	0.000	$DA_i > BOS_i$	7.437	0.000	$BOS_c \neq BOS_i$	1.390	0.165
$TTC_c > BOS_c$	6.196	0.000	$TTC_i > BOS_i$	5.065	0.000	$DA_c \neq DA_i$	2.246	0.025
$TTC_c \neq DA_c$	2.066	0.039	$DA_i \neq TTC_i$	2.735	0.006	$TTC_c \neq TTC_i$	2.556	0.011

Notes: Z-statistics and p-values are from proportion of t-tests.

Result 1 (Truth-telling)**.** *In the complete information setting, the proportion of truth-telling follows $TTC_c > DA_c > BOS_c$. Comparing each mechanism across information conditions, we find that truth-telling under TTC (BOS) increases (weakly increases) with more information, whereas truth-telling under DA decreases with more information.*

Support. *Table 4 reports proportion of t-tests for each pair of comparison across mechanisms under complete information (column 3), incomplete information (column 6), and across information conditions (column 9).*

By Result 1, we reject the null in favor of Hypothesis 1(a) under both complete and incomplete information. That is, the two strategy-proof mechanisms each induce greater proportion of truth-telling than the non-strategy-proof BOS. While we do not anticipate any difference in truth-telling between the two strategy-proof mechanisms, we do find surprisingly significant differences under both complete ($TTC_c > DA_c$) and incomplete information ($DA_i > TTC_i$), albeit in opposite directions, leading us to reject Hypothesis 1(b). As a result of these unexpected differences, we also reject Hypothesis 1(c) that information has no effect on the likelihood of truth-telling for the strategy-proof mechanisms. Specifically, we find that truth-telling under TTC (BOS) increases (weakly increases) with more information, whereas truth-telling under DA decreases with more information.

Compared to prior literature, our complete (incomplete) information setting corresponds to the full (partial) information setting in Pais & Pintér (2008), respectively. Our mechanism ranking for truth-telling under complete information is consistent with that under full information in Pais & Pintér (2008). Furthermore, both studies find an increase in truth-telling under TTC from incomplete to complete information, although Pais & Pintér (2008) find no change in truth-telling under BOS or DA between these two information conditions.

Next, we report aggregate performance of the mechanisms, including efficiency and stability. Following Chen & Sönmez (2006) and Calsamiglia et al. (2010), we take advantage of the one-shot implementation in our experiments and use the recombinant estimation technique (Mullin & Reiley, 2006). Based on the theoretical properties of the three mechanisms, we formulate the following hypothesis:

Hypothesis 2 (Efficiency)**.** *(a) The expected per capita payoff will be greater under TTC than under either BOS or DA. (b) The expected per capita payoff under either DA or TTC remains the same when more information is provided.*

Again, we are agnostic about the extent to which efficiency under BOS might be affected by changes in information conditions, as it might be context dependent. Nor do we make predictions regarding the efficiency comparisons between BOS and DA. Table 5 reports the recombinant estimation of expected per capita payoffs in the complete (upper panel) and incomplete information setting (lower panel). The recombinant estimation in the incomplete information setting is generated from the CS 2006 data using the same single tie-breaker as that in our complete information treatment. Table 6 presents the t-statistics and p-values from t-tests for efficiency comparisons across mechanisms and information conditions.

Table 5: Recombinant estimation of expected per capita payoffs

Mechanism	Mean ($\hat{\mu}$)	Var. (σ^2)	Covar. (ϕ)	Asym. Var. (var($\hat{\mu}$))	St. dev.
BOS_c	11.742	0.045	0.001	0.017	0.129
DA_c	11.759	0.060	0.001	0.023	0.152
TTC_c	12.255	0.029	0.001	0.014	0.120
BOS_i	11.150	0.034	0.001	0.011	0.104
DA_i	11.820	0.060	0.002	0.029	0.170
TTC_i	11.379	0.058	0.001	0.025	0.159

Notes: For a given player, the number of recombination is 200,000. Therefore, the sample size is 14,400,000 after recombinations. The recombinant estimations for both the complete and incomplete information settings are generated using the same single tie-breaker used in our experiment.

Table 6: Comparing expected per capita payoffs across mechanisms and information conditions

Complete Information			Incomplete Information			Complete vs. Incomplete		
(1) Hypotheses	(2) t-stat	(3) p-value	(4) Hypotheses	(5) t-stat	(6) p-value	(7) Hypotheses	(8) t-stat	(9) p-value
$TTC_c > BOS_c$	2.921	0.002	$TTC_i > BOS_i$	1.207	0.114	$BOS_c \neq BOS_i$	3.580	0.000
$TTC_c > DA_c$	2.566	0.005	$TTC_i > DA_i$	-1.893	0.971	$DA_c \neq DA_i$	0.270	0.787
$BOS_c \neq DA_c$	0.084	0.933	$BOS_i \neq DA_i$	3.364	0.001	$TTC_c \neq TTC_i$	4.400	0.000

Notes: T-statistics and p-values are from t-tests.

Result 2 (Efficiency). *Under complete information, the expected per capita payoff follows $TTC_c > DA_c \sim BOS_c$. Comparing each mechanism across information conditions, we find that efficiency under both TTC and BOS increases with more information, whereas efficiency under DA remains unchanged.*

Support. *Table 6 reports t-tests for each pair of comparison across mechanisms under complete information (column 3), incomplete information (column 6), and across information conditions (column 9). We report one-sided (two-sided) p-values for one-sided (two-sided) hypotheses, respectively.*

By Result 2, we reject the null in favor of Hypothesis 2(a) under complete information; however, we fail to reject the null under incomplete information. That is, under complete information, TTC achieves greater efficiency than either DA or BOS, whereas under incomplete information, it fails to outperform DA or BOS. Furthermore, we reject Hypothesis 2(b) under TTC but not under DA. In sum, we find that efficiency under both TTC and BOS increases with more information, which is consistent with Result 1 where participants are more likely to tell the truth under complete information. In contrast, additional information has no effect on efficiency under DA.

Our efficiency ranking under complete information is consistent with that under full information in Pais & Pintér (2008) ($TTC_c > DA_c \sim BOS_c$). However, they do not find any statistically significant change in efficiency from partial to full information conditions.

In addition to efficiency, we measure mechanism stability by computing the number of students who have the possibility to block per group, again using recombinant estimation. For simplicity, we call this measure the average number of blocking pairs. Based on the theoretical properties of the three mechanisms, we formulate the following hypothesis:

Hypothesis 3 (Stability). *(a) The average number of blocking pairs per group will be lower under DA than under either TTC or BOS. (b) The average number of blocking pairs per group under either DA or TTC remains the same when more information is provided.*

In this case, we are silent on the comparison between TTC and BOS, as well as stability under BOS across information conditions, as these comparisons depends on the specific environment. Table 7 reports the results of recombinant estimation in the number of justified envy per group under complete (upper panel) and incomplete information (lower panel). The results under incomplete information is generated from the CS 2006 data, using the same single tie-breaker as in our complete information setting.

Table 7: Recombinant estimation of average number of blocking pairs per group

Mechanism	Mean ($\hat{\mu}$)	Var. (σ^2)	Covar. (ϕ)	Asym. Var. (var($\hat{\mu}$))	St. dev.
BOS_c	9.354	4.935	0.088	1.589	1.261
DA_c	2.906	0.523	0.012	0.210	0.459
TTC_c	10.624	3.737	0.085	1.524	1.234
BOS_i	14.323	4.449	0.098	1.757	1.326
DA_i	2.500	1.250	0.035	0.629	0.793
TTC_i	12.461	5.839	0.095	1.718	1.311

Notes: For a given player, the number of recombination is 200,000. Therefore, the sample size is 14,400,000 after recombinations. The recombinant estimations for both the complete and incomplete information settings are generated using the same single tie-breaker used in our experiment.

Result 3 (Stability)**.** *Under both complete and incomplete information, the average number of blocking pairs per group follows $DA < BOS \sim TTC$. Comparing each mechanism across information conditions, we find that the average number of blocking pairs per group under BOS decreases with more information, whereas that under either DA or TTC is largely unchanged.*

Support. *Table 8 reports t-tests for each pair of comparison across mechanisms under complete information (column 3), incomplete information (column 6), and across information conditions (column 9). We report one-sided (two-sided) p-values for one-sided (two-sided) hypotheses, respectively.*

Table 8: Comparing average number of blocking pairs per group across mechanisms and information conditions

Complete Information			Incomplete Information			Complete vs. Incomplete		
(1) Hypotheses	(2) t-stat	(3) p-value	(4) Hypotheses	(5) t-stat	(6) p-value	(7) Hypotheses	(8) t-stat	(9) p-value
$BOS_c > DA_c$	4.807	0.000	$BOS_i > DA_i$	7.655	0.000	$BOS_c \neq BOS_i$	2.716	0.007
$TTC_c > DA_c$	5.861	0.000	$TTC_i > DA_i$	6.503	0.000	$DA_c \neq DA_i$	0.444	0.657
$TTC_c \neq BOS_c$	0.720	0.472	$TTC_i \neq BOS_i$	0.999	0.318	$TTC_c \neq TTC_i$	1.020	0.308

Notes: T-statistics and p-values are from t-tests.

By Result 3, we reject the null in favor of Hypothesis 3(a). That is, DA is more stable than either TTC or BOS under both complete and incomplete information. However, we fail to reject Hypothesis 3(b) that the two strategy-proof mechanisms generate the same number of blocking pairs across information conditions. Lastly, unpredicted by theory, we find that BOS becomes more stable with more information.

Compared to the prior experimental literature on school choice, our stability ranking among the three mechanisms under incomplete information is identical to that in the untruncated treatment in Calsamiglia et al. (2010) as well as that in the partial information treatment in Pais & Pintér (2008). In comparison, our stability ranking among the three mechanisms under complete information is directionally consistent with the corresponding ranking under full information (DA $\geq$ TTC) in Pais & Pintér (2008), although the latter is not statistically significant. Lastly, our finding that the stability of BOS improves with more information is again directionally consistent with the corresponding result in Pais & Pintér (2008), who report a 33 percentage point increase in the proportion of stable outcomes under BOS from partial to full information, but this difference is statistically insignificant in their study.

6. CONCLUSION

In this paper, we investigate the performance of three influential school choice mechanisms under complete information, and compare that with the performance of the same three mechanisms under incomplete information (Chen & Sönmez, 2006). Overall, we find that information has significant effects on the performance of school choice mechanisms.

Specifically, we find that, under complete information, TTC outperforms both DA and BOS in terms of truth-telling and efficiency, whereas DA is more stable than either TTC or BOS regardless of information conditions. Compared to the incomplete information setting in Chen & Sönmez (2006), of the two strategy-proof mechanisms, more information increases both truth-telling and efficiency under TTC, and reduces truth-telling under DA. In comparison, more information increases truthful preference revelation, stability and efficiency under BOS.

Findings from this paper and prior school choice experiments point to several policy implications. First, in real life implementations, truth-telling is not verifiable. However, a designer can advise students and parents to reveal their preferences truthfully if the mechanism is strategy-proof. We also see that, regardless of the information condition, a strategy-proof mechanism always outperforms the manipulable BOS in truth-telling, stability or efficiency. Therefore, a strategy-proof mechanism, such as DA or TTC, should be preferred to the manipulable BOS.

Second, of the two strategy-proof mechanisms, which one should be chosen? Our results suggest that the answer depends on whether the policy-makers put more weight on stability or efficiency. If stability is valued above efficiency, DA should be chosen. Otherwise, TTC is the clear choice. In practice, we see that both mechanisms have been chosen as a replacement for BOS.

Lastly, unpredicted by theory, we find that information provision improves the aggregate performance of both TTC and BOS. In real life, information acquisition is likely to be costly. Under this more realistic scenario, information provision by education authorities is likely to have even greater welfare gains (Chen & He, 2016). Therefore, our results suggest that education authorities should provide more information about the environment to improve the performance of either the manipulable BOS or a strategy-proof alternative, such as the TTC.

Supplement: Experimental Instructions

The complete instructions for subject #1 under BOS (Mechanism B) are shown here. Instructions for all other subjects are identical except the individual portion of the payoff table. Instructions for DA (Mechanism G) and TTC (Mechanism T) are identical to those for BOS except the "Allocation Method" and "An Example" sections, hence only those sections are shown here.

Instructions - Mechanism B

This is an experiment in the economics of decision making. The instructions are simple, and if you follow them carefully and make good decisions, you might earn a considerable amount of money. In this experiment, we simulate a procedure to allocate students to schools. The procedure, payment rules, and student allocation method are described below. Do not communicate with each other during the experiment. If you have questions at any point during the experiment, raise your hand and the experimenter will help you.

Procedure

- There are 36 participants in this experiment. You are participant #1.

- In this simulation, 36 school slots are available across seven schools. These schools differ in size, geographic location, specialty, and quality of instruction in each specialty. Each school slot is allocated to one participant. There are three slots each at schools A and B, and six slots each at schools C, D, E, F and G.
- **Your payoff** amount depends on the school slot you hold at the end of the experiment. Payoff amounts are outlined in the following table. These amounts reflect the desirability of the school in terms of location, specialty and quality of instruction.

Slot received at School:	A	B	C	D	E	F	G
Payoff to Participant #1 (in dollars)	13	16	9	2	5	11	7

  The table is explained as follows:

  - You will be paid $13 if you hold a slot at school A at the end of the experiment.
  - You will be paid $16 if you hold a slot at school B at the end of the experiment.
  - You will be paid $9 if you hold a slot at school C at the end of the experiment.
  - You will be paid $2 if you hold a slot at school D at the end of the experiment.
  - You will be paid $5 if you hold a slot at school E at the end of the experiment.
  - You will be paid $11 if you hold a slot at school F at the end of the experiment.
  - You will be paid $7 if you hold a slot at school G at the end of the experiment.

  ***NOTE* different participants have different payoff tables.** That is, payoff by school is different for different participants.
- During the experiment, each participant first completes the Decision Sheet by indicating school preferences. Note that you need to rank all seven schools in order to indicate your preferences.
- After all participants have completed their Decision Sheets, the experimenter collects the Sheets and starts the allocation process.
- Once the allocations are determined, the experimenter informs each participants of his/her allocation slot and respective payoff.

The payoff table for all 36 students:
Allocation Method

- In this experiment, participants are defined as belonging to the following school districts.

  Participants #1 – #3 live within the school district of school A,
  Participants #4 – #6 live within the school district of school B,
  Participants #7 – #12 live within the school district of school C,
  Participants #13 – #18 live within the school district of school D,
  Participants #19 – #24 live within the school district of school E,
  Participants #25 – #30 live within the school district of school F,
  Participants #31 – #36 live within the school district of school G.

Student ID	Schools						
	A	B	C	D	E	F	G
1	**13**	16	9	2	5	11	7
2	**16**	13	11	7	2	5	9
3	**11**	13	7	16	2	9	5
4	16	**13**	11	5	2	7	9
5	11	**16**	2	5	13	7	9
6	16	**13**	7	9	11	2	5
7	13	16	**9**	5	11	7	2
8	16	9	**11**	2	13	7	5
9	16	13	**2**	5	9	7	11
10	16	7	**9**	5	2	11	13
11	7	16	**11**	9	5	2	13
12	13	16	**9**	11	2	7	5
13	9	16	2	**13**	11	5	7
14	16	5	2	**9**	7	13	11
15	13	16	9	**11**	2	7	5
16	16	13	11	**5**	9	7	2
17	13	16	5	**7**	2	9	11
18	16	13	5	**9**	7	11	2
19	11	16	7	5	**13**	9	2
20	16	13	7	9	**5**	2	11
21	13	16	2	7	**9**	11	5
22	16	11	7	2	**9**	5	13
23	16	13	7	2	**5**	11	9
24	16	13	11	5	**9**	2	7
25	13	16	2	5	11	**9**	7
26	16	13	5	9	7	**2**	11
27	7	11	5	2	13	**9**	16
28	16	13	7	2	11	**5**	9
29	7	11	16	13	2	**9**	5
30	16	9	7	2	5	**11**	13
31	11	16	7	2	5	9	**13**
32	13	9	16	2	5	7	**11**
33	13	16	11	9	7	5	**2**
34	16	11	2	7	5	13	**9**
35	7	16	2	5	11	13	**9**
36	16	13	5	7	9	2	**11**

- In addition, for each school, a separate **priority order** of the students is determined as follows:
  - **Highest Priority Level:** Participants who rank the school as their first choice AND who also live within the school district.
  - **2nd Priority Level:** Participants who rank the school as their first choice BUT who do not live within the school district.
  - **3rd Priority Level:** Participants who rank the school as their second choice AND who also live within the school district.
  - **4th Priority Level:** Participants who rank the school as their second choice BUT who do not live within the school district.

    ⋮ ⋮

  - **13th Priority Level:** Participants who rank the school as their seventh choice AND who also live within the school district.
  - **Lowest Priority Level:** Participants who rank the school as their seventh choice BUT who do not live within the school district.
- The ties between participants at the same priority level are broken using a fair lottery. This means each participant has an equal chance of being the first in the line, the second in the line, $\cdots$, as well as the last in the line. To determine this fair lottery, a participant will be asked to draw 36 ping pong balls from an urn, one at a time. Each ball has a number on it, corresponding to a participant ID number. The sequence of the draw determines the order in the lottery.
- Therefore, to determine the priority order of a student for a school:
  - The first consideration is how highly the participant ranks the school in his/her Decision Sheet,
  - The second consideration is whether the participant lives within the school district or not, and
  - The last consideration is the order in the fair lottery.
- Once the priorities are determined, slots are allocated in seven rounds.

Round 1.
a. An application to the first ranked school in the Decision Sheet is sent for each participant.
b. Each school accepts the students with higher priority order until all slots are filled. These students and their assignments are removed from the system. The remaining applications for each respective school are rejected.

Round 2. a. The rejected applications are sent to his/her second ranked school in the Decision Sheet.

b. If a school still has available slots remaining from Round 1, then it accepts the students with higher priority order until all slots are filled. The remaining applications are rejected.

⋮ ⋮

Round 6. a. The application of each participant who is rejected by his/her top five choices is sent to his/her sixth choice.

b. If a school still has slots available, then it accepts the students with higher priority order until all slots are filled. The remaining applications are rejected.

Round 7. Each remaining participant is assigned a slot at his/her last choice.

An Example:

We will go through a simple example to illustrate how the allocation method works.

Students and Schools: In this example, there are six students, 1–6, and four schools, Clair, Erie, Huron and Ontario.

Student ID Number: $1,2,3,4,5,6$	Schools: Clair, Erie, Huron, Ontario

Slots and Residents: There are two slots each at Clair and Erie, and one slot each at Huron and Ontario. Residents of districts are indicated in the table below.

School	Slot 1	Slot 2	District Residents
Clair	☐	☐	1 2
Erie	☐	☐	3 4
Huron	☐		5
Ontario	☐		6

Lottery: The lottery produces the following order.

$\boxed{1-2-3-4-5-6}$

Submitted School Rankings: The students submit the following school rankings:

	1st Choice	2nd Choice	3rd Choice	Last Choice
Student 1	Huron	Clair	Ontario	Erie
Student 2	Huron	Ontario	Clair	Erie
Student 3	Ontario	Clair	Erie	Huron
Student 4	Huron	Clair	Ontario	Erie
Student 5	Ontario	Huron	Clair	Erie
Student 6	Clair	Erie	Ontario	Huron

Priority: School priorities depend on: (1) how highly the student ranks the school, (2) whether the school is a district school, and (3) the lottery order.

Clair : Student 6 ranks Clair first. Students 1, 3 and 4 rank Clair second; among them, student 1 lives within the Clair school district. Students 2 and 5 rank Clair third. Using the lottery order to break ties, the priority order for Clair is 6-1-3-4-2-5.

$$\overbrace{6}^{\text{1st Choice}}\ \overbrace{\underbrace{1}_{\text{Resident}}\ \underbrace{3\quad 4}_{\text{Non-Resident}}}^{\text{2nd Choice}}\ \overbrace{\underbrace{2\quad 5}_{\text{Non-Resident}}}^{\text{3rd Choice}}\ \overbrace{\text{None}}^{\text{4th Choice}}$$

Erie : Student 6 ranks Erie second. Student 3 ranks Erie third. Students 1, 2, 4 and 5 rank Erie fourth; among them student 4 lives within the Erie school district. Using the lottery order to break ties, the priority for Erie is 6-3-4-1-2-5.

$$\overbrace{\text{None}}^{\text{1st Choice}}\ \overbrace{6}^{\text{2nd Choice}}\ \overbrace{3}^{\text{3rd Choice}}\ \overbrace{\underbrace{4}_{\text{Resident}}\ \underbrace{1\quad 2\quad 5}_{\text{Non-Resident}}}^{\text{4th Choice}}$$

Huron : Students 1, 2 and 4 rank Huron first. Student 5 ranks Huron second. Students 3 and 6 rank Huron fourth. Using the lottery order to break ties, the priority for Huron is 1-2-4-5-3-6.

$$\overbrace{\underbrace{1\quad 2\quad 4}_{\text{Non-Residents}}}^{\text{1st Choice}}\ \overbrace{5}^{\text{2nd Choice}}\ \overbrace{\text{None}}^{\text{3rd Choice}}\ \overbrace{\underbrace{3\quad 6}_{\text{Non-Resident}}}^{\text{4th Choice}}$$

Ontario : Students 3 and 5 rank Ontario first. Student 2 ranks Ontario second. Students 1, 4 and 6 rank Ontario third; among them student 6 lives within the Ontario school district. Using the lottery order to break ties, the priority for Ontario is 3-5-2-6-1-4.

$$\overbrace{\underbrace{3 \quad 5}_{\text{Non-Residents}}}^{\text{1st Choice}} \quad \overbrace{2}^{\text{2nd Choice}} \quad \overbrace{\underbrace{6}_{\text{Resident}} \quad \underbrace{1 \quad 4}_{\text{Non-Resident}}}^{\text{3rd Choice}}$$

Allocation: This allocation method consists of the following rounds.

Round 1 : Each student applies to his/her **first choice**: Students 1, 2 and 4 apply to Huron,
students 3 and 5 apply to Ontario and student 6 applies to Clair.

- School Clair accepts Student 6.
- School Huron accepts Student 1 and rejects Students 2,4.
- School Ontario accepts Student 3 and rejects Student 5.

Applicants		School		Accept	Reject		Slot 1	Slot 2
6	$\longrightarrow$	Clair	$\longrightarrow$	6		$\longrightarrow$	$\boxed{6}$	$\boxed{}$
	$\longrightarrow$	Erie	$\longrightarrow$			$\longrightarrow$	$\boxed{}$	$\boxed{}$
1, 2, 4	$\longrightarrow$	Huron	$\longrightarrow$	1	2, 4	$\longrightarrow$	$\boxed{1}$	—
3, 5	$\longrightarrow$	Ontario	$\longrightarrow$	3	5	$\longrightarrow$	$\boxed{3}$	—

Accepted students are removed from the subsequent process.

Round 2 : Each student who is rejected in Round 1 then applies to his/her **second choice**: Student 2 applies to Ontario, student 4 applies to Clair, and student 5 applies to Huron.

- No slot is left at Ontario, so it rejects student 2.
- Clair accepts student 4 for its last slot.
- No slot is left at Huron, so it rejects student 5.

Applicants		School		Accept	Reject		Slot 1	Slot 2
4	$\longrightarrow$	Clair	$\longrightarrow$	4		$\longrightarrow$	$\boxed{6}$	$\boxed{4}$
	$\longrightarrow$	Erie	$\longrightarrow$			$\longrightarrow$	$\boxed{}$	$\boxed{}$
5	$\longrightarrow$	Huron	$\longrightarrow$		5	$\longrightarrow$	$\boxed{1}$	—
2	$\longrightarrow$	Ontario	$\longrightarrow$		2	$\longrightarrow$	$\boxed{3}$	—

Round 3 : Each student who is rejected in Rounds 1-2 applies to his/her **third choice**: Students 2 and 5 apply to Clair.

- No slot is left at Clair, so it rejects students 2 and 5.

Applicants		School		Accept	Reject		Slot 1	Slot 2
2, 5	⟶	Clair	⟶		2, 5	⟶	$\boxed{6}$	$\boxed{4}$
	⟶	Erie	⟶			⟶	$\boxed{}$	$\boxed{}$
	⟶	Huron	⟶			⟶	$\boxed{1}$	—
	⟶	Ontario	⟶			⟶	$\boxed{3}$	—

Round 4 : Each remaining student is assigned a slot at his/her **last choice**: Students 2 and 5 receive a slot at Erie.

Applicants		School		Accept	Reject		Slot 1	Slot 2
	⟶	Clair	⟶			⟶	$\boxed{6}$	$\boxed{4}$
2, 5	⟶	Erie	⟶	2, 5		⟶	$\boxed{2}$	$\boxed{5}$
	⟶	Huron	⟶			⟶	$\boxed{1}$	—
	⟶	Ontario	⟶			⟶	$\boxed{3}$	—

Based on this method, the final allocations are:

Student	1	2	3	4	5	6
School	Huron	Erie	Ontario	Clair	Erie	Clair

You will have 15 minutes to go over the instructions at your own pace, and make your decisions. Feel free to earn as much cash as you can. Are there any questions?

Decision Sheet - Mechanism B

- Recall: You are participant _#1_ and you live within the school district of School _A_.
- Recall: **Your payoff** amount depends on the school slot you hold at the end of the experiment. Payoff amounts are outlined in the following table.

School:	A	B	C	D	E	F	G
Payoff in dollars	13	16	9	2	5	11	7

You will be paid \$13 if you hold a slot of School A at the end of the experiment.
You will be paid \$16 if you hold a slot of School B at the end of the experiment.
You will be paid \$9 if you hold a slot of School C at the end of the experiment.
You will be paid \$2 if you hold a slot of School D at the end of the experiment.
You will be paid \$5 if you hold a slot of School E at the end of the experiment.
You will be paid \$11 if you hold a slot of School F at the end of the experiment.
You will be paid \$7 if you hold a slot of School G at the end of the experiment.

Please write down your ranking of the schools (A through G) from your first choice to your last choice. Please rank ALL seven schools.

1st choice	2nd choice	3rd choice	4th choice	5th choice	6th choice	last choice

Your I.D : #1 **Your Name** (print):________________________

This is the end of the experiment for you. Please remain seated until the experimenter collects your Decision Sheet. Thank you.

The lottery, as well as all participants' rankings will be entered into a computer after the experiment. The experimenter will inform each participants of his/her allocation slot and respective payoff once it is computed.

Session Number : 1 Mechanism 1 Payoff Matrix 1

Instructions - Mechanism G

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Allocation Method

- In this experiment, participants are defined as belonging to the following school districts.
  - Participants #1 - #3 live within the school district of school A,
  - Participants #4 - #6 live within the school district of school B,
  - Participants #7 - #12 live within the school district of school C,
  - Participants #13 - #18 live within the school district of school D,
  - Participants #19 - #24 live within the school district of school E,
  - Participants #25 - #30 live within the school district of school F,
  - Participants #31 - #36 live within the school district of school G.
- A priority order is determined for each school. Each participant is assigned a slot at the **best possible** school reported in his/her Decision Sheet that is consistent with the priority order below.
- The priority order for each school is separately determined as follows:
  - **High Priority Level:** Participants who live within the school district.

    Since the number of High priority participants at each school is equal to the school capacity, each High priority participant is guaranteed an assignment which is at least as good as his/her district school based on the ranking indicated in his/her Decision Sheet.
  - **Low Priority Level:** Participants who do not live within the school district.

    The priority among the Low priority students is based on their respective order in a fair lottery. This means each participant has an equal chance of being the first in the line, the second in the line, $\cdots$, as well as the last in the line. To determine this fair lottery, a participant will be asked to draw 36 ping pong balls from an urn, one at a time. Each ball has a number on it, corresponding to a participant ID number. The sequence of the draw determines the order in the lottery.
- Once the priorities are determined, the allocation of school slots is obtained as follows:
  - An application to the first ranked school in the Decision Sheet is sent for each participant.

- Throughout the allocation process, a school can hold no more applications than its number of slots.

  If a school receives more applications than its capacity, then it rejects the students with lowest priority orders. The remaining applications are retained.
- Whenever an applicant is rejected at a school, his application is sent to the next highest school on his Decision Sheet.
- Whenever a school receives new applications, these applications are considered together with the retained applications for that school. Among the retained and new applications, the lowest priority ones in excess of the number of the slots are rejected, while remaining applications are retained.
- The allocation is finalized when no more applications can be rejected.

  Each participant is assigned a slot at the school that holds his/her application at the end of the process.

An Example:

We will go through a simple example to illustrate how the allocation method works.

Students and Schools: In this example, there are six students, 1-6, and four schools, Clair, Erie, Huron and Ontario.

Student ID Number: $1,2,3,4,5,6$	Schools: Clair, Erie, Huron, Ontario

Slots and Residents: There are two slots each at Clair and Erie, and one slot each at Huron and Ontario. Residents of districts are indicated in the table below.

School	Slot 1	Slot 2	District Residents
Clair	☐	☐	1 2
Erie	☐	☐	3 4
Huron	☐		5
Ontario	☐		6

Lottery: The lottery produces the following order.

$$\boxed{1-2-3-4-5-6}$$

Submitted School Rankings: The students submit the following school rankings:

	1st Choice	2nd Choice	3rd Choice	Last Choice
Student 1	Huron	Clair	Ontario	Erie
Student 2	Huron	Ontario	Clair	Erie
Student 3	Ontario	Clair	Erie	Huron
Student 4	Huron	Clair	Ontario	Erie
Student 5	Ontario	Huron	Clair	Erie
Student 6	Clair	Erie	Ontario	Huron

Priority : School priorities first depend on whether the school is a district school, and next on the lottery order:

	Resident	Non-Resident
Priority order at Clair:	**1, 2**	– 3 – 4 – 5 – 6
Priority order at Erie:	**3, 4**	– 1 – 2 – 5 – 6
Priority order at Huron:	**5**	– 1 – 2 – 3 – 4 – 6
Priority order at Ontario:	**6**	– 1 – 2 – 3 – 4 – 5

The allocation method consists of the following steps:

Step 1 : Each student applies to his/her **first choice**: students 1, 2 and 4 apply to Huron,
students 3 and 5 apply to Ontario, and student 6 applies to Clair.

- Clair holds the application of student 6.
- Huron holds the application of student 1 and rejects students 2 and 4.
- Ontario holds the application of student 3 and rejects student 5.

Applicants		School		Hold	Reject
6	$\longrightarrow$	Clair	$\longrightarrow$	$\boxed{6}\,\boxed{}$	
	$\longrightarrow$	Erie	$\longrightarrow$	$\boxed{}\,\boxed{}$	
1, 2, 4	$\longrightarrow$	Huron	$\longrightarrow$	$\boxed{1}$ –	2, 4
3, 5	$\longrightarrow$	Ontario	$\longrightarrow$	$\boxed{3}$ –	5

Step 2 : Each student rejected in Step 1 applies to his/her next choice: student 2 applies to Ontario, student 4 applies to Clair, and student 5 applies to Huron.

- Clair considers the application of student 4 together with the application of student 6, which was on hold. It holds both applications.
- Huron considers the application of student 5 together with the application of student 1, which was on hold. It holds the application of student 5 and rejects student 1.
- Ontario considers the application of student 2 together with the application of student 3, which was on hold. It holds the application of student 2 and rejects student 3.

Hold	New applicants		School		Hold	Reject
$\boxed{6}\boxed{}$	4	$\longrightarrow$	Clair	$\longrightarrow$	$\boxed{6}\boxed{4}$	
$\boxed{}\boxed{}$		$\longrightarrow$	Erie	$\longrightarrow$	$\boxed{}\boxed{}$	
$\boxed{1}$ –	5	$\longrightarrow$	Huron	$\longrightarrow$	$\boxed{5}$ –	1
$\boxed{3}$ –	2	$\longrightarrow$	Ontario	$\longrightarrow$	$\boxed{2}$ –	3

Step 3 : Each student rejected in Step 2 applies to his/her next choice: Students 1 and 3 apply to Clair.

- Clair considers the applications of students 1 and 3 together with the applications of students 4 and 6, which were on hold. It holds the applications of students 1 and 3 and rejects students 4 and 6.

Hold	New applicants		School		Hold	Reject
$\boxed{6}\boxed{4}$	1, 3	$\longrightarrow$	Clair	$\longrightarrow$	$\boxed{1}\boxed{3}$	4, 6
$\boxed{}\boxed{}$		$\longrightarrow$	Erie	$\longrightarrow$	$\boxed{}\boxed{}$	
$\boxed{5}$ –		$\longrightarrow$	Huron	$\longrightarrow$	$\boxed{5}$ –	
$\boxed{2}$ –		$\longrightarrow$	Ontario	$\longrightarrow$	$\boxed{2}$ –	

Step 4 : Each student rejected in Step 3 applies to his/her next choice: Student 4 applies to Ontario and student 6 applies to Erie.

- Ontario considers the application of student 4 together with the application of student 2, which was on hold. It holds the application of student 2 and rejects student 4.
- Erie holds the application of student 6.

Hold	New applicants		School		Hold	Reject
1 3		⟶	Clair	⟶	1 3	
	6	⟶	Erie	⟶	6	
5 –		⟶	Huron	⟶	5 –	
2 –	4	⟶	Ontario	⟶	2 –	4

Step 5 : Each student rejected in Step 4 applies to his/her next choice: student 4 applies to Erie.

- Erie considers the application of student 4 together with the application of student 6, which was on hold. It holds both applications.

Hold	New applicants		School		Hold	Reject
1 3		⟶	Clair	⟶	1 3	
6	4	⟶	Erie	⟶	6 4	
5 –		⟶	Huron	⟶	5 –	
2 –		⟶	Ontario	⟶	2 –	

No application is rejected at Step 5. Based on this method, the final allocations are:

Student	1	2	3	4	5	6
School	Clair	Ontario	Clair	Erie	Huron	Erie

Instructions - Mechanism T

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Allocation Method

- In this experiment, participants are defined as belonging to the following school districts.
  - Participants #1 - #3 live within the school district of school A,
  - Participants #4 - #6 live within the school district of school B,
  - Participants #7 - #12 live within the school district of school C,
  - Participants #13 - #18 live within the school district of school D,

- Participants #19 - #24 live within the school district of school E,
- Participants #25 - #30 live within the school district of school F,
- Participants #31 - #36 live within the school district of school G.

- Each participant is first tentatively assigned to the school within his/her respective district. Next, Decision Sheet rankings are used to determine mutually beneficial exchanges between two or more participants. The order in which these exchanges are considered is determined by a fair lottery. This means each participant has an equal chance of being the first in the line, the second in the line, $\cdots$, as well as the last in the line. To determine this fair lottery, a participant will be asked to draw 36 ping pong balls from an urn, one at a time. Each ball has a number on it, corresponding to a participant ID number. The sequence of the draw determines the order in the lottery.

- The specific allocation process is explained below.
  - Initially all slots are available for allocation.
  - All participants are ordered in a queue based on the order in the lottery.
  - Next, an application to the highest ranked school in the Decision Sheet is submitted for the participant at the top of the queue.
    * If the application is submitted to his district school, then his tentative assignment is finalized (thus he is assigned a slot at his district school). The participant and his assignment are removed from subsequent allocations. The process continues with the next participant in line.
    * If the application is submitted to another school, say school S, then the first participant in the queue who tentatively holds a slot at School S is moved to the top of the queue directly in front of the requester.
  - Whenever the queue is modified, the process continues similarly: An application is submitted to the highest ranked school with available slots for the participant at the top of the queue.
    * If the application is submitted to his district school, then his tentative assignment is finalized. The process continues with the next participant in line.
    * If the application is submitted to another school, say school S, then the first participant in the queue who tentatively holds a slot at school S is moved to the top of the queue directly in front of the requester.

This way, each participant is guaranteed an assignment which is at least as good as his/her district school based on the preferences indicated in his/her Decision Sheet.

– A mutually-beneficial exchange is obtained when a cycle of applications are made in sequence, which benefits all affected participants, e.g., I apply to John's district school, John applies to your district school, and you apply to my district school. In this case, the exchange is completed and the participants as well as their assignments are removed from subsequent allocations.

– The process continues until all participants are assigned a school slot.

An Example:

We go through a simple example to illustrate how the allocation method works.

Students and Schools: In this example, there are six students, 1-6, and four schools, Clair, Erie, Huron and Ontario.

Student ID Number: 1, 2, 3, 4, 5, 6	Schools: Clair, Erie, Huron, Ontario

Slots and Residents: There are two slots each at Clair and Erie, and one slot each at Huron and Ontario. Residents of districts are indicated in the table below.

School	Slot 1	Slot 2	District Residents
Clair	☐	☐	1 2
Erie	☐	☐	3 4
Huron	☐		5
Ontario	☐		6

Tentative assignments: Students are tentatively assigned slots at their district schools.

School	Slot 1	Slot 2	
Clair	$\boxed{1}$	$\boxed{2}$	Students 1 and 2 are **tentatively assigned** a slot at Clair;
Erie	$\boxed{3}$	$\boxed{4}$	Students 3 and 4 are **tentatively assigned** a slot at Erie;
Huron	$\boxed{5}$	–	Student 5 is **tentatively assigned** a slot at Huron;
Ontario	$\boxed{6}$	–	Students 6 is **tentatively assigned** a slot at Ontario.

Lottery: The lottery produces the following order.

$$\boxed{1-2-3-4-5-6}$$

Submitted School Rankings: The students submit the following school rankings:

	1st Choice	2nd Choice	3rd Choice	Last Choice
Student 1	Huron	Clair	Ontario	Erie
Student 2	Huron	Ontario	Clair	Erie
Student 3	Ontario	Clair	Erie	Huron
Student 4	Huron	Clair	Ontario	Erie
Student 5	Ontario	Huron	Clair	Erie
Student 6	Clair	Erie	Ontario	Huron

This allocation method consists of the following steps:

Step 1 : A fair lottery determines the following student order: 1-2-3-4-5-6. Student 1 has ranked Huron as his top choice. However, the only slot at Huron is tentatively held by student 5. So student 5 is moved to the top of the queue.

Step 2 : The modified queue is now 5-1-2-3-4-6. Student 5 has ranked Ontario as his top choice. However, the only slot at Ontario is tentatively held by student 6. So student 6 is moved to the top of the queue.

Step 3 : The modified queue is now 6-5-1-2-3-4. Student 6 has ranked Clair as her top choice. The two slots at Clair are tentatively held by students 1 and 2. Between the two, student 1 is ahead in the queue. So student 1 is moved to the top of the queue.

Step 4 : The modified queue is now 1-6-5-2-3-4. Remember that student 1 has ranked Huron as his top choice. A cycle of applications is now made in sequence in the last three steps: student 1 applied to the tentative assignment of student 5, student 5 applied to the tentative assignment of student 6, and student 6 applied to the tentative assignment of student 1. These mutually beneficial exchanges are carried out: student 1 is assigned a slot at Huron, student 5 is assigned a slot at Ontario, and student 6 is assigned a slot at Clair. These students as well as their assignments are removed from the system.

Step 5 : The modified queue is now 2-3-4. There is one slot left at Clair and two slots left at Erie. Student 2 applies to Clair, which is her top choice between the two schools with remaining slots. Since student 2 tentatively holds a slot at Clair, her tentative assignment is finalized. Student 2 and her assignment are removed from the system.

Step 6 : The modified queue is now 3-4. There are two slots left at Erie. Student 3 applies to Erie, which is the only school with available slots. Since Student 3 tentatively holds a slot at Erie, her tentative assignment is finalized. Student 3 and her assignment are removed from the system.

Step 7 : The only remaining student is student 4. There is one slot left at Erie. Student 4 applies to Erie for the last available slot. Since Student 4 tentatively holds a slot at Erie, his tentative assignment is finalized. Student 4 and his assignment are removed from the system.

Final assignment Based on this method, the final allocations are:

Student	1	2	3	4	5	6
School	Huron	Clair	Erie	Erie	Ontario	Clair

Illustration

	Queue	Available Slots	The top student in the queue applies to a school.	At the end of the step
Step 1	1-2-3-4-5-6	Clair Clair Erie Erie Huron Ontario	1 applies to her 1st choice Huron, which is tentatively assigned to 5.	5 comes to the top. 1-2-3-4-**5**-6
Step 2	5-1-2-3-4-6	Clair Clair Erie Erie Huron Ontario	5 applies to her 1st choice Ontario which is tentatively assigned to 6.	6 comes to the top. 5-1-2-3-4-**6**
Step 3	6-5-1-2-3-4	Clair Clair Erie Erie Huron Ontario	6 applies to her 1st choice Clair, which is tentatively assigned to 1 and 2.	1 comes to the top. 6-5-**1**-2-3-4
Step 4	1-6-5-2-3-4	Clair Clair Erie Erie Huron Ontario	A cycle happens in the last 3 steps.	1 gets a slot at Huron. 5 gets a slot at Ontario. 6 gets a slot at Clair.
Step 5	2-3-4	Clair Erie Erie	2 applies to her 3rd choice Clair, because her 1st and 2nd choices (Huron and Ontario) are no longer available.	2 gets a slot at Clair, because she is a resident in Clair.
Step 6	3-4	Erie Erie	3 applies to Erie which is still available.	3 gets a slot at Erie, because he is a resident in Erie.
Step 7	4	Erie	4 applies to Erie.	4 gets a slot at Erie, because she is a resident in Erie.

Final assignment Based on this method, the final allocations are:

Student	1	2	3	4	5	6
School	Huron	Clair	Erie	Erie	Ontario	Clair

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ON NON-COOPERATIVE FOUNDATION AND IMPLEMENTATION OF THE NASH SOLUTION IN SUBGAME PERFECT EQUILIBRIUM VIA RUBINSTEIN'S GAME

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ABSTRACT

In this paper we provide an exact non-cooperative foundation of the Nash solution via a unique (weakly) subgame perfect equilibrium payoff vector in a two-person bargaining game, which is a modification of the well-known alternate offer game by Rubinstein (1982). We also discuss the extent to which our exact and approximate supports of the Nash solution allow an implementation of the Nash solution in (weakly) subgame perfect equilibrium. We show that a sound interpretation as an implementation can only be found in very rare cases where the domains of players' preferences are extremely restricted.

Keywords: Nash program, non-cooperative foundation, implementation.

JEL Classification Numbers: C7, C71, C72.

We are grateful to the editor for suggesting to us to relate our present results to the history of the underlying problem and to the broad literature on bargaining and on implementation. We also thank an associate editor and anonymous referees for valuable feedback.

1. INTRODUCTION

The complex title of this article describes precisely its contents and goals, but it hides the enormous importance of the underlying problem of allocating justly the available resources among a population of individual agents.

Our problem in its most simple form is the problem of fair division of some divisible object among two persons. Here "fair" is a non-technical term whose formal specification depends on the situation and on potential property rights of the negotiating persons. Accordingly, either distribution or exchange may best describe the activity to be analyzed.

Although this fundamental problem of bilateral negotiation is still at the heart of economics and game theory, it has a very long history. This is competently and transparently described by Dos Santos Ferreira (2002) who, referring to Stuart (1982) and Burnet (1900), traces back modern treatments of bilateral exchange and bargaining to Aristotle's (ca. 335 B.C.) *Nichomachean Ethics*. He argues convincingly that the underlying ideas about proportionality and the arithmetic and geometric means of modern axiomatic bargaining solutions can be traced back to Aristotle's analysis.

Dos Santos Ferreira (2002, p. 568) considers "The Nichomachean Ethics in which Aristotle presents his analysis of bilateral exchange" as "undoubtedly one of the most influential writings in the whole history of economic thought" that "through the commentaries of Albertus Magnus and ... of his pupil Thomas Aquinas ... was one of the main sources of the Scholastic doctrine of just prices." He then follows this influence via Turgot (1766, 1769), Marx (1867), Menger (1871), and Edgeworth (1881) to the modern treatments, in particular the seminal contributions by Nash (1950, 1953) and Rubinstein (1982) underlying our present analysis. Shubik (1985) mentions the 'horse market model' of Böhm-Bawerk (1891) which became the forerunner of assignment games, as another 19th century work concerned with bilateral exchange. The contract curve offered by Edgeworth and the price interval of Böhm-Bawerk reflect the problem of indeterminacy inherent in those early approaches that was only solved by Zeuthen (1930) and Hicks (1932).

Harsanyi (1956) compared the modelling of bilateral bargaining before and after the appearance of the theory of games (see Bishop, 1963) and found Zeuthen's approach, which he presented in the language of game theory, superior to that of Hicks and demonstrated that Zeuthen's solution coincides with the solution provided by Nash (1950, 1953), who defined and axiomatically characterized his bargaining solution.

The first contributions to an analysis of bilateral bargaining via strategic non-cooperative games were independently presented by Ståhl (1972) and Krelle (1976), who provided a model of finite horizon alternate offers consecutive bargaining. That model was extended to infinite horizon sequential bargaining with discounting payoffs in the seminal article of Rubinstein (1982).

While cooperative axiomatic and non-cooperative strategic game theoretic models are based on quite different implicit assumptions about legal and institutional environments, an interesting question arises as to the relationship between the solutions to the bargaining problem offered by each approach. That question belongs to what, based on some short passages in Nash's work, has been termed the *Nash Program* in the literature (Binmore & Dasgupta, 1987a). According to Reinhard Selten in a private communication it is Robert Aumann who first had used that expression.

A first contribution to the Nash program was provided by Nash (1953) himself when he compared the Nash solution with the payoffs of his so called *simple demand game*. The continuum of Nash equilibria of this game, however, cannot be used as a support for the Nash solution, which corresponds to just one of them. Therefore Nash used a sequence of increasingly less distorted smooth games that converges to the simple demand game and whose set of infinite Nash equilibria converges to a unique (in the words of van Damme, 1991) *essential Nash equilibrium* of the simple demand game with the payoffs of the Nash solution. This analysis provided the first *approximate* non-cooperative support of the Nash solution. While Nash's analysis underpinning this result is vague and incomplete, later contributions including van Damme (1991) rendered more precise arguments.

As to an *exact* as opposed to an *approximate* support, Binmore & Dasgupta (1987b) close their article with the passage: "Finally, it is necessary to comment on the fact that none of the non-cooperative bargaining models which have been studied implement the Nash bargaining solution exactly. In each case, the implementation is *approximate* (or exact only in the limit)."

Notice that here "implementation" is meant as a non-technical alternative for *support* or *foundation* and is different from the more challenging concept *implementation* in the mechanism design literature. Both, *exact support* and *exact implementation* of the Nash solution will be analyzed in this paper based on a modification of Rubinstein's game.

In contradiction to the above quotation of Binmore & Dasgupta (1987b), the first *exact* non-cooperative support for the Nash solution to our knowledge was provided by van Damme (1986) (see also Naeve-Steinweg, 1999) via a unique Nash equilibrium payoff vector of a meta-bargaining game with specific subsets of the set of bargaining solutions building the players' strategy sets. With slight modifications of the sets of solutions admissible as strategies, Chun (1985) and Naeve-Steinweg (2002) proved analogous results for the Kalai-Smorodinsky solution. Admitting all bargaining solutions as strategies, Trockel (2002a) provided evidence in support of the Nash solution. Using a quite different approach, i.e., a Walrasian modification of Nash's simple demand game, Trockel (2000) proved the existence of a unique Nash equilibrium support for the Nash solution. An exact support for the Nash solution by a unique subgame perfect equilibrium payoff vector can be found in Howard (1992).

The relation between the Nash solution and the Rubinstein game was clarified in a seminal article by Binmore et al. (1986), based on an earlier article by Binmore (1980) published in Binmore & Dasgupta (1987a). Using two different two-person cooperative bargaining games generated via different types of utility functions (*time discounting* versus *risk* versus von-Neumann-Morgenstern) each imposed on the basic dynamic model of Rubinstein (1982), they provide approximations of the two respective Nash solution payoff vectors by the two unique subgame perfect equilibrium outcomes.

In order to proceed from the Nash program aspect of the Nash solution support by non-cooperative equilibria to the implementation of the Nash solution in some equilibrium, one needs to clarify the relation between the Nash program and implementation of solutions of cooperative games (rather than just of social choice rules)! Strictly speaking, a solution can possibly be implemented in some equilibrium only if it can be identified (which already implies a *restricted domain*) with some social choice rule, and in the special context of bargaining games with traditional point-valued rather than set-valued solutions with some *social choice function*.

As stated above, the Nash program has its roots in some passages of Nash's work. It is discussed in great detail by Serrano (2004) who notes: "Similar to the micro-foundations of the macroeconomics, which aim to bring closer the two branches of economic theory, the Nash program is an attempt to bridge the gap between the two counterparts of game theory (cooperative and non-cooperative). This is accomplished by investigating non-cooperative procedures that yield cooperative solutions as their equilibrium outcomes."

He then quotes the following passage from Harsanyi (1974): "Nash (1953) has suggested that we can obtain a clear understanding of the alternative solution concepts proposed for cooperative games and can better identify and evaluate the assumptions to make about the players' bargaining behavior if we reconstruct them as equilibrium points in suitably defined bargaining games, treating the latter formally as non-cooperative games."

The relation between implementation theory and the Nash program has been analyzed and discussed in Serrano (1997, 2004), Bergin & Duggan (1999), and Trockel (2000, 2002a,b, 2003). As our modification of the Rubinstein game applies to diverse variants of the Rubinstein model, we will work with a particularly simple and transparent special version that allows it to interpret the discount factor in both ways discussed in Binmore et al. (1986), namely as an indicator of either players' impatience or their risk aversion. Further, we discuss the impact of our exact and also the approximate support results on implementability of the Nash solution. This mechanism theoretical aspect is highly relevant for applications of axiomatic bargaining solutions as discussed in Binmore et al. (1986) and Gerber & Upmann (2006).

The rest of this paper is organized as follows: In Section 2, some basic notions of bargaining theory are introduced. Section 3 presents our version of the Rubinstein game. In Section 4 we introduce a proposition on weakly subgame perfect support of the Nash solution. In Section 5 the concept of weakly subgame perfect equilibrium is presented. In Section 6, we establish the existence of a subgame perfect equilibrium support for the Nash solution. In Section 7 we discuss possible implications of our results on implementing (a social choice function representing) the Nash solution in (weakly) subgame perfect equilibrium.

2. BASIC CONCEPTS AND NOTATION

We use the following two different types of games, namely two-person cooperative bargaining games and two-person non-cooperative games in extensive form, briefly extensive games. The definition of the latter ones is quite intricate though their illustrations via game trees are very intuitive. We shall use this notion as treated in Myerson (1991, chapter 2) or in Mas-Colell et al. (1995, chapter 7).

As to cooperative bargaining games, we use the following:

Definition 1. *A two-person bargaining game is a pair* (U,d) *where* $d \in U \subset$

$\mathbb{R}^2_+$ *and* U *is non-empty, convex, compact and there exists an* $x \in U$ *such that* $x >> d$. *The set of two-person bargaining games is denoted by* $\mathbb{B}$.

Definition 2. *A bargaining solution is a mapping*

$$\begin{aligned} L : \mathbb{B} &\longrightarrow \mathbb{R}^2 \\ (U,d) &\longmapsto L(U,d) \in U \end{aligned}$$

If we can associate any $(U,d) \in \mathbb{B}$ with some extensive game $G^{U,d}$ whose subgame perfect equilibrium payoff vectors coincide with $L(U,d)$, then the game $G^{U,d}$ *supports* the solution $L(U,d)$ of (U,d) by subgame perfect equilibrium. Such a support provides an exact non-cooperative foundation for the solution L in the sense of the Nash program (Binmore et al., 1986; Serrano, 2004). Exact non-cooperative foundations for the Nash solution have been provided in van Damme (1986), Naeve-Steinweg (1999), Howard (1992), Naeve (1999), Trockel (2000, 2002b).

In the present paper, we want to present an exact non-cooperative foundation for the Nash Solution based on the Rubinstein game.

The relevant notion of a subgame perfect Nash equilibrium due to Selten (1965) is defined as a Nash equilibrium of an extensive game which induces a Nash equilibrium in any subgame.

3. THE RUBINSTEIN GAME

The Rubinstein infinite horizon strategic bargaining model with the two players' alternating offers is concerned with how to divide a unit of some perfectly divisible good with a resulting allocation for the two players. This game introduced by Rubinstein (1982) was meant to analyze "what 'will be' the agreed contract, assuming that both parties behave rationally?" No link to axiomatic cooperative bargaining or even the Nash solution is indicated or, at least it appears so, intended. Discount factors δ_1, δ_2 are assumed to be fixed for both players. Possible consequences for the subgame perfect equilibrium regarding a relation to the Nash solution if δ_1, δ_2 are close to 1 are not an issue.

It was Binmore (1980) who related a dynamic version of Nash's simple demand game that he called "modified Nash demand game II" to the (asymmetric) Nash solution, by approximating it by unique subgame perfect equilibrium payoff vectors of his strategic games where the discount factors δ_1, δ_2 come close to 1. There are various versions of the original model of Rubinstein (1982) which has finite horizon predecessors in Ståhl (1972) and Krelle (1976).

The most general version is used in Osborne & Rubinstein (1994, chapter 7) where the set of feasible agreements X is a non-empty compact, connected set of some Euclidean space. In the introduction of Binmore et al. (1986), the set X represents "physical outcomes" building the two players' "possible agreements". The formal model in their section 2 "strategic bargaining models" specifies this set as $\bar{X} = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$. In both cases, inefficient feasible agreements are principally possible. Rubinstein (1982) uses $X = [0,1]$ where the "split the pie" assumption excludes $x_1 + x_2 < 1$ and implies efficiency of the agreements.

Binmore (1980) implicitly expresses the feasible set of alternatives as the set of payoff vectors in the utility image of some unspecified outcome set and explicitly assumes the set U of feasible payoff vectors to be a non-empty compact, convex set in $\mathbb{R}^2$. In this framework, the feasible proposal pairs (x_1, x_2) of the two players are payoff vectors and thus are directly comparable to the Nash solution point of (U, d). Here $d \in U$ is the disagreement payoff vector of the cooperative bargaining problem.

In the framework of the other approaches based on X, histories without any agreement at any time are mapped by the players' utility functions π_1 and π_2 onto such a $d \in U$, where U is $\pi(X) = \{(\pi_1(x), \pi_2(x)) \mid x \in X\}$.

As shown and in fact exploited in Binmore et al. (1986), the same underlying X may lead via different sets of players' utility functions to different sets U and accordingly different Nash solution points $N(U, d)$. Binmore et al. (1986) presents two detailed versions of strategic bargaining á la Rubinstein: one with time preferences and impatient players, the other one with exogenous risk of breakdown of negotiations with risk averse players who have von Neumann-Morgenstern utility functions. In both models, the subgame perfect equilibrium payoff vectors converge to the respective Nash solution points of the induced utility possibility set U, where $\delta \in (0,1)$ converges to 1.

We shall use a particularly simple and transparent version of the strategic bargaining that simultaneously allows both interpretations, namely impatience or risk aversion of players as represented by δ as a discount factor as a probability of continuation of the negotiation process. This model is essentially that of Binmore (1980) and of Example 125.1 in Osborne & Rubinstein (1994). That will not affect the validity of our analysis for the more complex versions mentioned above. The interpretation of the discount factors we choose will have, however, a crucial impact on the application of our results to implementability in the mechanism theoretic sense. A reference for this model is also the collection of sections 6.7.1, 6.7.2 and 10.1.2 in Peters (2015).

Let $X = [0,1]$ represent the pie to be split among the two players. The payoff vector resulting from a division $x = (x_1, x_2)$ with $x_1 + x_2 = 1$ is determined by the utility functions $u_i : X \to \mathbb{R}$ with $u_i(x_i) = x_i$ for all $i = 1,2$. Discounting utilities with $\delta \in (0,1)$ results in a payoff vector $\delta^t x \in U$ for an agreement $x \in X$ at time $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We assume $U = \triangle := \{(z_1, z_2) \in \mathbb{R}^2_+ \mid z_1 + z_2 = 1\}$ and $d = u(0,0) = 0 \in \mathbb{R}^2$.

For notational convenience, we shall identify players' proposals $x_1, y_2 \in X$ with the payoff vectors $(x_1, 1 - x_1)$ and $(1 - y_2, y_2) \in \triangle = U$. These identifications of $\triangle$ with X via the two projections on the first and second coordinate, respectively, allow us to speak of proposals in X or in $\triangle$ without creating confusion.

We treat only the symmetric case with the discount factor δ being the same for both players. The extension to the asymmetric case is possible like in the quoted literature and is straightforward.

In contrast to the finite horizon version of Ståhl (1972), in the Rubinstein game backward induction cannot be used for determining subgame perfect equilibria.

In our specific "split the pie" framework, there exists a "unique (not just essentially unique)" subgame perfect equilibrium (Osborne & Rubinstein, 1994, p. 125). This unique subgame perfect Nash equilibrium σ^*_δ is characterized as follows:

> $\sigma^*_{\delta,1}$: At $t \in 2\mathbb{N}_0$, propose $x^*_\delta := (\frac{1}{1+\delta}, \frac{\delta}{1+\delta}) \in U$; at $t \in 2\mathbb{N}_0 + 1$, accept any proposal $z \in U$ of player 2 if and only if $z_1 \geq \delta x^*_{\delta,1}$.
>
> $\sigma^*_{\delta,2}$: At $t \in 2\mathbb{N}_0 + 1$, propose $y^*_\delta := (\frac{\delta}{1+\delta}, \frac{1}{1+\delta}) \in U$; at $t \in 2\mathbb{N}_0$, accept any proposal $z \in U$ of player 1 if and only if $z_2 \geq \delta y^*_{\delta,2}$.

The parameters x^*_δ and y^*_δ build the unique solution of the two equations $x_2 = \delta y_2$, $y_1 = \delta x_1$, for $x, y \in U$.

The stationarity of these equilibrium strategies is a result rather than an assumption (Osborne & Rubinstein, 1994, p. 126).

It can be easily verified that the Nash products $x^*_{\delta,1} x^*_{\delta,2}$ and $y^*_{\delta,1} y^*_{\delta,2}$ of x^*_δ and y^*_δ are the same. As both points are on the efficient boundary of U, this is also true in the more general case where the u_i's are not identity functions; see for instance Figure 311.1 in Osborne & Rubinstein (1994) with δ converging to 1, both of x^*_δ and y^*_δ converge to $z^* := N(U,0)$, the Nash solution point of $(U,0)$.

The choice of $X = [0,1]$ like in Rubinstein (1982) used also in Example 120.1 of Osborne & Rubinstein (1994), is less natural than the $\bar{X} := \{(x_1,x_2) \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\}$ chosen in Binmore et al. (1986), if one wants to compare the subgame perfect equilibrium payoffs with the Nash solution of cooperative games for classes considered usually in the literature. There U is generally a compact, convex (often strictly convex) set with $d \in U$. Our special case deals only with the efficient boundaries of such sets and $d = (0,0)$ does not satisfy $d_1 + d_2 = 1$. Making a proposal in our model corresponds to making a Pareto efficient proposal in the general case. In fact nothing relevant would change, if we replaced $(U,0) = (X,0)$ by $(\bar{X},0)$.

4. AN EXACT NON-COOPERATIVE FOUNDATION

Denote the extensive form game of Rubinstein with discount factor $\delta \in (0,1)$ described in the previous section by G^δ and its subgame perfect equilibrium payoff vector by $\hat{z}^\delta$. Notice that the limit cases of $\delta = 0$ and $\delta = 1$ correspond to the ultimatum game and the Nash simple demand game, respectively.

In G°, the whole cake goes to the proposer in the unique subgame perfect equilibrium. In G^1, every Nash equilibrium payoff vector of the Nash simple demand game can be realized via some subgame perfect equilibrium. This discontinuity of the subgame perfect equilibrium correspondence at G^1 excludes its use for an exact support of the Nash solution.

It is our goal in this article to define a game that is based on the Rubinstein game and can play the role of a missing limiting game that turns the approximate support of the Nash solution due to Binmore et al. (1986) into an exact one.

In order to gain some intuition for the game G to be defined, we consider a game G_δ for an arbitrary $\delta \in (0,1)$ defined as follows: At stage 0, player 1 proposes some $x \in \triangle$, then player 2 either accepts, in which case the play ends with paying out the proposed payoffs, or she rejects with her only alternative move by which she decides that the Rubinstein game G^δ has to be played with player 1 starting as the proposer. Obviously, G_δ has the same unique subgame perfect equilibrium payoff vector as G^δ.

Next, we get the same result via replacing G_δ by $\hat{G}_\delta$ which we define as follows: At stage 0, player 1 proposes $x \in \triangle$. Player 2 then reacts by either accepting, which yields the end of the play and payoff vector x, or she reacts by choosing some $\rho \in (0,\delta]$ which means that G^{ρ} has to be played. Again it is clear that the unique subgame perfect equilibrium is $\hat{z}^{\delta}$.

In all those games, the respective $\hat{z}^{\delta}$ can be reached in two different ways: Either by the proposal $x := \hat{z}^{\delta}$ of player 1 being accepted by player 2, or by the rejection of player 2 via the unique subgame perfect equilibrium of G^{δ}.

What happens if we replace the choice set $(0,\delta]$ used in $\hat{G}_{\delta}$ by $(0,1)$? Let us denote the game resulting from doing so by $\hat{G}$. The Nash equilibrium payoff vector $z^* = \lim_{\delta\to 1} \hat{z}^{\delta}$ can be realized in $\hat{G}$ only as an accepted proposal. No choice of $\rho \in (0,1)$ prescribing the play of $\hat{G}_{\rho}$ and its unique subgame perfect equilibrium payoff vector $\hat{z}^{\rho}$ could possibly justify a rejection of the proposal z^*. But unfortunately, this equilibrium fails to be subgame perfect! Off the equilibrium path, any proposal $x \neq z^*$ would be to the disadvantage of player 1 or could be rejected by player 2 via a suitable choice of $\rho \in (0,1)$. But as there is no optimal way to choose such ρ, the game G does not have any subgame perfect equilibrium.

We can establish, however, that z^* is the unique *weakly subgame perfect* equilibrium payoff vector of $\hat{G}$. And we will argue that the use of weakly subgame perfect equilibria secures the credibility of threats sufficiently well in order to justify this concept.

Moreover, we shall for convenience constrain ourselves to the countable set of $\delta_k \in (0,1)$ with $\delta_k := k/(k+1)$, $k \in \mathbb{N}$. Then $\lim_{k\to\infty} \delta_k = 1$. Accordingly, we denote G^{δ_k} and $\hat{z}^{\delta_k}$ by G^k and $\hat{z}^k$, respectively.

Although our main result Theorem 1 will provide a subgame perfect equilibrium support for the Nash solution, we believe that our Proposition 1 below is also an interesting support result by itself. It builds a basis for proving Theorem 1.

We shall now introduce first our extensive game G, denoted by $G(=G^{U,d})$, and then the concept of a weakly subgame perfect equilibrium that coincides with that of a subgame perfect equilibrium on finite games.

At round 0, one of the two players of the bargaining game (U,d) is selected randomly with probability $1/2$ to make a proposal $z \in \triangle$. After the first player makes a proposal, the other reacts by choosing an element $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

If she chooses 0, the proposal is accepted and the payoffs will be realized. If she chooses $k \in \mathbb{N}$, the proposal is rejected and the game G^k will be played, whose unique subgame perfect equilibrium payoff vector $\hat{z}^k$ at $t = 1$ results in the discounted payoff vector $\delta_0\hat{z}^k$ at $t = 0$. In order to simplify the notation, we assume, w.l.o.g., $\delta_0 := 1$.

Like the Rubinstein games G^k, $k \in \mathbb{N}$, the game G has an infinite number of Nash equilibria, among them the one where every player always chooses the proposal $N(U,d)$ and always accepts this proposal and rejects any other one. Any other bargaining solution can be supported by Nash equilibria of G in an analogous way. This is essentially the situation in *Nash's simple demand game*.

Our final solution in the theorem will be based on subgame perfect equilibrium. But, as in contrast to the G^k, $k \in \mathbb{N}$, the game G does not have any subgame perfect equilibrium, we shall first prove a proposition where we use the (weaker) concept of weak subgame perfectness due to Trockel (2011).

Definition 3. *A Nash equilibrium of an extensive game is weakly subgame perfect when it induces some Nash equilibrium in every subgame in which a Nash equilibrium exists.*

We shall very briefly discuss this concept in section 5. Here we will use it in order to state and prove our first non-cooperative support result.

Proposition 1. *For the bargaining game* $(U,d) = (U,0)$*, the extensive game* $G(= G^{U,d})$ *as defined above has an infinity of weakly subgame perfect equilibria with identical equilibrium path and equilibrium payoff vector* $z^* = N(U,d)$*.*

Proof. The proof consists of several steps:

- One type of Nash equilibria is defined by the following rule for both players:
  As the proposer choose z^*, as the follower accept exactly those proposals that are at least as good as z^*. In any Rubinstein subgame G^k, play according to the unique subgame perfect equilibrium.

  It is obvious that no other proposal nor any other reaction to a proposal can constitute an advantageous unilateral deviation for any player. In these Nash equilibria, z^* is realized in the first round.

- There does not exist any subgame perfect equilibrium in this game.
  This follows from the fact that G has subgames without Nash equilibria, namely those that directly follow any proposal $z \in \triangle$ that offers to the other player a payoff smaller than her coordinate of z^*. Although this player should reject, there is no optimal $k \in \mathbb{N}$ to do so.

- The Nash equilibria described in the first step are weakly subgame perfect.
  The trivial subgame G has Nash equilibria according to the first step. Each of those induces on every Rubinstein subgame G^k, $k \in \mathbb{N}$, a (subgame perfect) Nash equilibrium. All subgames starting at proposals that offer the other player a payoff at least as high as her coordinate of z^* have "acceptance" as the optimal, hence Nash equilibrium choice. Off the equilibrium path, these equilibria induce subgame perfect equilibria of Rubinstein games G^k, $k \in \mathbb{N}$.

  The only remaining subgames are those without equilibria described in the second step. So the Nash equilibria described in the first step are weakly subgame perfect.

- Any other Nash equilibrium fails to be weakly subgame perfect.
  In order to establish this claim, consider a Nash equilibrium payoff vector $\widetilde{z} \neq z^*$.

  There are two possible ways how $\widetilde{z}$ may have been realized:

  a) as an accepted proposal in the first round,
  b) as the result of a subgame G^k that started right after a first proposal has been rejected.

  W.l.o.g., let player 1 be the first proposer in the first round, and hence in every G^k if it is played after rejection of player 2.

  Case (a): If player 1 proposes $\widetilde{z}$ with $\widetilde{z}_2 < z_2^*$ and player 2 accepts, this cannot possibly be a part of a Nash equilibrium, since player 2 could just reject with a sufficiently large k and ensure herself $\widehat{z}_2^k$ arbitrary close to $N_2(U,d)$, and $\widehat{z}_2^k > \widetilde{z}_2$.

  If player 1 proposes $\widetilde{z}$ with $\widetilde{z}_2 > z_2^*$ and player 2 accepts, player 1 could have improved by proposing z^* unless player 2 rejects z^*. So only if player 2's strategy contains rejection of z^*, then $\widetilde{z}$ could possibly be a Nash equilibrium payoff vector. But rejecting z^* is only possible via choosing some G^k, $k \in \mathbb{N}$. As in each G^k, the unique subgame perfect equilibrium payoff for player 2 would be $\widehat{z}_2^k < z_2^* < \widetilde{z}_2$, none of them would justify rejection of z^*. Therefore, the payoff vector $\widetilde{z}$ can possibly result only from a Nash equilibrium that is not weakly subgame perfect and satisfies $\delta_0 \widetilde{z}_2 = \widetilde{z}_2 > \widehat{z}_2^k = \delta_0 \widehat{z}_2^k$.

Case (b): Suppose $\widetilde{z}$ is a weakly subgame perfect equilibrium payoff vector of G. Then $\widetilde{z} = \delta_0 \widehat{z}^k = \widehat{z}^k$ for some $k \in \mathbb{N}$. But then player 2 could improve by choosing $k+1$ with discounted payoff vector $\delta_0 \widehat{z}^{k+1} = \widehat{z}^{k+1}$. This implies that $\widetilde{z}$ is not a Nash equilibrium payoff vector of G, yielding a contradiction.

□

5. REMARKS ON WEAKLY SUBGAME PERFECT EQUILIBRIA

In contrast to the games G^k, $k \in \mathbb{N}$, the game G has an infinity of weakly subgame perfect equilibria. How bad is this? There is no coordination problem involved as long as both players stay on the equilibrium path and the equilibrium payoff is uniquely determined. So the multiplicity of those equilibria appears to be harmless, in particular as subgame perfect equilibria also may have multiple ways of behavior off the equilibrium path.

So the criticism could only be based on the lack of credible threats to reject, because there is no optimal way of rejecting! But from a decision theoretical point of view, this criticism is dubious. If there is a choice between money amounts $\{-50, 1, 2, \ldots, 10\}$, we take it for granted that -50 is rejected (via accepting 10). If the choice is among $\{-50\} \cup \mathbb{N}$, do we think that -50 will be accepted just because there is no best alternative? In real life, we avoid very bad or worst cases even if we are unable to do that in an optimal way.

But as this is a controversial point, we will provide a modified non-cooperative support result via subgame perfect equilibria in the next section.

6. SUBGAME PERFECT EXACT SUPPORT

When attempting to base an exact subgame perfect equilibrium foundation for the Nash solution based on Rubinstein's games G^k, $k \in \mathbb{N}$, the dilemma is the appearance of those subgames starting right after an initial proposal that do not have any Nash equilibrium. There are two possible ways that one may consider:

1. Add a best alternative to the set $\mathbb{N}_0$. We did not see any natural way to do so. We might end up with no or a multiplicity of subgame perfect equilibria. Anyway, we did not follow this approach.

2. Stop those subgames starting right after the first proposals from being subgames. In order to do so, we modify our original game G in the following way: in the beginning, the proposer is chosen randomly with probability $1/2$, but both players do not observe that random choice. So each player has probability $1/2$ that she is the chosen proposer and $1/2$ that she has to react to her opponent's proposal. Accordingly, both players' strategies have to contain full descriptions of what they would propose and how they would react to any possible proposal $z \in \triangle$.

   We follow that approach and define now an extensive game $\widetilde{G}$, denoted by $\widetilde{G}(= \widetilde{G}^{(U,d)})$ in detail.

In the beginning, the referee throws a fair coin in order to decide who of the two players starts with a proposal $z \in \triangle$. But the result of this random choice is not observed by the players. Then, not knowing whether they will start with a proposal or a rejection to the proposal, but knowing that any Rubinstein game played after a rejection starts with another proposal of the proposer, both players simultaneously and independently choose a pair consisting of a proposal and a reaction function on the set of possible proposals.

On this basis, we define now the extensive game $\widetilde{G}$ as follows:

At stage 0, the following things happen:

1. The referee throws a fair coin in order to randomly but privately determine which player will act as the proposer.

2. Both players, not knowing which of them will act as the proposer, simultaneously submit pairs (x_1, f_1), $(x_2, f_2) \in U \times \mathbb{N}_0^U$ to the referee, where x_i are their proposals, f_1 and f_2 are reaction functions to their opponent's proposal.

3. The referee informs both players on their randomly determined roles. The proposer whose role is now common knowledge is w.l.o.g. player 1.

At stage 1, the game either ends if $f_2(x_1) = 0$ or continues with stage 2 if $f_2(x_1) = k \in N$. In this case, player 1 starts at stage 2 with a proposal $x_1 \in U$ in the Rubinstein game G^k. The rest of the game is just playing this Rubinstein game G^k.

The specific structure of $\widetilde{G}$ at stage 0 has two consequences that are crucial for our Theorem:

First, it guarantees that the players' choices of proposals together with their reaction functions build actions at $t = 0$ in non-singleton information sets. Therefore, no subgame starts with such actions. Secondly, the full information of both players about their respective roles as proposer and reactor before the start of actions at stage 2 prevents the annoying situation that all G^k, $k \in N$ would start only at non-singleton information sets. In that case, the only subgame of $\widetilde{G}$ would be $\widetilde{G}$ itself.

Clearly, $N(U,d)$ would still be a subgame perfect equilibrium payoff vector. But the whole plethora of Nash equilibria would become subgame perfect ones, too!

Under the aspect of trying to support the Nash solution, we would essentially be back to Nash's simple demand game. Notice that in $\widetilde{G}$, the Rubinstein subgames G^k of G, $k \in \mathbb{N}$ will reappear twice: once via a rejection of player 2 and once as a rejection of player 1 in that part of $\widetilde{G}$ that follows the non-realized choice of player 2 as the proposer. Only those G^k, $k \in \mathbb{N}$ following a rejection by player 2 are potentially effective for the outcome of the game. But also in the other (now irrelevant) Rubinstein games, following rejections by player 1 of proposals by player 2, subgame perfectness of $\widetilde{G}$ requires the players to play subgame perfect equilibria.

It is impossible to just cancel that at stage 1 irrelevant part of the game tree as it is relevant for the players' choices at stage 0 before they are informed about their respective later roles.

This modification of our original game G is illustrated in the following two figures. Figure 1 and 2 are equivalent stylized illustrations of stages 0 and 1 of $\widetilde{G}$.

Notice that these figures are only schematic illustrations of the game $\widetilde{G}$ in its first stages $t = 0,1$ rather than complete game trees. The infinite action sets for both players are represented in these figures only by one typical action for each player, namely (x_1, f_1), (x_2, f_2). In the figures G^0 is the degenerate game consisting of the singleton set $\{0\}$ representing acceptance of a proposal.

The construction in Figure 2 is similar to the way in which Sudhölter et al. (2000) define the canonical extensive form for the battle of sexes game. It has precisely the intended effect in our present context. The one-player subgames without optimal actions in the reduced game have vanished now.

The only remaining subgames of the modified game $\widetilde{G}$ are $\widetilde{G}$ itself and the Rubinstein games G^k, $k \in \mathbb{N}$.

Figure 1.

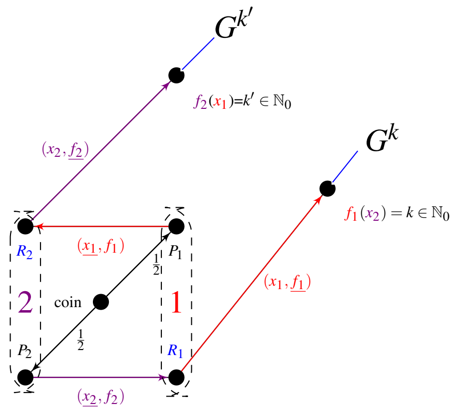
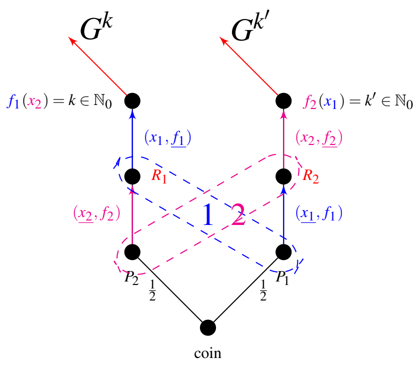


Figure 2.



In $\widetilde{G}$, any subgame perfect equilibrium is a pair $(x_1, f_{1,k^1}; x_2, f_{2,k^2})$ with $(x_1, x_2) = z^*$ and $f_{i,k^i} : U \to \mathbb{N}_0$ such that

$$f_{i,k^i}(x) = \begin{cases} 0 & : x_i \geq z_i^* \\ k^i & : x_i < z_i^* \end{cases}$$

and $k^i \in \{k \in \mathbb{N} \mid \widehat{z}_i^k > x_i\}$, $i = 1, 2$.

After the first moves of both players, the game ends either in its equilibrium with payoff vector z^* or, else, in some game G^k, $k \in \mathbb{N}$, where the unique subgame perfect equilibrium is induced. The multiplicity of subgame perfect equilibria arises from the various k^i that may be chosen for f_{i,k^i}, $i = 1, 2$. But the equilibrium path is unique.

We can formulate the modified version of our support result now as follows:

Theorem 1. *For the bargaining game* $(U,d) = (U,0)$*, the extensive game* $\widetilde{G}(= \widetilde{G}^{(U,d)})$ *as defined above has an infinity of subgame perfect equilibria with identical equilibrium path and equilibrium payoff vector* z^**.*

Proof. This theorem is in fact a corollary to the Proposition.

All Rubinstein games G_k, $k \in \mathbb{N}$ occur as subgames in $\widetilde{G}$. There is no subgame anymore however, that starts after a proposal with actions $k \in \mathbb{N}_0$. The k chosen after a proposal x_1 of player 1 is now determined by the simultaneous choice (x_2, f_2) of player 2 at both points of her information set at $t = 0$ via $k := f_2(x_1)$. $k' := f_1(x_2)$ has been eliminated from further consideration by the referee's random choice of player 1 as the proposer. Accordingly, a lack of an optimal way of rejecting a proposal cannot destroy the subgame perfectness of a weakly subgame perfect Nash equilibrium. As at $t = 0$, there do not exist singleton information sets for the players, the games $\widetilde{G}$ and G_k, $k \in \mathbb{N}$ build all subgames of $\widetilde{G}$.

Now, we define a Nash equilibrium of $\widetilde{G}$ literally as in the first step of the proof of our Proposition. Any weakly subgame perfect equilibrium of G becomes (or corresponds to) subgame perfect equilibria of $\widetilde{G}$. Any other Nash equilibrium payoff vector could in G only result from non weakly subgame perfect behavior in some G^k, thus in $\widetilde{G}$ only from violating subgame perfectness in that G^k. This proves our Theorem. □

Remark 1. *The subgame perfect Nash equilibria of* $\widetilde{G}$ *are in fact even sequential Nash equilibria! The beliefs of both players can be expressed by probabilities on their non-singleton information sets. Whatever probabilities there may be, however, they do not have any influence on their decisions at these information sets, as every choice* (x_i, f_i), $i = 1,2$ *is intended to be optimal at each point of the information set, respectively. Only the parts of those pairs* (x_i, f_i), $i = 1,2$, *relevant at the different points of the information sets differ.*

7. FROM SUPPORT TO IMPLEMENTATION

The step from a non-cooperative support of a cooperative solution to a mechanism theoretic implementation is not trivial and requires some care and in fact certain assumptions. In Serrano (1997), we find the following passage:

> "The Nash Program and the abstract theory of implementation are often regarded as unrelated research agendas. Indeed, their goals

> are quite different: while the former attempts to gain additional support for cooperative solutions based on the specification of certain non-cooperative games, the latter tries to help an incompletely informed designer implement certain desirable outcomes. However, it is misleading to think that their methodologies cannot be reconciled. A common critics that is raised against the mechanisms in the Nash program is that they are not performing real 'implementations' since their rules depend on the data of the underlying problem (say the characteristic function) that the designer is not supposed to know."

Bergin & Duggan (1999) also emphasize the importance of independence of the game rules expressed by a mechanism of the players' preference profiles. And it is crucial now that payoffs are in utilities representing preferences rather than in money.

The very fact that the presence of an outcome space is an additional ingredient in mechanism theory as compared to the Nash program indicates that these two can hardly be considered "equivalent", as claimed, for instance in Dagan & Serrano (1998)(abstract). Detailed treatments of the relation between the Nash program and the implementation theory can be found in Bergin & Duggan (1999), Trockel (2000, 2002a) and Serrano (2004).

The conditions which are necessary in order to have a non-cooperative support that automatically provides an implementation in some equilibrium concepts are often satisfied in models in the literature (Moulin, 1984; Howard, 1992). However, strictly speaking, there *solution based social choice rules* are implemented (Trockel, 2003).

This holds also true in principle for the model of Rubinstein (1982), as used in Binmore et al. (1986) and Osborne & Rubinstein (1994). However, the situation there is different. In a strict sense, these results do not provide a non-cooperative implementation for the Nash solution on a given prespecified class of two person cooperative bargaining games. They rather just define such an implementation for the classes of those bargaining games generated by their game forms together with their different types of utility functions. From a puristic point of view, there is missing an axiomatization of the Nash solution on those classes. Clearly, this solution is still well defined as the maximizer of the Nash product. In fact, the section 3 in Binmore et al. (1986) has the heading *Nash solution as an approximation to the equilibria*. This terminology differs from the one prevailing in the literature on non-cooperative foundation where one thinks of supporting the Nash solution by equilibria of non-cooperative games, exactly or by approximation. In fact, the equilibria do approximate the Nash solution.

The Nash program is based on two passages in Nash (1951) and Nash (1953). While the first one may be interpreted as giving a higher priority to the strategic than to the axiomatic approach to bargaining, the second one from the introduction in Nash (1953) emphasizes the equal importance of both approaches:

> "We give two independent derivations of our solution of the two-person cooperative game. In the first, the cooperative game is reduced to a non-cooperative game. To do this, one makes the players' steps of negotiation in the cooperative game become moves in the non-cooperative model. Of course, one cannot represent all possible bargaining devices as moves in the non-cooperative game. The negotiation process must be formalized and restricted, but in such a way that each participant is still able to utilize all the essential strengths of his position.
>
> The second approach is by the axiomatic method. One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely. The two approaches to the problem, via the negotiation model or via the axioms, are complementary; each helps to justify and clarify the other."

A *justification* in Nash's sense of the Nash bargaining solution concept on a specified *whole class of bargaining games* via the non-cooperative approach as opposed to just as the Nash solution point of *one specific bargaining game* requires intuitively that each game in that class can be generated or represented by a game in a family of "similar" strategic games. Such kind of *uniform* support for a whole class leads naturally to a common game form underlying that family of strategic games which is already an important step towards implementation.

Binmore et al. (1986) by using the terms *time-preference Nash solution* and *von Neumann-Morgenstern Nash solution* stresses the fact that their two models provide approximate supports for the Nash solution on *different sets* of bargaining games.

The other direction of the Nash program, namely *justification* and *clarification* of the Rubinstein approach, is not so obvious. This model is defined with *discount factors* as important ingredients, technically and conceptually. But as soon as the discounting is taken serious rather than almost neglected the Nash solution of the induced cooperative bargaining games may differ significantly from the subgame perfect equilibrium payoff vectors of the strategic games.

As far as *implementation* is concerned, this fact is not disturbing, however. What in implementation theory has to be implemented is a social choice rule. And the implementation is not conceived as a *justification* or *clarification* like in the Nash program. Rather the social choice rule is the inherently justified solution method for a social choice problem with a specified outcome set. Its implementation in a strategic equilibrium concept represents the idea of realizing something already accepted as socially desirable via strategic interaction in the society according to certain rules, namely a *mechanism* or *game form*.

In situations where solutions of certain games can be identified with social choice rules, or as Hurwicz (1994) termed them, *desirability correspondences*, non-cooperative foundation may extend to implementation.

In our context, the Nash solution has to play the role of the social choice rule. One can easily see that the search for a natural and adequate outcome set needed for implementation leads to different results for the various versions of sequential bargaining in Binmore (1980), Binmore et al. (1986) or Osborne & Rubinstein (1994). Consequently, the generated utility spaces or cooperative bargaining games may be quite different.

In this context, it is important to notice that our use of the payoff set U (identified with the underlying $X = [0,1]$) in this paper on which the negotiation is modeled rather than on the set $\bar{X} = \{x \in \mathbb{R}^2_+ | x_1 + x_2 \leq 1\}$ is just for convenience and simplicity of presentation, like the treatment in Binmore (1980). It could as well have been formulated via the models in Binmore et al. (1986) like for instance in Osborne & Rubinstein (1994, Proposition 310.3).

From the conceptual point of view, there is a fundamental difference between the two models in Binmore et al. (1986) and therefore also in the two interpretations of the factor δ in the results of our present paper as far as the implementability problem is concerned.

If the Nash solution should be implemented on a class of bargaining games resulting from players who are really impatient then the time preference Nash solution cannot even approximately be implemented because the discounting factor cannot be made close to 1 by the designer. So the time preference Rubinstein approach could lead to implementation of the Nash solution only on a class of bargaining games generated by the strategic games with players whose idiosyncratic discount factors would have to be close enough to 1. Moreover, because of symmetry of the Nash solution, these discount factors would have to be the same for any pair of players negotiating. That means that practically we can forget about approximate implementation in that context.

A similar argumentation holds for those bargaining games in the second model of Binmore et al. (1986) that are generated when players are strongly risk averse. Only uniform weak enough risk aversion would allow implementation of the Nash solution for that induced class of bargaining games.

A third possibility is to think of a pool of risk neutral perfectly patient players. In that case, the second model of Binmore et al. (1986) allows to give the breakdown probabilities as instruments into the hands of the designer. This fits well the structure of our games G and $\tilde{G}$. Only then we would get arbitrarily close to implementation of the Nash solution on the induced class of cooperative bargaining games. Only in that case our two present results could be transformed into conceptually meaningful implementation results. Yet, this domain of the Nash social choice rule would be very specific and very small.

Our exact non-cooperative foundation results for the Nash bargaining solution that we have presented in this article are, as we mentioned earlier, not the only ones in the literature.

An approximate support via a modification of Rubinstein's game works also for $n > 2$; see Moulin (1984).

As to the tasks of *clarification* and *justification*, it is interesting to point out to the similarity of insights into the meaning of the Nash solution provided by the Rubinstein alternating offer approach and by the non-cooperative strategic unique Nash equilibrium support and implementation in Trockel (2000) based on *Walrasian* payoff functions.

In the latter one, this Walrasian property of the Nash solution (Trockel, 1996) represents perfect competition that simulates an abundance of *outside options* for both players in a game protecting them from exploitation by their respective opponents. In the Rubinstein game with almost negligible discounting and in our modified games, it is the *infinity of future options*, (almost) equally valuable, that creates the same effect. That suggests the interpretation of implementing the Nash solution as a sort of surrogate for sufficient competitive pressure.

When the implementation is not an issue but only a non-cooperative foundation that helps to *justify* the Nash bargaining solution and to *clarify* its meaning then our results will work equally well (but as we think not better) as the approximate results based on the Rubinstein game. The advantage of our modified Rubinstein game forms lies in the fact that they somehow make the subgame perfect equilibrium payoff function continuous at $\delta = 1$.

While the Rubinstein games with δ converging to 1 induce a limit for the associated sequence of subgame perfect equilibrium payoffs, there is no associate *limit model* whose subgame perfect payoff equilibrium vectors would confirm this result. As we have shown in section 4, our game G is the limiting game for a sequence of modified versions of Rubinstein games having the same subgame perfect equilibrium outcomes as these.

We would like to conclude the paper with one remark: Totally analogous results to our Proposition and our Theorem can be proved via using Ståhl's rather than Rubinstein's model. In the games G^k, k would have to be replaced by a double index (k,l), where k represents the discount factor δ, and l represents the number of stages of a (finite horizon!) Ståhl game.

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AN ASCENDING MULTI-ITEM AUCTION WITH FINANCIALLY CONSTRAINED BIDDERS

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ABSTRACT

Several heterogeneous items are to be sold to a group of potentially budget-constrained bidders. Every bidder has private knowledge of his own valuation of the items and his own budget. Due to budget constraints, bidders may not be able to pay up to their values and typically no Walrasian equilibrium exists. To deal with such markets, we propose the notion of 'equilibrium under allotment' and develop an ascending auction mechanism that always finds such an equilibrium assignment and a corresponding system of prices in finite time. The auction can be viewed as a novel generalization of the ascending auction of Demange et al. (1986) from settings without financial constraints to settings with financial constraints. We examine various strategic and efficiency properties of the auction and its outcome.

Keywords: Ascending auction, budget constraint, equilibrium under allotment.

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1. INTRODUCTION

Auctions are typically the most efficient mechanism for the allocation of private goods and have been used since antiquity for the sale of a variety

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of items. The academic study of auctions grew out of the pioneering work of Vickrey (1961) and has blossomed into an enormously important area of economic research over the last few decades. Standard auction theory assumes that all potential bidders are able to pay up to their values on the items for sale. However in reality buyers may face budget constraints for a variety of reasons and therefore may be unable to afford what the items are worth to them. As stressed by Maskin (2000) in his Marshall lecture, the consideration of financial constraints on buyers is particularly relevant and important in many developing countries, where auctions are used to privatize state assets for the promotion of efficiency, competition and development, but entrepreneurs may often be financially constrained. Financial constraints not only occur in developing countries but also in developed nations. In particular, Che & Gale (1998) have discussed a variety of situations where financial constraints may arise, ranging from an agent's moral hazard problem, business downturns and financial crises, to the acquisition decisions in many organizations which delegate to their purchasing units but impose budget constraints to control their spending, and to the case of salary caps in many professions where budget constraints are used to relax competition.

Financial constraints can pose a serious obstacle to the efficient allocation of resources. For instance, financial constraints seem to have played an important role in the outcome of auctions for selling spectrum licenses conducted in US (see McMillan, 1994; Salant, 1997) and in European countries (see Illing & Klüh, 2003). In this paper, we study a general model in which a number of (indivisible) items are sold to a group of financially constrained bidders. Each bidder wants to consume at most one item. When no bidder faces a financial constraint, the model reduces to the well-known assignment model as studied by Koopmans & Beckmann (1957), Shapley & Shubik (1972), Crawford & Knoer (1981), Leonard (1983), and Demange et al. (1986) among others. Each bidder has private information about his values for the items and his budget and is unwilling to reveal such information for strategic reasons. In particular the auctioneer (seller) does not know the values and budgets of the bidders. It is well-known that even when a single item is auctioned, it is generally impossible to specify a mechanism which achieves full market efficiency when bidders face budget constraints. Of course, this observation also holds when there are multiple items for sale. Even worse, when bidders face financial constraints, a Walrasian equilibrium typically fails to exist,[1] and allocation

[1] Besides budget constraints, price rigidities or fixed prices can also cause the failure of Walrasian

mechanisms that perform well when no bidders face budget constraints, if applied, often result in highly inefficient outcomes.

A natural question is therefore whether an allocation mechanism can be designed that yields a reasonably efficient assignment of the items and a corresponding price system that supports the assignment.[2] In this paper we propose a general solution concept of equilibrium under allotment that gives a sufficiently efficient assignment of items and a supporting system of prices. More importantly we develop a dynamic auction mechanism that yields an equilibrium under allotment in finite time. The proposed auction can be seen as a novel generalization of the well-known ascending auction of Demange et al. (1986) (DGS auction in short) from settings without financial constraints to settings with financial constraints. It works as follows: the auctioneer starts with the seller's reservation price vector, that specifies the lowest admissible price for each item, and each bidder responds with the set of items demanded at those prices. The auctioneer adjusts prices upwards for a minimal set of overdemanded items until a system of prices is reached for which either a set of items is underdemanded or there is neither underdemand nor overdemand. In the first case precisely one item is assigned to a bidder that demanded that item at the previous price system. This bidder with item leaves the auction, while the remaining bidders are requested to report their demands for the remaining items at the previous prices of these items. We will show that at these prices either there is overdemand for some of the remaining items or there are a set of prices with neither overdemand nor underdemand. In the first situation the auctioneer continues by adjusting prices upwards for a minimal set of overdemanded items, until again a system of prices is reached for which either a set of items is underdemanded or there is neither underdemand nor overdemand. In case of underdemand again one item is assigned and the auction continues with the remaining bidders and items. As soon as there is neither underdemand nor overdemand, an equilibrium has been reached for all the remaining items.

An attractive feature of the auction is that it only requires the bidders to report their demands at price vectors along a finite path rather than their values or budgets. This property is very useful and practical, because businessmen

equilibrium; see for instance Talman & Yang (2008), Hatfield et al. (2012, 2016), Andersson & Svensson (2014, 2016), Andersson et al. (2015), and Herings (2015).

[2] It is impossible to achieve a Pareto-optimal outcome, because Walrasian equilibria simply may not exist due to budget constraints.

are in general reluctant to reveal their values, costs, or budgets. This also gives an explanation of why dynamic auctions like English and Dutch auctions are more popular than sealed-bid auctions like the Vickrey auction; see e.g., Rothkopf et al. (1990), Perry & Reny (2005), and Bergemann & Morris (2007). We show that when bidders face no budget constraints, the proposed auction reduces to the well-known DGS auction and thus maintains the DGS auction's strategic properties. In this case, the auction finds a Walrasian equilibrium and it is in the best interest of every bidder to bid truthfully. Moreover in the case where there are budget constraints, the auction might end up with an outcome in which a bidder does not receive his most preferred item given the prices at which the items are sold, because this item has been sold to some other bidder. A bidder that does not receive his most preferred item finds himself constrained on his ability to bid for that item. As shown in Borgs et al. (2005) it may be impossible to design truthful-bidding multi-unit auctions in the case of budget-constrained bidders. Indeed, a bidder that finds himself constrained in the outcome of the auction, might be able to attain a better outcome by misreporting his demands when he has an information advantage over other bidders. However, in the case of at most two items we demonstrate that bidders who receive their most preferred item will have no incentive to manipulate the auction. This property seems similar in essence to those found in the matching literature;[3] see Dubins & Freedman (1981) and Roth & Sotomayor (1990). Another salient feature of the auction is that when a bidder feels himself constrained in his ability to influence the assignment of a particular item, then the price of this item equals the budget of another bidder who is actually assigned with this item and thus pays his full budget. We further demonstrate that the assignment of items and a system of prices generated by the auction yield a Pareto efficient allocation of the items and the money, when no bidder finds himself constrained.

This paper connects directly with a number of papers concerned with auction design under budget constraints. The existing literature concentrates on *sealed-bid auctions* for selling a single item to many bidders, or two items to two bidders. In contrast we propose a dynamic auction for selling multiple items to many financially constrained bidders. Rothkopf (1977) is among the first to study some issues concerning sealed-bid auctions with budget

[3] In the marriage matching, men have no incentive to manipulate in the men-proposing deferred acceptance procedure provided that women are honest, because every man is matched with his best possible partner.

constrained bidders. He investigates how such constraints may affect the best bids of a bidder. Palfrey (1980) analyzes a price discriminatory sealed-bid auction in a multiple item setting under budget constraints and gives a complete characterization of a Nash equilibrium in the case of two items or less and two bidders or less. Pitchik & Schotter (1988) study the equilibrium bidding behavior in sequential auctions for the sale of two items with budget constrained bidders. Che & Gale (1996, 1998) focus on single item auctions with budget constraints under incomplete information. They show that when bidders are subject to financial constraints, the well-known revenue equivalence theorem does not hold. In particular, Che & Gale (1998) provide several conditions under which first-price auctions yield higher expected revenue and social surplus than second-price auctions; see also Krishna (2002) and Klemperer (2004). Laffont & Roberts (1996) characterize an optimal sealed-bid auction in a single item setting under financial constraints.

Maskin (2000) studies the performance of second-price auctions and all-pay auctions and proposes a constrained-efficient sealed-bid auction for the sale of a single item when bidders are financially constrained. Zheng (2001) examines a single-object, first-price sealed-bid auction where budget constrained bidders have the possibility of defaulting on their bids. He shows that budget constraints and default risk together can have a highly significant impact on seller's profit, bidding behavior, and the likelihood of bankruptcy. Benoît & Krishna (2001) investigate simultaneous ascending auctions and sequential auctions for the sale of two items with budget constrained bidders. They compare the performance of both types of auctions when the two items are complements or substitutes; see also Krishna (2002). While concentrating on package auctions without budget constraints, Ausubel & Milgrom (2002) also briefly discuss the case of budget constraints and use the notion of core as a solution under the assumption that every bidder has strict preferences over finite choices. Quintero Jaramillo (2004) shows that a seller can benefit from offering small credit subsidies in an auction with financially constrained bidders. Brusco & Lopomo (2008, 2009) consider simultaneous ascending auctions of two identical objects and two bidders and show that even the slightest possibility of financial constraints may cause significant inefficiencies. Pitchik (2009) studies a sealed-bid sequential auction for selling two items to two bidders with budget constraints and incomplete information. Hafalir et al. (2012) examine a sealed-bid Vickrey auction for selling a divisible good that achieves near Pareto efficiency, weaker than Pareto efficiency. Talman

& Yang (2015) develop a dynamic auction for the assignment market with budget-constrained bidders that always finds a core allocation of the underlying economy, thus resulting in a Pareto efficient outcome.

This paper is organized as follows. Section 2 presents the model. Section 3 introduces the notion of equilibrium under allotment and other basic concepts and results. Section 4 describes and illustrates the ascending auction. Section 5 discusses the convergence of the auction process. Section 6 examines the outcome of the auction. Section 7 derives strategic and efficiency properties. Section 8 concludes. The appendix of the paper contains most of the proofs.

2. THE MODEL

A seller or auctioneer has n indivisible items for sale to a set of m financially constrained bidders. Let $N = \{1,\ldots,n\}$ denote the set of the items for sale and $M = \{1,2,\cdots,m\}$ the set of bidders. In addition to the n real items there is a *dummy item*, denoted by 0. The dummy item 0 can be assigned to any number of bidders simultaneously, any real item $j \in N$ can be assigned to at most one bidder. The seller has for each real item $j \in N$ a nonnegative reservation price $c(j)$ below which the item will not be sold. By convention, the reservation price of the dummy good is known to be $c(0) = 0$. A price vector $p \in \mathbf{R}^{n+1}_{+}$ gives a price $p_j \geq 0$ for each item $j \in N \cup \{0\}$. A price vector $p \in \mathbf{R}^{n+1}_{+}$ is *feasible* if $p_j \geq c(j)$ for every $j \in N$ and $p_0 = 0$. Every bidder $i \in M$ attaches a (possibly negative) monetary value to each item in $N \cup \{0\}$ given by the valuation function $V^i\colon N \cup \{0\} \to \mathbf{R}$. Also by convention, the value of the dummy item for every buyer i is known to be $V^i(0) = 0$. It should be noted that a set $S \subseteq N$ of real items gives value $V^i(S) = \max_{j \in S} V^i(j)$ to bidder i, i.e., bidder i can utilize only one item and thus will never buy more than one real item. So, this is the well known assignment model as studied by Koopmans & Beckmann (1957), Shapley & Shubik (1972), Crawford & Knoer (1981), Leonard (1983), and Demange et al. (1986).

In this paper we generalize this standard model by considering the situation where each bidder i is initially endowed with a nonnegative amount of m^i units of money. Bidders are not allowed to have deficits on their money balances, so no bidder can afford an item j with a price p_j higher than his initial amount of money m^i. This means that unlike in the standard assignment model, the bidders are financially constrained by their initial money holdings m^i, $i \in M$. Since a bidder i is never willing to pay more than his valuation $V^i(j)$ for any item j, his budget m^i is never binding when $m^i \geq \max_{j \in N} V^i(j)$. We say that bidder i is *financially constrained* if $m^i < \max_{j \in N} V^i(j)$, i.e., the valuation of bidder i for at least one item exceeds what he can afford, and that bidder i faces *no financial constraint* otherwise.

All values $V^i(j)$, $j \neq 0$, and m^i are private information and thus only bidder i knows his own values $V^i(j)$, $j \neq 0$ and his own budget m^i. Further it is assumed that all seller's reservation prices, and all valuations and money amounts of the bidders are integer values.

The utility of a bidder i possessing item j and money amount $x_i \geq 0$ is given by

$$U^i(j, x_i) = V^i(j) + x_i - m^i,$$

i.e., the utility is equal to the value of the item j plus the difference between his amount of money x_i and his initial amount m^i. So, $U^i(0, m^i) = 0$, i.e., the utility of bidder i is normalized to zero when he gets the dummy item 0 and his initial amount of money m^i. The utility of bidder i who buys item $j \in N \cup \{0\}$ against price $p_j \leq m^i$ is thus given by

$$U^i(j, m^i - p_j) = V^i(j) - p_j.$$

A *feasible assignment* π assigns to every bidder $i \in M$ precisely one item $\pi(i) \in N \cup \{0\}$ such that no real item $j \in N$ is assigned to more than one bidder. Note that a feasible assignment may assign the dummy good to several bidders and that a real item $j \in N$ is *unassigned* at π if there is no bidder i such that $\pi(i) = j$. Let $N_\pi = \{j \in N \mid j \neq \pi(i) \text{ for all } i \in M\}$, i.e, N_π is the set of real items that are not assigned to any bidder in π. A feasible assignment π^* is *socially efficient* if

$$\sum_{i \in M} V^i(\pi^*(i)) + \sum_{j \in N_{\pi^*}} c(j) \geq \sum_{i \in M} V^i(\pi(i)) + \sum_{j \in N_\pi} c(j)$$

for every feasible assignment π, so a socially efficient assignment maximizes the total value that can be obtained from allocating the items over all agents.

For feasible price vector $p \in \mathbf{R}^{n+1}_+$, the budget set of bidder i is given by

$$B^i(p) = \{j \in N \cup \{0\} \mid p_j \leq m^i\},$$

i.e., the budget set of bidder i at price system p is the set of all affordable items at p. Given a feasible price vector $p \in \mathbf{R}^{n+1}_+$, the demand set of bidder i is defined by

$$D^i(p) = \{j \in B^i(p) \mid V^i(j) - p_j = \max_{k \in B^i(p)} (V^i(k) - p_k)\},$$

thus $D^i(p)$ is the collection of most preferred items at p by i within his budget set, i.e., an item $j \in N \cup \{0\}$ is in the demand set $D^i(p)$ if and only if it can be afforded at p and maximizes the surplus $V^i(k) - p_k$ over all affordable items k. When the demand set contains multiple items, then at the given prices of the items the bidder is indifferent between any two items in his demand set. Note that for any feasible p, the demand set $D^i(p) \neq \emptyset$, because $p_0 = 0 \leq m^i$ and thus the dummy item is always in the budget set $B^i(p)$. In fact this means that the bidder has always the possibility not to buy any real item.

A pair (p, π) of a feasible price vector p and a feasible assignment π is said to be *admissible* if $p_{\pi(i)} \leq m^i$ for all $i \in M$, i.e., every bidder i can afford to buy the item $\pi(i)$ assigned to him. Note that every admissible pair (p, π) yields the corresponding allocation (π, x) with $x_i = m^i - p_{\pi(i)} \geq 0$.

Definition 2.1. *A Walrasian equilibrium* (**WE**) *is an admissible pair* (p^*, π^*) *such that*
(a) $\pi^*(i) \in D^i(p^*)$ *for all* $i \in M$,
(b) $p^*_j = c(j)$ *for every unassigned item* $j \in N_{\pi^*}$.

If (p^*, π^*) is a WE, p^* is called a (Walrasian) equilibrium price vector and π^* a (Walrasian) equilibrium assignment. Because all values and money amounts are integer and the seller's reservation prices are nonnegative integers, it follows that if there exists an equilibrium price vector $p^* \in \mathbf{R}^{n+1}_+$, there must be an integral equilibrium price vector $p \in \mathbf{Z}^{n+1}_+$. Therefore we can restrict ourselves to the set $\mathbf{Z}^{n+1}_+$ of nonnegative integer price vectors.

From Shapley & Shubik (1972) it is well known that in a situation without financial constraints a Walrasian equilibrium exists and every equilibrium assignment is socially efficient. To find an equilibrium some revealing mechanism is needed, because all valuations $V^i(j)$, $j \neq 0$, are private information. The well-known auctions proposed by Crawford & Knoer (1981) and Demange et al. (1986) are such mechanisms. In the remaining of this paper we call the auction introduced in the latter paper the DGS auction. In this literature the notion of overdemanded set of real items is used. A set $S \subseteq N$ of real items is *overdemanded* at a price vector $p \in \mathbf{R}^{n+1}$ if the number of bidders who demand goods only from this set is greater than the number of items in that set.

See the Appendix for a further discussion of this notion. The DGS auction is an ascending auction in which the auctioneer starts with the reservation price vector $p \in \mathbb{Z}_+^{n+1}$ given by $p_0 = 0$ and $p_j = c(j)$, $j \in N$. Then each bidder is required to report his demand set $D^i(p)$. When there is an overdemanded set of goods, the price of every item j in a *minimal* overdemanded set (i.e., no strict subset of this overdemanded set is overdemanded) is increased by one and the bidders have to resubmit their demands at this new price vector. The auction stops as soon as there are no overdemanded sets anymore. It is well-known that the DGS auction for the assignment model without financial constraints stops in a finite number of price adjustments with a unique minimal equilibrium price vector $p^{\min} \in \mathbb{Z}_+^{n+1}$, i.e., (i) there exists a feasible assignment π^* such that (p^{min}, π^*) constitutes a Walrasian equilibrium and (ii) it holds that $p \geq p^{min}$ for any other equilibrium price system $p \in \mathbf{R}_+^{n+1}$. Since the minimum Walrasian price vector corresponds to the Vickrey-Clarke-Groves payments (see Leonard, 1983), the DGS auction has truthful bidding in equilibrium. Also note that in the single item case, the DGS auction reduces to the English auction.

In case of financial constraints a Walrasian equilibrium is not guaranteed to exist, and when it does exist, the equilibrium (assignment) need not be socially efficient. The latter can be easily seen from an example with two bidders and one item. When $V^1(1) > V^2(1) > c(1) = 0$, then social efficiency requires to assign the item to bidder 1. Now, suppose that $m^1 < \min(m^2, V^2(1))$. Then there exists a Walrasian equilibrium, but at any equilibrium the item is assigned to bidder 2 at some (integer) price p_1, $m^1 < p_1 \leq \min(m^2, V^2(1))$. So, all equilibria are socially inefficient.

The following example further shows that financial constraints may cause not only the nonexistence of a Walrasian equilibrium but also the failure of the DGS auction.[4]

Example 1. Consider a market with three bidders ($i = 1,2,3$) and two real items ($j = 1,2$). The values of the bidders are shown in Table 1 and the seller's reservation price vector is $C = (c(0), c(1), c(2)) = (0,0,0)$.

Case 1 (No Budget Constraints). Then this market has a (unique) equilibrium assignment $\pi = (\pi(1), \pi(2), \pi(3)) = (0,2,1)$. The set of equilibrium

[4] It should be noted that the DGS auction was not designed for the current setting with budget constraints but for the settings without budget constraints.

prices is given by

$$\{p \in \mathbf{R}^3 \mid p_0 = 0,\ 5 \le p_1 \le 6,\ 4 \le p_2 \le 5 \text{ with } p_1 = p_2 + 1\}.$$

The two equilibrium integer price vectors (for the real items) are $p^{\min} = (5,4)$ and $p^{\max} = (6,5)$. The DGS auction will find the equilibrium $(\pi, p^{\min})$, realizing a social value of 11 and a revenue of 9 to the seller.

Case 2 (Budget Constraints). Let $(m^1, m^2, m^3) = (4,3,8)$ be the budgets of the bidders. Observe that all budgets are totally different across bidders. We show that the budget constraints fail the existence of a Walrasian equilibrium. Suppose to the contrary that there would be a Walrasian equilibrium price vector $p = (p_0, p_1, p_2)$. Clearly, we should have that $p_1 \le 6$ and $p_2 \le 5$, since otherwise no bidder demands a real item. When $p_1 = p_2 + 1$ we have that $D^3(p) = \{1,2\}$. Further, when $p_1 = p_2 + 1 > 4$, then $D^1(p) = D^2(p) = \{0\}$ and there is underdemand, whereas when $p_1 = p_2 + 1 < 4$, then $D^1(p) = \{1\}$ and $D^2(p) = \{2\}$ and there is overdemand. So, there is no equilibrium with $p_1 = p_2 + 1$. When $p_1 < p_2 + 1$, then we have that $D^3(p) = \{1\}$. So, for equilibrium we must have that $p_1 > 4 = m^1$, otherwise also bidder 1 wants to have item 1. However, then $p_2 \ge p_1 - 1 > 3 = m^2$ and thus there is no demand for item 2. So, there cannot be an equilibrium with $p_1 < p_2 + 1$. Similarly, when $p_1 > p_2 + 1$, it holds that $D^3(p) = \{2\}$, implying that $p_2 > 3 = m^2$, otherwise also bidder 2 wants to have item 2. Then $p_1 > p_2 + 1 > 4 = m^1$ and thus there is no demand for item 1. Again there is no Walrasian equilibrium with $p_1 > p_2 + 1$. Hence, a Walrasian equilibrium does not exist. When one applies the DGS auction, first p_1 is increased from 0 to 1 and then both prices of the real items are increased simultaneously from $(1,0)$ to $(4,3)$. At each of these price systems p (with $p_0 = 0$) there is overdemand for both real items because, $D^1(p) = \{1\}$, $D^2(p) = \{2\}$ and $D^3(p) = \{1,2\}$. However, at the next update we have $p = (0,5,4)$ (with $p_0 = 0$ the price of the dummy item) and the demand sets are $D^1(p) = \{0\}$, $D^2(p) = \{0\}$ and $D^3(p) = \{1,2\}$. So, at prices $p_1 = 4$, $p_2 = 3$, each of the three bidders demands at least one of the items. As a result, the set $\{1,2\}$ is a minimal overdemanded set and, according to the DGS auction, both prices are increased by one. However, at $p_1 = 5$ and $p_2 = 4$, only bidder 3 demands just one of the items (he is indifferent between both items). So, at these prices the seller wants to sell both items, but only one of the items is demanded. It shows that the DGS auction fails to allocate the items. At $p = (0,3,4)$ there is overdemand, while at the next update there is underdemand. $\square$

Table 1: Bidders' values on each item.

Items	0	1	2
Bidder 1	0	5	0
Bidder 2	0	0	5
Bidder 3	0	6	5

Example 1 demonstrates clearly why under financial constraints an equilibrium does not need to exist. Without budget constraint, a bidder withdraws his demand for a real item when the price of the item becomes higher than the bidder's valuation. However, at the price equal to the valuation, the bidder is indifferent between the real and the dummy item (i.e., not buying an item). So, when at this price the real item belongs to the demand set, then also the dummy item belongs to it and the seller can choose between allocating the real item or the dummy item to the bidder. With budget constraint, by contrast, there are two possibilities that a bidder withdraws his demand. The first one is, as before, because the price rises above his valuation of the item. In this case, the dummy item is also in the demand set when the price is equal to the valuation. However, the second possibility is that the price is going to exceed the budget. Then, it might happen that at price equal to the budget, the bidder prefers the real item above every other item (and so the real item is the single item is his demand set), while the demand set does not contain the item anymore when the price is increased by only one. In the example this happens when the price system goes from $(4,3)$ to $(5,4)$. At $p_1 = 4$ the first bidder strictly prefers the first item to any other item (including the dummy item), while at $p_1 = 5$ the first item is not affordable anymore and bidder 1 only demands the dummy item (the same holds for bidder 2 with respect to item 2). So, with budget constraints it is possible that an overdemanded item (or set of items) becomes underdemanded when the price (prices) rises with only one unit. Because of this discontinuity of the demand sets, the Walrasian equilibrium fails to exist. Note that without budget constraints the change from overdemand to underdemand cannot happen, because then the bidder is indifferent between a real item and the dummy item when the price is equal to the reservation value.

The change from overdemand to underdemand in Case 2 of Example 1 is also the reason why the DGS auction fails to work properly. Rather than follow the DGS auction precisely (the auction requires to increase the prices of all items in a minimal overdemanded set), one might consider the possibility to rise only one of the prices at $(4,3)$. However, this is not of any help. For instance, when only p_1 increases from 4 to 5, then at $(5,3)$ there is no demand for item 1, whereas both bidders 2 and 3 demand item 2. So, item 1 is underdemanded and item 2 is still overdemanded. Then increasing p_2 from 3 to 4, gives again the situation as described in the example. Similarly, when first p_2 is increased from 3 to 4, then at $(4,4)$ there is no demand for item 2, whereas both bidders 1 and 3 demand item 1. So, anyway the procedure ends up with prices $(5,4)$ at which bidders 1 and 2 demand the dummy item and bidder 3 is indifferent between the two real items. Of course, it is then possible to assign either item 1 or item 2 to bidder 3. In the first case, bidder 3 pays 5 to the seller who keeps item 2, realizing a social value of 6. In the second case, bidder 3 pays 4 to the seller who keeps item 1, realizing a social value of 5. Both assignments result in a loss of efficiency, because bidders 1 and 2 are willing to pay for the unassigned item, but they don't receive it. This brings us to the central issue of this paper: the design of an auction for markets with financially constrained bidders.

3. EQUILIBRIUM UNDER ALLOTMENT

A possible way out of market situations in which the Walrasian equilibrium does not exist and thus the DGS auction cannot work properly is as follows: as soon as underdemand appears, one may allot an item from the chosen minimal overdemanded set at the previous price system to one of the bidders who demanded that item at that price system, for instance, by having a lottery between these bidders. The bidder to whom the item is allotted, has to pay the price of the item at the previous price system. Of course, allotting the item to one of these bidders implies that the item cannot be assigned to the others who demanded also the item at the same price. So, the auctioneer can only accept one of the bids but has to decline all other equal bids. In Case 2 of Example 1 the auctioneer might accept one of the bids at price system with $p_1 = 4$ and $p_2 = 3$, for instance, by allotting item 2 to bidder 2 against $p_2 = 3$. Then bidder 2 leaves the auction with item 2 and the auction continues with the bidders 1 and 3 and item 1, resulting in a price $p_1 = 5$ at which only bidder

3 demands item 1. This outcome yields *a social value of* 11 *and a revenue of* 8 *to the seller*, resulting in a much better outcome than the one given at the end of the previous section. However, note that this outcome can only sustain because the bid of bidder 3 for item 2 has been declined. In summary, this procedure generates the outcome (p^*, π^*) where $p^* = (p_0^*, p_1^*, p_2^*) = (0,5,3)$ and $\pi^* = (\pi^*(1), \pi^*(2), \pi^*(3)) = (0,2,1)$. Observe that at prices p^*, bidder 1 gets his best-liked item 0 and pays nothing, bidder 2 gets his best-liked item 2 and pays $p_2^* = 3$ equal to his budget $m^2 = 3$, whereas bidder 3 gets item 1 (second-best) rather than his most-preferred item 2, and pays $p_1^* = 5$. So, bidder 3 finds himself rationed at this outcome on item 2, and bidder 2 who receives item 2 pays his full budget $m^2 = 3$.

The reasoning above gives us a clue to the introduction of an equilibrium under allotment and the design of a dynamic auction. The necessity to decline bids of some bidders while accepting an equal bid of one bidder induces a situation of rationing. After all, any bidder who leaves the auction with a net surplus lower than the net surplus that could have been obtained from an item j when paying the same price as what the bidder paid to which the item was allotted, feels himself a posterior rationed on the demand of such an item j. To explore this observation, we adapt the Walrasian equilibrium by incorporating the concept of an *allotment scheme* $R = (R^1, \cdots, R^m)$ where, for $i \in M$, the vector $R^i \in \{0,1\}^{n+1}$ is a *rationing* vector yielding which goods bidder i can demand and for which goods offers of bidder i will be declined. That is, $R^i_j = 1$ means that bidder i is allowed to demand good j, while $R^i_j = 0$ means that bidder i is not allowed to demand good $j \in N$. When $R^i_j = 0$, we say that bidder i is *rationed* on his demand for item j. If a bidder is not rationed on any item, we say that he is *unrationed*. Since the dummy item is always available for every bidder i, we have that $R^i_0 = 1$ for all i. Given a rationing vector R^i with $R^i_j = 0$ for item j, the vector R^i_{-j} denotes the same R^i but allows bidder i to demand item j by ignoring $R^i_j = 0$. At a feasible price vector p and an allotment scheme $R = (R^1, \cdots, R^m)$, the demand set of bidder $i \in M$ is given by

$$
\begin{aligned}
D^i(p, R^i) \quad = \quad & \{ j \in N \mid R^i_j = 1,\ p_j \le m^i \text{ and} \\
& V^i(j) - p_j = \max\nolimits_{\{k \in N \cup \{0\} \mid p_k \le m^i \text{ and } R^i_k = 1\}} (V^i(k) - p_k) \}.
\end{aligned}
$$

We now introduce the notion of *equilibrium under allotment* for the assignment model with financially constrained bidders.

Definition 3.1. *An* equilibrium under allotment (p,π,R) *on a market with financially constrained bidders consists of an admissible pair* (p,π) *and an allotment scheme R such that*

(i) $\pi(i) \in D^i(p,R^i)$ *for all* $i \in M$;

(ii) $p_j = c(j)$ *for any unassigned item* $j \in N_\pi$;

(iii) *If* $R^i_j = 0$ *for some* i, *then* (a) $j \in D^i(p,R^i_{-j})$ *and* (b) *there exists* $h \in M \setminus \{i\}$ *with* $\pi(h) = j$ *and* $m^h = p_j$.

Conditions (i) and (ii) correspond to Conditions (a) and (b) of the definition of the Walrasian equilibrium and are straightforward. In (iii) conditions on the allotment scheme are specified.[5] First, (iiia) says that any rationing is binding, i.e., a bidder that is rationed on some item, demands the item if the rationing on that item is dropped. Second, (iiib) states that rationing on an item can only occur if the item is sold to some other bidder and that this bidder pays his full budget for the item and thus cannot afford a higher price. Together the conditions imply that it is impossible to drop any of the rationings and that in an equilibrium under allotment the seller extracts all the money from the buyer that is assigned a rationed item. In an equilibrium under allotment the prices of the unrationed items are fully competitive. However, the prices of items for which some of the bidders are rationed are not competitive prices in the sense that at these prices there is still overdemand for these items. However, as Example 1 shows, rising these prices results in underdemand and henceforth items with prices above the reservation prices of the seller but nevertheless unsold. When there is no rationing in the equilibrium, i.e., $R^i_j = 1$ for all $i \in M$ and $j \in N$, the equilibrium under allotment is simply a Walrasian equilibrium.

Parallel to the well-known equilibrium existence theorem of Shapley & Shubik (1972) on the assignment market *without financial constraints*, we can establish the following existence theorem on the assignment market *with financial constraints*.

Theorem 3.2. *The assignment model with financially constrained bidders has at least one equilibrium under allotment.*

[5] These conditions may be seen as the counterparts of standard rationing conditions in fix-price literature, see e.g., Drèze (1975) and van der Laan (1980).

In the next Section we design an ascending auction that always finds an equilibrium under allotment, thus providing a constructive proof for Theorem 3.2. To describe the auction and prove its convergence, we introduce the notions of overdemanded and underdemanded sets and give some of its properties.

For a set of real items $S \subseteq N$, and a price vector $p \in \mathbf{R}^{n+1}_{+}$, define the *lower inverse demand* set of S at p by

$$D^{-}_{S}(p) = \{i \in M \mid D^{i}(p) \subseteq S\},$$

i.e., this is the set of bidders who demand only items in S. Note that S is a subset of real items, so any bidder i in the lower inverse demand set does not demand the dummy item and thus has a strict positive surplus $V^{i}(j) - p_{j}$ for any item j in his demand set $D^{i}(p)$. We also define the *upper inverse demand* of S at p by

$$D^{+}_{S}(p) = \{i \in M \mid D^{i}(p) \cap S \neq \emptyset\},$$

i.e., this is the set of bidders that demand at least one of the items in S. Clearly, the lower inverse demand set is a subset of the upper inverse demand set. Let $|A|$ stand for the cardinality of a finite set A.

Definition 3.3.
1. *A set of real items $S \subseteq N$ is overdemanded at price vector $p \in \mathbf{R}^{n+1}_{+}$ if $|D^{-}_{S}(p)| > |S|$. An overdemanded set S is said to be minimal if no strict subset of S is overdemanded.*
2. *A set of real items $S \subseteq N$ is underdemanded at price vector $p \in \mathbf{R}^{n+1}_{+}$ if* (i) $S \subseteq \{j \in N \mid p_{j} > c(j)\}$ *and* (ii) $|D^{+}_{S}(p)| < |S|$. *An underdemanded set S is said to be minimal if no strict subset of S is an underdemanded set.*

The notion of minimal overdemanded set is due to Demange et al. (1986) and the notion of minimal underdemanded set can be found in Mishra & Talman (2006) and is used in a slightly different way by Sotomayor (2002). We further say that an item j is *overpriced* if $\{j\}$ is a minimal underdemanded set, i.e., no bidder has item j in his demand set. So, a minimal underdemanded set S either contains at least two (not overpriced) items, or has an overpriced item as its single element.

In the next three lemmas we give some properties, the proofs of the lemmas are relegated to the Appendix. The first lemma states that for every nonempty subset S of a minimal overdemanded set O at p, the number of bidders in the lower inverse demand set $D^{-}_{O}(p)$ that demand at least one item of S is at least equal to the number of items in S plus the difference between $|D^-_O(p)|$ and $|O|$ and thus is at least one more than the number of items in S.

Lemma 3.4. *Let O be a minimal overdemanded set of items at a price vector p. Then, for every nonempty subset S of O, we have*

$$|\{i \in D^-_O(p) \mid D^i(p) \cap S \neq \emptyset\}| \geq |S| + |D^-_O(p)| - |O|.$$

The next corollary follows immediately.

Corollary 3.5. *For every item in a minimal overdemanded set O at p, there are at least two bidders in $D^-_O(p)$ (actually the number is $|D^-_O(p)| - |O| + 1 \geq 2$) demanding that item.*

The next lemma shows that the number of bidders in the upper inverse demand set of a minimal underdemanded set is precisely one less than the number of items in the set and that any bidder in the upper inverse demand set demands at least two items from the minimal underdemanded set.

Lemma 3.6. *Let U be a minimal underdemanded set of items at a price vector p. Then $|D^+_U(p)| = |U| - 1$ and the demand set $D^i(p)$ of every bidder $i \in D^+_U(p)$ contains at least two elements of U.*[6]

Mishra & Talman (2006, Th. 1) establishes the next result for the case without financial constraints. In fact, this result holds true no matter whether there are financial constraints or not.

Lemma 3.7. *There is a Walrasian equilibrium at $p \in \mathbf{R}^{n+1}_+$ if and only if at p no set of items is overdemanded and no set of items is underdemanded.*

4. AN ASCENDING AUCTION

In this section we introduce an ascending auction which extends the DGS auction to the current setting with financial constraints. The auction starts with $p_j = c(j)$ for each real item $j \in N$ and $p_0 = 0$. In the first round the prices of the items in some minimal overdemanded set are increased. In the DGS

[6] This lemma, the lemma and corollary above were already introduced in the first draft of this paper (van der Laan & Yang, 2008) and have found applications elsewhere; see e.g., Andersson et al. (2013) and Andersson et al. (2015).

auction for the model without financial constraints this continues as long as there is overdemand. As soon as there is no overdemand, the auction ends up with an equilibrium price system and an assignment. However, as Example 1 has shown, in case of financial constraints it might happen that an increase of the prices of the items in a (minimal) overdemanded set results in a situation with underdemand. To deal with such situations, in the modified auction precisely one item is allocated each time when a price increase results in an underdemanded set. Roughly speaking, the auctioneer starts by announcing the seller's reservation prices of the real items and requires the bidders to respond with their demand sets. If there is overdemand without any underdemanded set of items, then the prices of the items in a minimal overdemanded set are increased with one and the bidders are required again to report their demand sets. This continues until a situation is reached in which there is either an underdemanded set of items, or there is neither overdemand nor underdemand. When the first case happens, then precisely one of the items in the chosen minimal overdemanded set at the previous price system is sold against its price in this system to one of the bidders who had the item in his demand set; see Step 4 in the auction below for a detailed description. This bidder with the item leaves the market, after which the auctioneer recalls the previous prices for the remaining items and requires the remaining bidders to resubmit their demands for the remaining items at these prices. This continues until either all items are sold subsequently or a situation is reached at which there is neither overdemand nor underdemand. Then there is an equilibrium for the remaining items and bidders.

At each round t of the auction a new price system p^t is announced with the vector of the seller's reservation prices $p^1 = C = (c(0), c(1), \cdots, c(n)) \in \mathbb{Z}^{n+1}$ at the first round $t = 1$. During the auction process the set of bidders and the set of items are shrinking, so accordingly these sets and also the notions of price vector, demand set and (minimal) overdemanded and underdemanded sets all have to be adapted. We denote by N^t and M^t the set of real items and the set of bidders respectively that are still involved at round t, meaning that the set of items $N \setminus N^t$ has been assigned to the set of bidders $M \setminus M^t$ before round t. Accordingly, p^t is a vector of $|N^t| + 1$ nonnegative integer prices with $p^t_0 = 0$ the price of the dummy item and p^t_j the price of real item j, $j \in N^t$. Correspondingly, the budget set and the demand set of some bidder $h \in M^t$ at round t are given by

$$B^h(p^t) = \{j \in N^t \cup \{0\} \mid p^t_j \leq m^h\},$$

and

$$D^h(p^t) = \{j \in B^h(p^t) \mid V^h(j) - p_j^t = \max_{k \in B^h(p^t)} (V^h(k) - p_k^t)\}.$$

Note again that $0 \in B^h(p^t)$ for every p^t and thus $B^h(p^t)$ is never empty.

The Ascending Auction

Step 1 (Initialization): Set $t := 1$, $p^t := C$, $N^t := N$ and $M^t := M$. Go to Step 2.

Step 2: Every bidder $i \in M^t$ reports his demand set $D^i(p^t) \subseteq N^t \cup \{0\}$. If there exists an underdemanded set at p^t, go to Step 4. Otherwise, go to Step 3.

Step 3: If there is no overdemanded set at p^t, then go to step 5. Otherwise, the auctioneer chooses a minimal overdemanded set $O^t \subseteq N^t$ of items. Then set $p_j^{t+1} := p_j^t + 1$ for every $j \in O^t$, $p_j^{t+1} := p_j^t$ for every $j \in (N^t \setminus O^t) \cup \{0\}$, $M^{t+1} := M^t$ and $N^{t+1} := N^t$. Set $t := t+1$ and return to Step 2.

Step 4: Let $U^t \subseteq N^t$ be a minimal underdemanded set. Then take some item $k \in U^t \cap O^{t-1}$ and bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O^{t-1}\}$ such that $k \in D^h(p^{t-1})$ and $k \notin D^h(p^t)$. Assign item k to bidder h against price p_k^{t-1}. Set $M^{t+1} := M^t \setminus \{h\}$ and $N^{t+1} := N^t \setminus \{k\}$. If $N^{t+1} = \emptyset$, the auction stops, otherwise let $p_j^{t+1} := p_j^{t-1}$ for all $j \in N^{t+1} \cup \{0\}$. Set $t := t+1$ and return to Step 2.

Step 5: There is a feasible assignment π^t for N^t, M^t, such that (p^t, π^t) is a Walrasian equilibrium for N^t, M^t. Item $\pi^t(i) \in N^t \cup \{0\}$ is assigned to bidder $i \in M^t$ against price p_k^t, $k = \pi(i)$, and the auction stops.

We now explain each step in more detail and then provide an example to illustrate how the auction actually operates. In Step 1, the auctioneer announces a set of items for sale and sets the starting prices equal to the reservation prices.

In Step 2, each bidder is asked to report his demand set for the available items at the current prices. Based on the reported demands from the bidders, the auctioneer checks if there is any underdemanded set of items. If so, then

Step 4 will be performed. Otherwise, the auctioneer goes to Step 3 and checks whether there is any overdemanded set of items. If not, the auction goes to Step 5. In case there is overdemand, the auctioneer chooses a minimal overdemanded set of items and goes to the next round. In this round the price of every item in the chosen minimal overdemanded set is increased by one unit, the price of any other item remains constant and Step 2 will be performed again.

In Step 4, the auctioneer first chooses a minimal underdemanded set. Then she selects precisely one item, say item k, that belonged to the minimal overdemanded set that was chosen in Step 2 at the previous round $t-1$ and that also belongs to the minimal underdemanded set at the current round t. This item k is assigned to a bidder h satisfying (i) his demand set at $t-1$ was a subset of the minimal overdemanded set, (ii) who demanded the item k at the previous round $t-1$, and (iii) who does not demand item k anymore at the current round t. This bidder h pays the price p_k^{t-1} of item k at the previous round and leaves the auction with the item k. When no real items are left, the auction stops. Otherwise, the auction moves to the next round $t+1$ with the remaining items and bidders and all the remaining items are set equal to the prices in round $t-1$. Step 2 will be performed again. When the auction reaches Step 5, then according to Lemma 3.7 a Walrasian equilibrium has been reached for the remaining set of items and bidders and the auction terminates.

It should be noted that there will be at least one remaining bidder when the auction returns to Step 2 from Step 4. Clearly, this is true when the number of bidders m is larger than the number of items n, because in Step 4 always precisely one bidder leaves with one item. When $m \leq n$, it might happen that at certain round the auction returns from Step 4 to Step 2 with only one bidder. Obviously then overdemand cannot occur in Step 2. In the next section we prove that underdemand can never occur in Step 2 when the auction returned from Step 4 in the previous round. So, when after Step 4 the auction returns to Step 2 with precisely one bidder, then neither underdemand nor overdemand can occur and the auction goes to Step 5.

Example 2. Consider a market with five bidders (1, 2, 3, 4, 5) and four real items (1, 2, 3, 4). The initial endowment vector of money is given by $m = (m^1, m^2, m^3, m^4, m^5) = (3,4,3,5,4)$ and bidders' values are given in Table 2. The seller's reservation price vector is given by

$$C = (c(0), c(1), c(2), c(3), c(4)) = (0,2,2,2,2).$$

Table 2: Bidders' values on each item.

Items	0	1	2	3	4
Bidder 1	0	4	8	5	7
Bidder 2	0	7	6	8	3
Bidder 3	0	5	5	9	7
Bidder 4	0	9	4	6	2
Bidder 5	0	6	5	4	10

Without financial constraints this market has a unique socially efficient assignment $\pi^* = (\pi^*(1), \pi^*(2), \pi^*(3), \pi^*(4), \pi^*(5)) = (2, 0, 3, 1, 4)$, yielding total social value $\sum_{i \in M} V^i(\pi^*(i)) = 36$. The ascending DGS auction finds a minimal equilibrium price vector $p^* = (0, 7, 6, 8, 6)$ and the socially efficient allocation π^* within a finite number of rounds. The seller's revenue generated by the auction is 27.

In the current situation with financial constraints, the bidders cannot afford to buy items at these minimal equilibrium prices. To find an equilibrium under allotment we apply the new ascending auction described above. The price vectors, demand sets and other relevant data generated by the auction are given in Table 3. Since $p_0^t = 0$ for all t, these prices are deleted from the vectors p^t in the second column of Table 3. In the first seven rounds the auction operates in the same way as the DGS auction. Both auctions start at round $t = 1$ with price vector $p^1 = (0, 2, 2, 2, 2)$ (Step 1). Then, in Step 2, bidders report their demand sets: $D^1(p^1) = \{2\}$, $D^2(p^1) = \{3\}$, $D^3(p^1) = \{3\}$, $D^4(p^1) = \{1\}$ and $D^5(p^1) = \{4\}$. There is no underdemand and the auction goes to Step 3. The set $S = \{3\}$ is a minimal overdemanded set and the auctioneer adjusts p^1 to $p^2 = (0, 2, 2, 3, 2)$, after which the process returns to Step 2. Proceeding with alternating Steps 2 and 3, both auctions generate at round 6 price vector $p^6 = (0, 3, 3, 4, 4)$. At this price vector there is overdemand for the items 1 and 2 (there are three bidders for the two items) and, according to Step 3, the prices of the items 1 and 2 are increased. However, at the new price vector $p^7 = (0, 4, 4, 4, 4)$, there is no demand anymore for item 2, i.e., item 2 is overpriced. Now the DGS auction breaks down without reaching an equilibrium. In fact, due to the financial constraints a Walrasian equilibrium does not exist. Of course, in this final round 7 of the DGS auction the auctioneer can still decide to allocate item 1 to the unique bidder 4 having

1 in his demand set, item 3 to the unique bidder 2 and item 4 to the unique bidder 5. However, item 2 is not allocated and the remaining bidders 1 and 3 don't get any real item. The resulting allocation gives a total value of $V^2(3)+V^4(1)+V^5(4)+c(2)=29$ and is not socially efficient. The seller's revenue from this ad-hoc termination of the DGS auction is only 12 and her total revenue is $12+c(2)=14$.

When faced with the overpriced item 2 at round 7, in the new auction the auctioneer continues with Step 4 and assigns item 2 randomly to one of the bidders 1 and 3. Note that both bidders demand item 2 at p^6 and that their demand sets $D^h(p^6)$, $h=1,3$, are subsets of the minimal overdemanded set $O^6=\{1,2\}$. Further both bidders do not demand item 2 at p^7. Suppose item 2 is assigned to bidder 1. Then this bidder pays $p_2^6=3$ and leaves the auction with item 2. Then round 8 starts with $M^8=\{2,3,4,5\}$ and $N^8=\{1,3,4\}$, the auctioneer adjusts p^7 to $p^8=(p_0,p_1,p_3,p_4)=(0,3,4,4)$ (with the same prices as in round 6 for the three remaining real items), and the process returns to Step 2. At p^8, item 1 is (a minimal) overdemanded (set) and its price is increased to $p_1^9=4$. At round 9 there is neither overdemand nor underdemand and the auction goes to Step 5, in which the dummy item 0 is assigned to bidder 3 (who pays nothing) and the items 1, 3 and 4 to the bidders 4 at $p_1^9=4$, 2 at $p_2^9=4$, and 5 at $p_5^9=4$ respectively. This assignment and these prices form a Walrasian equilibrium for the sets $N^9=\{1,3,4\}$ of real items and $M^9=\{2,3,4,5\}$ of bidders that are still available in round 9.

The final price system $p^*=(p_0,p_1,p_2,p_3,p_4)=(0,4,3,4,4)$ and assignment $\pi^*=(\pi(1),\pi(2),\pi(3),\pi(4),\pi(5))=(2,3,0,1,4)$ form an equilibrium under allotment with allotment scheme $R^*=(R^1,R^2,R^3,R^4,R^5)$, where $R_2^{*3}=0$ and $R_j^{*i}=1$ for all $(i,j)\neq(3,2)$. This equilibrium yields a total value of $\sum_{i\in M}V^i(\pi(i))=35$, which is slightly less than the value 36 of the Walrasian equilibrium allocation. Recall that there is no Walrasian equilibrium at all in this example due to budget constraints. At (p^*,π^*,R^*), the bidders $1,2,4$ and 5 get their most preferred item. However, bidder 3 gets the dummy item, but prefers and can afford item 2, but this item has been allotted to bidder 1. When in round 7 item 2 should have been assigned to bidder 3 instead of bidder 1, the auction would realize a total value of 32. In both cases the seller's revenue from the auction is 15, which is also her total revenue, because all items are sold. □

Table 3: The data generated by the auction in Example 4.

Round	p^t	N^t	M^t	$D^1(p^t)$	$D^2(p^t)$	$D^3(p^t)$	$D^4(p^t)$	$D^5(p^t)$	O^t
1	$(2,2,2,2)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{3\}$	$\{3\}$	$\{1\}$	$\{4\}$	$\{3\}$
2	$(2,2,3,2)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,3\}$	$\{3\}$	$\{1\}$	$\{4\}$	$\{1,3\}$
3	$(3,2,4,2)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,2,3\}$	$\{4\}$	$\{1\}$	$\{4\}$	$\{4\}$
4	$(3,2,4,3)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,2,3\}$	$\{4\}$	$\{1\}$	$\{4\}$	$\{4\}$
5	$(3,2,4,4)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,2,3\}$	$\{2\}$	$\{1\}$	$\{4\}$	$\{2\}$
6	$(3,3,4,4)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{1\}$	$\{4\}$	$\{1,2\}$
7	$(4,4,4,4)$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{0\}$	$\{3\}$	$\{0\}$	$\{1\}$	$\{4\}$	
8	$(3,4,4)$	$\{1,3,4\}$	$\{2,3,4,5\}$		$\{1,3\}$	$\{1\}$	$\{1\}$	$\{4\}$	$\{1\}$
9	$(4,4,4)$	$\{1,3,4\}$	$\{2,3,4,5\}$		$\{3\}$	$\{0\}$	$\{1\}$	$\{4\}$	

5. CONVERGENCE

In this section we show that the auction is well-designed, i.e., all steps are feasible and the auction stops in finitely many rounds. The proofs of all lemmas of this section are given in the Appendix.

First, observe that each time when Step 4 is performed an item is assigned to some of the bidders and both the set of bidders and the set of items decrease by one. So, when $m \leq n$, at each round t we have that $|M^t| \leq |N^t|$. We show that in this case the auction always stops in Step 5. When $m > n$, then at each round t we have that $|M^t| > |N^t|$. In this case the auction stops either in Step 4 when $N^{t+1} = \emptyset$ or in Step 5. In the first case all items are assigned sequentially in a number of n Steps 4, in the latter case the auction reaches a round in which there is neither overdemand nor underdemand. Then, according to Lemma 3.7, there is a Walrasian equilibrium for the sets of remaining items and bidders, showing the feasibility of Step 5. Clearly, also the Steps 1-3 are feasible. So to show feasibility, we only need to consider Step 4.

The auction starts in Step 1 with all prices equal to the seller's reservation prices. At this starting price system there is no underdemand, because by Definition 3.3.2 an item can only be underdemanded when its price is above its seller's reservation price. So, at the starting price vector p^1 in round $t = 1$, either the auction goes to Step 5 and stops, or there is overdemand. In the latter case, a sequence of alternating Steps 2 and Steps 3 is performed with in each Step 3 an increase of the prices of all items in a minimal overdemanded set, until there is neither underdemand nor overdemand and the auction goes to Step 5, or items become underdemanded and the auction goes to Step 4. So, when in some round t, the auction goes to Step 4 for the first time, then in round $t-1$ the prices in some minimal overdemanded set, say O^{t-1}, were increased. We prove that this holds in any round t in which the auction goes to Step 4, i.e., when there is underdemand in some round t, then there was overdemand at round $t-1$ and thus, when the auction reaches Step 4 in round t, then in round $t-1$ the prices of the items in some minimal overdemanded set O^{t-1} were increased. In Step 4 an item k in the intersection of some minimal underdemanded set U^t and the set O^{t-1} is selected and assigned to a bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O^{t-1}\}$ satisfying $k \in D^h(p^{t-1}) \setminus D^h(p^t)$. The next two lemmas state that there indeed exist such an item k and bidder h.

Lemma 5.1. *Let U be a minimal underdemanded set at prices p^t in some round t and let O be the chosen minimal overdemanded set at the previous* *round $t-1$. Then $U \cap O \neq \emptyset$.*

Lemma 5.2. *Let U be a minimal underdemanded set at prices p^t in some round t and let O be the chosen minimal overdemanded set at the previous round $t-1$. Then there exist item k and bidder h satisfying the requirements of Step 4.*

In the special case of $U^t = \{k\}$ with $k \in O^{t-1}$, i.e., the single item k in U^t is overpriced at p^t, we have that no bidder is demanding k at p^t. Hence, any bidder h with $D^h(p^{t-1}) \subseteq O^{t-1}$ and having item k in his demand set $D^h(p^{t-1})$ can be selected. Note that according to Corollary 3.5, there are at least two of such bidders.

The next lemma shows that any time when in some round $t+1$ the auction enters Step 2 after in round t an item k has been assigned to some bidder h by Step 4, there will be no underdemand of items. So, when the auction arrives in Step 2 after Step 4, then the auction goes always to Step 3. Then, either there is neither overdemand nor underdemand and the auction goes to Step 5 (and stops), or there is overdemand and the prices of items in some minimal overdemanded set are increased. This guarantees that any time when the auction goes to Step 4, prices in some minimal overdemanded set were increased in the previous round. Recall that when in round $t+1$ Step 2 is reached from Step 4, the price vector p^{t+1} is equal to the price vector p^{t-1}, except that some item k has been deleted.

Lemma 5.3. *Let U be a minimal underdemanded set in round t that appears after in round $t-1$ the prices of the items in a minimal overdemanded set O were increased, and let, in Step 4, $k \in U \cap O$ be the item assigned to some bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O\}$ such that $k \in D^h(p^{t-1})$, but $k \notin D^h(p^t)$. When the auction proceeds to round $t+1$ and returns to Step 2, then there will be no underdemanded set of items.*

The final lemma states that when in Step 4 an item has been assigned, the new set of bidders M^{t+1} cannot become empty. This is obvious when the number of bidders is bigger than the number of items. However, it also holds when the set of bidders is at most equal to the number of items. The reason is that when the auction returns from Step 4 to Step 2 with precisely one bidder, the auction goes to Step 5 and stops.

Lemma 5.4. *When in some round t an item k is assigned to some bidder $h \in M^t$ at Step 4 of the auction, then $|M^{t+1}| \geq 1$.*

Now we come to present the convergence theorem for the auction.

Theorem 5.5. *The auction terminates with a feasible assignment and a price system in a finite number of rounds.*

Proof. The auction starts in Step 1 with all prices equal to the seller's reservation prices and the auction goes to Step 2. Now, $p_j = c(j)$ for all j and thus, by definition, there cannot be underdemand and the auction goes to Step 3. When there is also no overdemand, the auction goes to Step 5 and stops. Otherwise, the prices of all items in a minimal overdemanded set are increased and the auction returns to Step 2. The auction continues with alternating Steps 2 and 3 until there is neither overdemand nor underdemand and the auction goes to Step 5 and stops, or underdemand arises for the first time. Since the value of any item to any bidder i is finite and any initial endowment m^i is also finite, one of these cases occurs within a finite number of rounds. When the auction goes to Step 4 and assigns an item k to some bidder h. By Lemmas 5.1 and 5.2 this step is feasible. After that the auction either stops in Step 4 because all items are assigned or, according to Lemma 5.4 returns to Step 2 with at least one remaining bidder. According to Lemma 5.3 there is no underdemand when the auction returns to Step 2 after Step 4. Hence, either there is neither overdemand nor underdemand and the auction goes to Step 5 and stops, or there is overdemand again. Then, similarly as above, within a finite number of rounds again one item is assigned in Step 4, or the auction goes to Step 5 and stops. Repeating this every time after the auction returns in Step 2 after Step 4, it follows that the auction terminates in finitely many rounds, because the number of items is finite.

When the auction stops in Step 4, all items are assigned to different bidders and the auction ends up with a feasible assignment and price system. When the auction stops in Step 5 in some round t, then according to Lemma 3.7 there is a Walrasian equilibrium assignment with respect to the set of items N^t and the set of bidders M^t. Together with the items that have been assigned already before in Step 4, this Walrasian assignment forms a feasible assignment for N and M. Hence the auction terminates with a feasible assignment and a price system in finitely many rounds. □

6. THE OUTCOME OF THE AUCTION

According to Theorem 5.5 the auction finds a feasible assignment in finitely many rounds. In this section we prove that the feasible assignment and the resulting price system induces an equilibrium under allotment. Let π^* be the assignment resulting from the auction, i.e., $\pi^*(i) = k$ for some $k \in N$ when bidder i was assigned an item in either Step 4 or Step 5, and $\pi^*(i) = 0$ otherwise; and let p^* be the resulting price vector, i.e., when item k is assigned, then p^*_k is the price at which item k is assigned to some bidder h, otherwise p^*_k is the price of the item in the round t in which the auction stops in Step 5. When an item k is assigned at Step 4, then the item is underdemand and thus $p^t_k > c(k)$; and because the auction starts with the reservation price vector C, we have $p^t_k \geq c(k)$ for all items $k \in N^t$ when in round t the auction stops at Step 5. Note that $p^t_0 = c(0) = 0$ for all t. Hence $p^*_k = p^{t-1}_k = p^t_k - 1 \geq c(k)$ when item k is assigned in round t by Step 4, $p^*_k = p^t_k \geq c(k)$ for any item k that is assigned in the final round t by Step 5 and $p^*_0 = c(0)$ and thus p^* is feasible. Further, when a bidder gets assigned an item in either Step 4 or 5, then the item is in his demand set and thus every bidder i can afford to buy the item $\pi^*(i)$ assigned to him. Hence (p^*, π^*) is admissible. We further define the allotment scheme R^* as follows. For $i \in M$, define R^{i*} by

$$R^{i*}_k = \begin{cases} 0 & \text{if } k \in A^i, \\ 1 & \text{otherwise}, \end{cases} \tag{6.1}$$

where $A^i = \{j \in N \setminus \pi^*(i) \mid p^*_j \leq m^i \text{ and } V^i(j) - p^*_j > V^i(\pi^*(i)) - p^*_{\pi^*(i)}\}$.

Theorem 6.1. *The admissible pair (p^*, π^*) and the allotment scheme R^* yield an equilibrium under allotment (p^*, π^*, R^*).*

Proof. We have shown above that (p^*, π^*) is an admissible pair. So, it remains to prove that the conditions (i)-(iii) of Definition 3.1 hold. To prove (i), first consider a bidder i that got assigned an item k in Step 4 at some round t against price p^{t-1}_k. Then according to Step 4,

$$k \in D^i(p^{t-1}) = \{j \in N^{t-1} \mid p^{t-1}_j \leq m^i,\, V^i(j) - p^{t-1}_j = \max_{\ell \in B^i(p^{t-1})} (V^i(\ell) - p^{t-1}_\ell)\}$$

where $B^i(p^{t-1}) = \{\ell \in N^{t-1} \cup \{0\} \mid p^{t-1}_\ell \leq m^i\}$. After item k has been assigned to bidder i in round t, the auction continues with Step 2 in round

$t+1$ with the remaining set of items $N^{t+1} = N^{t-1} \setminus \{k\}$. Since at any stage $\tau \geq t+1$, $p_j^{\tau} \geq p_j^{t-1}$ for all $j \in N^{t+1}$, it follows that

$$V^i(k) - p_k^* \geq V^i(j) - p_j^*, \text{ for all } j \in N^{t+1} \text{ with } p_j^* \leq m^i.$$

Further, observe that any $j \in N \setminus N^{t-1}$ has been assigned in some round $\tau \leq t-1$, before in round t the item k is assigned to bidder i. According to (6.1) we have that $R_j^{*i} = 0$ for all $j \in N \setminus N^{t-1}$ satisfying $p_j^* \leq m^i$ and $V^i(j) - p_j^* > V^i(k) - p_k^*$. Hence $k \in D^i(p^*, R^{*i})$. Second we consider a bidder i who was assigned item k in Step 5 in the final round t. Such a bidder i has item k in his demand set $D^i(p^t)$ with respect to the items in N^t. Again, for any $j \in N \setminus N^t$ that was assigned before in some round $\tau \leq t-1$, we have that $R_j^{*i} = 0$ when $p_j^* \leq m^i$ and $V^i(j) - p_j^* > V^i(k) - p_k^*$. Hence, also in this case we have that $k \in D^i(p^*, R^{*i})$.

To prove (ii), observe that when an item k is not assigned to a bidder i, then k belongs to the set N^t when the auction stops in Step 5 in the final round t. Then there is neither underdemand nor overdemand and, according to Lemma 3.7, then the auction ends with a Walrasian equilibrium allocation with respect to the remaining items in N^t and the remaining bidders in M^t. By definition of the Walrasan equilibrium we then have that $p_k^* = p_k^t = c(k)$ for any unassigned item k.

Condition (iiia) immediately follows from (6.1). Further, since there is a Walrasian equilibrium for the remaining items N^t and bidders M^t when in the final round t the auction stops in Step 5, it also follows from (6.1) that rationing only occurs for items that have been assigned in some Step 4 before the final round t. To show that the bidder who is assigned a rationed item pays his full budget for that item, again observe that, when for some item j we have that $\pi(h) = j$ and $R_j^{i*} = 0$ for some bidder $i \neq h$, then item j has been allocated in some Step 4 before the end of the auction. Let item j be allocated in some round t. Then item j was in a minimal overdemanded set O at p^{t-1} and for bidder h to which j is assigned it holds that (i) $h \in \{h' \in M^t \mid D^{h'}(p^t) \subseteq O\}$, (ii) $j \in D^h(p^{t-1})$ and (iii) $j \notin D^h(p^t)$. Since $p_k^t = p_k^{t-1} + 1$ for all $k \in O$ and $p_k^t = p_k^{t-1}$ for all $k \in N^t \setminus \{O\}$, it follows that $p_j^{t-1} = m^h$, otherwise j should still have been in the demand set of h at p^t. Hence $p_j^* = p_j^{t-1} = m^h$, which shows (iiib). □

Theorem 6.1 shows that the auction finds an equilibrium under allotment in a finite number of price adjustments. First, note that the associated allotment scheme is *endogenously generated.* Second, Theorem 6.1 immediately implies that the existence Theorem 3.2 of Section 3 is true: the assignment model with financially constrained bidders has an equilibrium under allotment. Since at an equilibrium under allotment trade takes place at non-Walrasian prices, the corresponding allocation is typically suboptimal.[7] Given this suboptimality principle, Example 2 in Section 3 has shown that our ascending auction can realize both a high total value and high revenue for the seller. Property (iiib) of the equilibrium definition also stresses that the seller extracts all the money from the buyer of an item, when other bidders feel themselves rationed for that item.

So far we have considered the case that some or all bidders may confront financial constraints. We have shown that the proposed ascending auction can handle such a situation and always finds an equilibrium under allotment. One may naturally ask whether the proposed auction can find a Walrasian equilibrium when no bidder faces a budget constraint. The following theorem demonstrates that this is indeed the case. This shows that the current auction is indeed an appropriate generalization of the DGS auction to the more complex situation where bidders have budget constraints.

Theorem 6.2. *If $m^i \geq \max_{j\in N} V^i(j)$ for all $i \in M$, then the auction for markets with financially constrained bidders coincides with the DGS auction and finds a Walrasian equilibrium with a minimal equilibrium price vector p^* in finitely many rounds.*

Proof. It is sufficient to show that the ascending auction never generates an underdemanded set of items. It is true in round 1 because the ascending auction starts with the reservation price vector C. Suppose that in some round t, there is no underdemanded set of items and O is the minimal overdemanded set of items chosen by the auctioneer as described in Step 3. We show that there will be no underdemanded set of items in round $t+1$.

We first prove that no subset S of the set O is underdemanded at p^{t+1}. Because $m^i \geq \max_{j\in N} V^i(j)$ and $0 \notin O$, every bidder $i \in D^-_O(p^t)$ who demands items from S at p^t will continue to demand the same items in S and may demand other items as well at p^{t+1}. It follows from Lemma 3.4 that the set S cannot

[7] It is known from the literature on equilibria under price rigidities that equilibria with rationing are typically not Pareto efficient, see e.g., Böhm & Müller (1977) and Herings & Konovalov (2009).

be underdemanded at p^{t+1}. Second, no subset S of $N^t \setminus O$ is underdemanded at p^{t+1}, because S is not underdemanded at p^t and the price of each item in $N^t \setminus O$ in round $t+1$ is the same as in round t and the price of each item in O is increased by one in round $t+1$. Combining the two reasonings for the case $S \subseteq O$ and $S \subseteq N^t \setminus O$, it follows that also any $S \subseteq N^t$ with $S \cap O \neq \emptyset$ and $S \cap (N^t \setminus O) \neq \emptyset$ is not underdemanded at p^{t+1}. So the ascending auction never goes to Step 4 and thus coincides exactly with the DGS auction. It is known that the DGS auction finds an equilibrium with the minimal equilibrium price vector. □

7. EFFICIENCY AND STRATEGIC ISSUES

7.1. Efficiency

We have seen that under financial constraints a Walrasian equilibrium may not exist. In Section 2 we also show by example that under financial constraints even if a Walrasian equilibrium exists, it need not be socially efficient. However, we will show that under financial constraints every Walrasian equilibrium is Pareto efficient. To discuss Pareto efficiency we first need to give the utilities of all agents, seller and bidders, at an allocation. An *allocation* is a pair (π, x) with π a feasible assignment and $x \in \mathbf{R}^m_+$ a nonnegative vector of money, assigning amount $x_i \geq 0$ of money to bidder i, $i \in M$. Everything that is not assigned to the bidders at allocation (π, x), is assigned to the seller. So at allocation (π, x) the seller receives the total amount of money $\sum_{i \in M} (m^i - x_i)$ from the bidders and keeps all unsold items for himself. It follows that allocation (π, x) yields utilities

$$U^i(\pi(i), x_i) = V^i(\pi(i)) + x_i - m^i, \;\; i \in M,$$

to the bidders and utility

$$U^s(\pi, x) = \sum_{i \in M} (m^i - x_i) + \sum_{j \in N_\pi} c(j)$$

to the seller, i.e., the utility of the seller is equal to the total amount of money he receives plus the sum of his reservation values of the unassigned items. Following the standard definition, we say an allocation (π^*, x^*) is *Pareto efficient* if there does not exist another allocation (π, x) such that

$$U^i(\pi(i), x_i) \geq U^i(\pi^*(i), x^*_i), \text{for all } \; i \in M \text{ and } U^s(\pi, x) \geq U^s(\pi^*, x^*)$$

with strict inequality for at least one of the agents.

If (p^*,π^*) is a WE, with p^* the (Walrasian) equilibrium price vector and π^* the (Walrasian) equilibrium assignment, the corresponding allocation (π^*,x^*) with $x_i^* = m^i - p^*_{\pi^*(i)}$ is called a (Walrasian) equilibrium allocation. It is well-known (see Mas-Colell et al., 1995) that for exchange economies with *divisible goods*, under certain conditions every Walrasian equilibrium allocation is Pareto efficient. However, the proof of the standard model does not apply to our model.

Theorem 7.1. *For the market model with financially constrained bidders, let (p^*,π^*) be an admissible pair. If (p^*,π^*) is a Walrasian equilibrium, then its corresponding equilibrium allocation (π^*,x^*) is Pareto efficient.*

Proof. Suppose that (π^*,x^*) is not Pareto efficient. Then there exists an allocation (π,x) such that

$$V^i(\pi(i)) + x_i - m^i \geq V^i(\pi^*(i)) + x_i^* - m^i = V^i(\pi^*(i)) - p^*_{\pi^*(i)}, \tag{7.2}$$

for every bidder $i \in M$, and for the seller

$$\sum_{i\in M}(m^i - x_i) + \sum_{j\in N_\pi} c(j) \geq \sum_{i\in M}(m^i - x_i^*) + \sum_{j\in N_{\pi^*}} c(j),$$

where at least one of these $m+1$ inequalities is strict. Define $q_j = c(j)$ for $j \in N_\pi$, $q_{\pi(i)} = m^i - x_i$ for every $i \in M$ with $\pi(i) \neq 0$ and $Q = \sum_{\{i\in M|\pi(i)=0\}}(m^i - x_i)$. Since $p_j^* = c(j)$ when $j \in N_{\pi^*}$, the seller's condition becomes

$$Q + \sum_{j\in N} q_j \geq \sum_{j\in N} p_j^*. \tag{7.3}$$

Since $\pi^*(i) \in D^i(p^*)$, $0 \in B^i(p^*)$ and $p_0^* = 0$, we have for every $i \in M$ that

$$V^i(\pi^*(i)) - p^*_{\pi^*(i)} \geq V^i(0) - p_0^* = 0, \;\; i \in M.$$

So, for every $i \in M$ with $\pi(i) = 0$ it follows from (7.2) that $x_i \geq m^i$ and thus $Q \leq 0$. Suppose $q_j > p_j^*$ for some $j \in N$. Since $q_j > p_j^* \geq c(j)$, and $q_h = c(h)$ when $h \in N_\pi$, we must have that $\pi(i) = j$ for some $i \in M$. So, in (π,x), bidder i receives item j and money amount $x_i \geq 0$. The latter inequality implies that $q_j \leq m^i$. So $p_j^* < q_j \leq m^i$, i.e., item j is in the budget set $B^i(p^*)$ of i at p^*. On the other hand

$$U^i(j,x_i) = V^i(j) + x_i - m^i = V^i(j) - q_j \geq V^i(\pi^*(i)) - p^*_{\pi^*(i)}$$

and thus

$$V^i(j) - p^*_j > V^i(j) - q_j \geq V^i(\pi^*(i)) - p^*_{\pi^*(i)},$$

which contradicts that $\pi^*(i) \in D^i(p^*)$. Hence $q_j \leq p^*_j$ for all j. With $Q \leq 0$, it follows from inequality (7.3) that $Q = 0$ (and thus $x_i = m^i$ for all i with $\pi(i) = 0$) and $q_j = p^*_j$ for all $j \in N_\pi$. So, the seller's inequality holds with equality.

Suppose that there is a bidder i with strict inequality, thus

$$V^i(\pi(i)) + x_i - m^i > V^i(\pi^*(i)) - p^*_{\pi^*(i)}. \tag{7.4}$$

Since $x_i = m^i$ and thus $V^i(\pi(i)) + x_i - m^i = 0$ if $\pi(i) = 0$, we must have that $\pi(i) \neq 0$. Then $m^i - x_i = q_{\pi(i)} = p^*_{\pi(i)}$ and the inequality (7.4) becomes

$$V^i(\pi(i)) - p^*_{\pi(i)} > V^i(\pi^*(i)) - p^*_{\pi^*(i)}.$$

Since $x_i \geq m^i - p^*_{\pi(i)} \geq 0$ and thus $p^*_{\pi(i)} = q_{\pi(i)} \leq m^i$, this again contradicts that $\pi^*(i) \in D(p^*)$. □

7.2. Strategic Issues

When the auction results in a Walrasian equilibrium, it also preserves the strategic properties of the DGS auction and thus truthful bidding is optimal for the bidders; see Leonard (1983). It should be noted, however, that without financial constraints in the DGS auction bidders only drop out for their bidding on an item when another item (maybe the dummy item) becomes more preferred. Under financial constraints it might also happen that a bidder drops out for an item because the price of the item rises above his budget. However, this does not affect the strategic properties of the auction as long as there is no underdemand. In conclusion, if underdemand never appears in Step 2, the auction behaves as the DGS auction, and no bidder has incentive to manipulate the auction.

In general, due to budget constraints a Walrasian equilibrium does not exist and our auction generates an equilibrium under allotment at which some bidders are rationed on their demands. Borgs et al. (2005) have demonstrated that it may be impossible to design truthful bidding multi-unit auctions with budget-constrained bidders. Indeed, it could be possible for a rationed bidder to attain a better outcome by misreporting his demands if this bidder knew all valuations and budgets of all other bidders and convinced that all other bidders would bid honestly. On the other hand, truthful bidding is optimal when the auction terminates with a Walrasian equilibrium. Observe in this case that at the outcome of the auction no bidder is rationed on his demand. We conjecture that this is still true in case of financially constrained bidders: for every unrationed bidder at the outcome of the auction it is in his best interest to bid truthfully. We prove this conjecture for the case of at most two real items.

Theorem 7.2. *For the market model with at most two items and many financially constrained bidders, let (p^*,π^*) be the outcome of the auction when bidders report truthfully, and let i be a bidder that does not find himself rationed in (p^*,π^*). Then there do not exist values $W^i(j)$, $j=1,2$, and outcome (q,ρ) when i reports his demands according to W^i, such that $U^i(\rho(i),m^i-q_{\rho(i)}) > U^i(\pi^*(i),m^i-p^*_{\pi^*(i)})$.*

Proof. We prove the case of two items, i.e., $N=\{1,2\}$. The case of one item can be shown similarly. Suppose that there exist $W^i(j)$, $j=1,2$, and (q,ρ) such that

$$U^i(\rho(i),m^i-q_{\rho(i)}) > U^i(\pi^*(i),m^i-p^*_{\pi^*(i)}). \tag{7.5}$$

For ease of notation, denote $\pi^*(i)=j$ and $\rho(i)=k$. If $p^*_k>m^i$, then $q_k<p^*_k$. When $p^*_k\le m^i$, then either $k=j$ and thus $q_k<p^*_k$ (because of inequality (7.5)) or $k\neq j$. In the latter case

$$U^i(k,m^i-q_k)=V^i(k)-q_k>U^i(j,m^i-p^*_j)=V^i(j)-p^*_j\ge V^i(k)-p^*_k,$$

because bidder i is rationed, and thus also in this case $q_k<p^*_k$. So, $q_k<p^*_k$ must hold.

Since $V^i(k)-q_k>V^i(j)-p^*_j$, we must have that $k\neq 0$. So, when reporting according to W^i, bidder i gets a real item. Without loss of generality, take $k=1$ and thus $q_1<p^*_1$. Suppose $q_2<p^*_2$. Then, by the feature of the ascending auction, the number of bidders h satisfying

$$D^h(q)\subseteq\{1,2\}$$

is at least equal to 3, because otherwise there are at most two bidders that demand a real item at q and the auction cannot reach an outcome in which both prices are higher. So, also when bidder i misreports his demands there are at least two other bidders that demand a real item from $\{1,2\}$. Since also at least two bidders can afford the prices p_1^* and p_2^*, the auction cannot terminate with price system q and assigning item 1 to bidder i.

It remains to consider the case that $q_2 \geq p_2^*$. Then under the true valuations, there has been some round t with p^t such that $p_1^t = q_1 < p_1^*$, $p_2^t \leq p_2^* \leq q_2$ and $p_1^\tau > p_1^t$ for all $\tau > t$ (thus the price of item 1 was higher in every round after t). Then at p^t either $\{1\}$ or $\{1,2\}$ was a minimal overdemanded set. In the first case there was at least one bidder $h \neq i$ that preferred item 1 to any other item. Since $q_2 \geq p_2^t$, also at q all these bidders prefer item 1 to any other item. Since item j was sold at p_1^* and thus at least one bidder has item 1 in his demand set at p^* and could afford p_1^*, also under W^i the auction cannot terminate with price system q and assigning item 1 to bidder i. Finally, in case $\{1,2\}$ was a minimal overdemanded set, then there were at least two bidders $h \neq i$ with $D^h(p^t) = \{1,2\}$. Then for all these bidders also $D^h(q) = \{1,2\}$ when $q_2 = p_2^t$ and $D^h(q) = \{1\}$ if $q_2 > p_2^t$. Since again a bidder paid $p_1^* > q_1$, also in this case the auction cannot terminate under W^i with price system q and assigning item 1 to bidder i. □

8. CONCLUDING REMARKS

In this paper we investigated a general and practical market model in which an auctioneer wants to sell a number of items to a group of financially constrained bidders. Every bidder demands at most one item and knows his valuation of the items and his budget information privately. The auctioneer does not know this private information unless bidders tell her. When bidders face budget constraints, a Walrasian equilibrium typically fails to exist. We proposed the notion of equilibrium under allotment to remedy the nonexistence of Walrasian equilibrium. An ascending auction has been developed which, starting with the seller's reservation price of each item, always finds an equilibrium under allotment in finite steps. This auction provides an effective allocation mechanism for handling situations with financially constrained bidders, generating high revenues for the seller and arguably efficient assignment of items. Another interesting feature of the auction is that it can extract all the money from those bidders who receive an item on which some other bidder is rationed. We have further shown that when no bidder is financially constrained, the proposed auction reduces to the auction of Demange et al. (1986) and thus preserves the strategic properties of the DGS auction. We have also examined the strategic and efficiency properties of the proposed auction and its outcome.

Finally it is worth noting that that Ausubel (2006), Gul & Stacchetti (2000), Kelso & Crawford (1982), Milgrom (2000), Sun & Yang (2009, 2014) have proposed dynamic auctions for more general environments in which each bidder may consume several items but has no budget constraint. It will be interesting but also significantly more difficult to study this more general situation with financially constrained bidders. Another important question is whether it is possible to design an efficient and strategy-proof dynamic multi-item auction with budget-constrained bidders.

9. APPENDIX

9.1. Proofs of the Lemmas of Section 3

Proof of Lemma 3.4. Since O is overdemanded at p, the constant $d = |D^-_O(p)| - |O|$ must be a positive integer. By definition the lemma holds (with equality) for $S = O$. For any nonempty strict subset S of O, define $D_S = \{i \in D^-_O(p) \mid D^i(p) \cap S \neq \emptyset\}$. Then we have

$$D^-_O(p) \setminus D_S = \{i \in D^-_O(p) \mid D^i(p) \subseteq O \setminus S\}.$$

Suppose to the contrary that $|D_S| < |S| + d$. Since $0 < |S| \leq |O| - 1$, we have that

$$\begin{aligned} |D^-_O(p) \setminus D_S| &= |D^-_O(p)| - |D_S| > |D^-_O(p)| - (|S| + d) = \\ &= |D^-_O(p)| - |S| - (|D^-_O(p)| - |O|) = |O| - |S| = |O \setminus S|. \end{aligned}$$

This means that the set $O \setminus S$ is overdemanded, contradicting the fact that O is a minimal overdemanded set. Hence, $|D_S| \geq |S| + d = |S| + |D^-_O(p)| - |O|$. □

Proof of Lemma 3.6. If $|U| = 1$, then U consists of an overpriced item and $|D^+_U(p)| = 0$. So, both statements are true.

For $|U| \geq 2$, denote $T = D^+_U(p)$. To prove the first part, suppose $|T| \leq |U| - 2$. Then take any element k of U and denote $T' = D^+_{U\setminus\{k\}}(p)$. Clearly, $T' \subseteq T$ and thus $|T'| \leq |T|$. Hence

$$|T'| \leq |T| \leq |U| - 2 = |U \setminus \{k\}| - 1$$

and thus $U \setminus \{k\}$ is underdemanded, contradicting the assumption that U is a minimal underdemanded set.

To prove the second part, suppose there is a bidder i having only one element of U in his demand set. Let k be this element. Then T' does not contain bidder $i \in T$. Hence $|T'| \leq |T| - 1$ and thus

$$|T'| \leq |T| - 1 = |U| - 2 = |U \setminus \{k\}| - 1,$$

showing that $U \setminus \{k\}$ is underdemanded. Again this contradicts the fact that U is a minimal underdemanded set. □

Proof of Lemma 3.7.[8] First, let (p,π) be a Walrasian equilibrium (p,π). Clearly, at p no set of items is overdemanded and no set of items is underdemanded.

To prove the other direction, let $M^1 = \{i \in M \mid 0 \notin D^i(p)\}$ and $N^1 = \{j \in N \mid p_j > c(j)\}$. First, consider any $T \subseteq M^1$ and let $D^T = \cup_{i\in T} D^i(p)$. Because D^T is not overdemanded, $|D^T| \geq |T|$. By the well-known theorem of Hall (1935), there exists a one-to-one mapping $\tau\colon M^1 \to N$ such that $\tau(i) \in D^i(p)$ for all $i \in M^1$. We can extend τ to a mapping from M to $N \cup \{0\}$ by setting $\tau(i) = 0$ for all $i \notin M^1$. Next, consider any $S \subseteq N^1$. Because S is not underdemanded, $|D^-_S(p)| \geq |S|$. Again by Hall's Theorem, there exists a one-to-one mapping $\rho : N^1 \to M$ such that $j \in D^{\rho(j)}(p)$ for all $j \in N^1$.

With respect to τ and ρ, denote $K = \{i \mid \tau(i) \in N^1\}$, $L = \{\tau(i) \mid i \in K\}$ and $Q = \{\rho(j) \mid j \in N^1 \setminus L\}$ and define the mapping $\pi\colon M \to N \cup \{0\}$ by

$$\pi(i) = \begin{cases} \tau(i), & \text{for } i \in M \setminus Q, \\ \rho^{-1}(i), & \text{for } i \in Q. \end{cases}$$

Clearly, $\pi(i) \in D^i(p)$ for all $i \in M$, and no real item is assigned by π to two different bidders, and for every item $j \in N^1$, there is a bidder i who demands the item at p and is assigned the item. This shows that (p,π) is a Walrasian equilibrium. □

9.2. Proofs of the lemmas of Section 5

In proofs of this subsection it should be noted that the sets $D^-_S(p^\tau)$ and $D^+_S(p^\tau)$ are defined with respect to the current set of bidders M^τ, for any set $S \subset N^\tau$ and for any $\tau = t-1,\ t$.

Proof of Lemma 5.1. Suppose to the contrary that $U \cap O = \emptyset$. Since U is underdemanded at round t, we have that $p^t_j > c(j)$ for any $j \in U$. Further, since $U \cap O = \emptyset$, we have for any $j \in U$ that $j \notin O$. Hence $p^t_j = p^{t-1}_j$ and thus also $p^{t-1}_j > c(j)$ for all $j \in U$. Since there is no underdemand in round $t-1$, it follows that $|D^+_U(p^{t-1})| \geq |U|$. Moreover, any bidder that demands some item $j \in U$ at p^{t-1}, also demands this item at p^t, because only prices of the items in O are increased. Hence $|D^+_U(p^t)| \geq |D^+_U(p^{t-1})| \geq |U|$ and thus U is not underdemanded at p^t, yielding a contradiction. Hence $U \cap O \neq \emptyset$. □

Proof of Lemma 5.2. Since O is overdemanded at p^{t-1}, we have $|D^-_O(p^{t-1})| > |O|$.

[8] This proof is much simpler than the original one given by Mishra and Talman (2006).

Now, consider the set $S = U \cap O$. By Lemma 5.1 this set is not empty. When $U = O$ and thus $S = O$, then by Lemma 3.6 there are $|U| - 1 = |O| - 1$ bidders demanding at least one item from U at p^t, because U is underdemanded at p^t. So, in this case there are at least two bidders in $D_O^-(p^{t-1})$ not demanding any item from $U = O$ anymore at price p^t. Select h from this set of bidders and select k from the set $D^h(p^{t-1})$ (recall that this set is never empty and does not contain any dummy item). Since $D^h(p^{t-1}) \subseteq O$ and for each bidder $h \in D_O^-(p^{t-1})$, this item k and this bidder h satisfy the requirements.

Next, consider the case that S is a strict subset of O. Denote $H = \{i \in D_O^-(p^{t-1}) \mid D^i(p^{t-1}) \cap S \neq \emptyset\}$. From Lemma 3.4 we have that

$$|H| \geq |S| + |D_O^-(p^{t-1})| - |O| \geq |S| + 1,$$

i.e., the number of bidders in $D_O^-(p^{t-1})$ that demand an item of S at p^{t-1} is at least one more than the number of items in S. Next, consider the set $T = U \setminus O$. Since there is no underdemand at p^{t-1} we have that

$$|D_T^+(p^{t-1})| \geq |T|.$$

Since $p_j^t = p_j^{t-1}$ for all $j \in T = U \setminus O$, any bidder that demands an item from T at p^{t-1}, is still demanding this item at p^t, so $D_T^+(p^{t-1}) \subseteq D_T^+(p^t)$. On the other hand, U is underdemanded at p^t, so

$$|D_U^+(p^t)| < |U|.$$

Further, observe that $H \cap D_T^+(p^{t-1}) = \emptyset$, since $H \subseteq D_O^-(p^{t-1})$ and the members of $D_O^-(p^{t-1})$ demand only items in O, whereas the members of $D_T^+(p^{t-1})$ demand at least one item from $T = U \setminus O$ at p^{t-1}. Therefore, the number of bidders in H that still demand items in S at p^t can be at most $|S| - 1$. Suppose not, i.e., the number is at least $|S|$. Then the number of bidders in $D_U^+(p^t)$ (demanding at least one item of U at p^t) is at least equal to $|S|$ plus the number of bidders in $D_T^+(p^{t-1})$, i.e.

$$|D_U^+(p^t)| \geq |S| + |T| = |U \cap O| + |U \setminus O| = |U|,$$

contradicting the fact that U is underdemanded. Hence there are at least two bidders in H that are no longer demanding items in $U \cap O$ at p^t. Select h as one of these bidders and k as one of the elements in the non-empty set $D^h(p^{t-1}) \cap S$. Then item k and bidder h satisfy the requirements. □

Proof of Lemma 5.3. First, observe that, by definition of the auction, $M^{t+1} = M^{t-1} \setminus \{h\}$, $N^{t+1} = N^{t-1} \setminus \{k\}$ and $N^{t+1} \neq \emptyset$ (otherwise the auction ends in Step 4). Further, $p_j^{t+1} = p_j^{t-1}$ for all $j \in N^{t+1}$. Denote $\tilde{O} = O \setminus \{k\}$. For $S \subseteq N^{t+1}$ we consider two cases, namely $S \subseteq \widetilde{O}$ and $S \setminus \widetilde{O} \neq \emptyset$. In the first case we have by Lemma 3.4 that at least $|S|+1$ members of the set $D_O^-(p^{t-1}) = \{i \in M^{t-1} \mid D^i(p^{t-1}) \subseteq O\}$ demanded at least one item of S in round $t-1$. Since $p_j^{t+1} = p_j^{t-1}$ for all $j \in N^{t+1}$, for any bidder i in M^{t+1} it holds that

$$D^i(p^{t+1}) = D^i(p^{t-1}) \setminus \{k\}$$

and thus any bidder $i \in M^{t+1} \cap D_O^-(p^{t-1}) = D_O^-(p^{t-1}) \setminus \{h\}$ that demanded an item of S at round $t-1$ is still demanding an item of S at round $t+1$. So, when h demanded an item of S at round $t-1$, the number of bidders of M^{t+1} demanding an item of S at round $t+1$ is at least $|S|$, otherwise the number is at least $|S|+1$. Hence S is not underdemanded.

For the second case $S \setminus \widetilde{O} \neq \emptyset$ we consider the partition of S given by $S^1 = S \cap \widetilde{O}$ and $S^2 = S \setminus \widetilde{O}$. Denote

$$K^1 = \{i \in D_O^-(p^{t-1}) \mid D^i(p^{t-1}) \cap S^1 \neq \emptyset\}$$

and

$$K^2 = \{i \in M^{t-1} \mid D^i(p^{t-1}) \cap S^2 \neq \emptyset\}.$$

Since $D^i(p^{t-1}) \subseteq O$ for all $i \in D_O^-(p^{t-1})$ and $S^2 \subseteq N^{t-1} \setminus O$, it follows that $K^1 \cap K^2 = \emptyset$. Since O is a minimal overdemanded set in round $t-1$ and there is no underdemand in round $t-1$, we have that S^1 is neither overdemanded nor underdemanded at p^{t-1}, because it is a strict subset of O. By Lemma 3.4 we have that at least $|S^1|+1$ members of $D_O^-(p^{t-1})$ demanded at least one item of S^1 in round $t-1$ and similarly as above it follows that at least $|S^1|$ members of $D_O^-(p^{t-1}) \setminus \{h\}$ are still demanding an item of S^1 at round $t+1$. Furthermore, none of these bidders belong to K^2, because $D_O^-(p^{t-1}) \cap K^2 = \emptyset$. Further $|K^2| \geq |S^2|$, because there is no underdemand at round $t-1$. Clearly, any member of K^2 is still demanding an item of S^2 at round $t+1$, because all prices of the remaining items in N^{t+1} are equal to the prices in round $t-1$. Therefore the number of bidders that demand at least one item of $S = S^1 \cup S^2$ is at least equal to

$$|S^1| + |K^2| \geq |S^1| + |S^2| = |S|$$

and thus S is not underdemanded in round $t+1$. □

Proof of Lemma 5.4. Each time when an item is assigned in Step 4, the number of items and the number of bidders decreases with one. Suppose that in some round t Step 4 is performed for the ℓth time. As long as $\ell < |M|-1$, we have that $M^{t+1} = |M| - \ell > 1$. Now, suppose that $\ell = |M|-1$. Then $|M^{t+1}| = 1$ and the auction returns to Step 2. According to Lemma 5.3, there is no underdemand in Step 2 and thus the auction goes to Step 3. However, because only one bidder is left, also overdemand cannot occur and thus the auction goes to Step 5 and terminates. □

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DISCRETE CONVEX ANALYSIS: A TOOL FOR ECONOMICS AND GAME THEORY

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ABSTRACT

This paper presents discrete convex analysis as a tool for use in economics and game theory. Discrete convex analysis is a new framework of discrete mathematics and optimization, developed during the last two decades. Recently, it has been recognized as a powerful tool for analyzing economic or game models with indivisibilities. The main feature of discrete convex analysis is the distinction of two convexity concepts, M-convexity and L-convexity, for functions in integer or binary variables, together with their conjugacy relationship. The crucial fact is that M-concavity in its variant is equivalent to the gross substitutes property in economics. Fundamental theorems in discrete convex analysis such as the M-L conjugacy theorems, discrete separation theorems and discrete fixed point theorems yield structural results in economics such as the existence of equilibria and the lattice structure of equilibrium price vectors. Algorithms in discrete convex analysis provide iterative auction algorithms for finding equilibria.

Keywords: Convex analysis, indivisibility, equilibrium, fixed point.

JEL Classification Numbers: C61, C65.

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1. INTRODUCTION

Convex analysis and fixed point theorems have played a crucial role in economic and game-theoretic analysis, for instance, in proving the existence of a competitive equilibrium and Nash equilibrium; see Debreu (1959), Arrow & Hahn (1971), and Fudenberg & Tirole (1991). Traditionally, in such studies, it is assumed that commodities are perfectly divisible, or mixed strategies can be used, or the space of strategies is continuous. However, this traditional approach cannot be equally applied to economic models which involve significant indivisibilities or to game-theoretic models where the space of strategies is discrete and mixed strategies are not useful. In this paper we present a new approach based on discrete convex analysis and discrete fixed point theorems which have been recently developed in the field of discrete mathematics and optimization and become a powerful tool for analyzing economic or game models with indivisibilities.

Discrete convex analysis (Murota, 1998, 2003) is a general theoretical framework constructed through a combination of convex analysis and combinatorial mathematics. The framework of convex analysis is adapted to discrete settings and the mathematical results in matroid/submodular function theory are generalized.[1] The theory extends the direction set forth in discrete optimization by Edmonds (1970), Frank (1982), Fujishige (1984), and Lovász (1983); see also Fujishige (2005, Chapter VII). The main feature of discrete convex analysis is the distinction between two convexity concepts for functions in integer or binary variables, M-convexity and L-convexity[2] together with their conjugacy relationship with respect to the (continuous or discrete) Legendre–Fenchel transformation. Roughly speaking, M-convexity is defined in terms of an exchange property and L-convexity by submodularity.

The application of discrete convex analysis to mathematical economics was initiated by Danilov et al. (1998, 2001) to show the existence of a Walrasian equilibrium in an exchange economy with indivisible goods (see also Murota, 2003, Chapter 11). The next stage of the interaction was brought about by the crucial observation of Fujishige & Yang (2003) that M-concavity in its variant

[1] The readers who are interested in general backgrounds are referred to Rockafellar (1970) for convex analysis, Schrijver (1986) for linear and integer programming, Korte & Vygen (2012) and Schrijver (2003) for combinatorial optimization, Oxley (2011) for matroid theory, and Fujishige (2005) and Topkis (1998) for submodular function theory.

[2] "M" stands for "Matroid" and "L" for "Lattice."

called M$^\natural$-concavity[3] is equivalent to the gross substitutability (GS) of Kelso & Crawford (1982). The survey papers by Murota & Tamura (2003a) and Tamura (2004) describe the interaction at the earlier stages of this development.

Concepts, theorems, and algorithms in discrete convex analysis have turned out to be useful in the modeling and analysis of economic problems. The M-L conjugacy corresponds to the conjugacy between commodity bundles and price vectors in economics. The conjugacy theorem in discrete convex analysis implies, for example, that a valuation (utility) function has the substitutes property (M$^\natural$-concavity) if and only if the indirect utility function is an L$^\natural$-convex function, where L$^\natural$-convexity is a variant of L-convexity.

One of the most successful examples of the discrete convex analysis approach is Fujishige and Tamura's model (Fujishige & Tamura, 2006, 2007) of two-sided matching, which unifies the stable matching of Gale & Shapley (1962) and the assignment model of Shapley & Shubik (1972). The existence of a market equilibrium is established by revealing a novel duality-related property of M$^\natural$-concave functions. Tamura's monograph (Tamura, 2009), though in Japanese, gives a comprehensive account of this model.

Another significant instance of the discrete convex analysis approach is the design and analysis of auction algorithms. Based on the Lyapunov function approach of Ausubel (2006) and Sun & Yang (2009), Murota et al. (2013a, 2016) shed a new light on a variety of iterative auctions by making full use of the M-L conjugacy theorem and L$^\natural$-convex function minimization algorithms. The lattice structure of equilibrium price vectors is obtained as an immediate consequence of the L$^\natural$-convexity of the Lyapunov function.

The contents of this paper are as follows:

Section 1: Introduction
Section 2: Notation
Section 3: M$^\natural$-concave set function
Section 4: M$^\natural$-concave function on $\mathbb{Z}^n$
Section 5: M$^\natural$-concave function on $\mathbb{R}^n$
Section 6: Operations for M$^\natural$-concave functions
Section 7: Conjugacy and L$^\natural$-convexity
Section 8: Iterative auctions
Section 9: Intersection and separation theorems
Section 10: Stable marriage and assignment game

3 "M$^\natural$" and "L$^\natural$" are read "em natural" and "ell natural," respectively.

Section 11: Valuated assignment problem
Section 12: Submodular flow problem
Section 13: Discrete fixed point theorem
Section 14: Other topics

Following the introduction of notations in Section 2, Sections 3 to 5 present the definition of M$^\natural$-concave functions and the characterizations of (or equivalent conditions for) M$^\natural$-concavity in terms of demand functions and choice functions. Section 6 shows the operations valid for M$^\natural$-concave functions, including the convolution operation used for the aggregation of utility functions. Section 7 introduces L$^\natural$-convexity as the conjugate concept of M$^\natural$-concavity, and Section 8 presents several iterative auctions. Section 9 deals with duality theorems of fundamental importance, including the discrete separation theorems and the Fenchel-type minimax relations. Section 10 is a succinct description of Fujishige and Tamura's model. Combinations of M$^\natural$-concave functions with graph/network structures are considered in Sections 11 and 12. Section 13 explains the basic idea underlying the discrete fixed point theorems. Finally Section 14 gives a brief discussion of other related topics.[4]

2. NOTATION

Basic notations are listed here.

- The set of all real numbers is denoted by $\mathbb{R}$, and the sets of nonnegative reals and positive reals are denoted, respectively, by $\mathbb{R}_+$ and $\mathbb{R}_{++}$. The set of all integers is denoted by $\mathbb{Z}$, and the sets of nonnegative integers and positive integers are denoted, respectively, by $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$. The sign $\forall$ means for all.

- We consistently assume $N = \{1, 2, \ldots, n\}$ for a positive integer n. Then 2^N denotes the set of all subsets of N, i.e., the power set of N.

- The characteristic vector of a subset $A \subseteq N = \{1, 2, \ldots, n\}$ is denoted by $\chi_A \in \{0, 1\}^n$. That is,

$$(\chi_A)_i = \begin{cases} 1 & (i \in A), \\ 0 & (i \in N \setminus A). \end{cases} \tag{2.1}$$

[4] For other applications, we refer to Murota (2000b, 2009), Katoh et al. (2013), and Simchi-Levi et al. (2014).

For $i \in \{1,2,\ldots,n\}$, we write χ_i for $\chi_{\{i\}}$, which is the ith unit vector. We define $\chi_0 = \mathbf{0}$ where $\mathbf{0} = (0,0,\ldots,0)$. We also define $\mathbf{1} = (1,1,\ldots,1)$.

- For a vector $x = (x_1, x_2, \ldots, x_n)$ and a subset $A \subseteq \{1,2,\ldots,n\}$, $x(A)$ denotes the component sum within A, i.e., $x(A) = \sum_{i \in A} x_i$.

- For two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, $x \leq y$ means the componentwise inequality. That is, $x \leq y$ is true if and only if $x_i \leq y_i$ is true for all $i = 1, 2, \ldots, n$.

- For two integer vectors a and b in $\mathbb{Z}^n$ with $a \leq b$, $[a,b]_{\mathbb{Z}}$ denotes the integer interval between a and b (inclusive), i.e., $[a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z}^n \mid a \leq x \leq b\}$.

- For two vectors x and y, $x \vee y$ and $x \wedge y$ denote the vectors of componentwise maximum and minimum. That is, $(x \vee y)_i = \max(x_i, y_i)$ and $(x \wedge y)_i = \min(x_i, y_i)$ for $i = 1, \ldots, n$.

- For a real number $z \in \mathbb{R}$, $\lceil z \rceil$ denotes the smallest integer not smaller than z (rounding-up to the nearest integer) and $\lfloor z \rfloor$ the largest integer not larger than z (rounding-down to the nearest integer). This operation is extended to a vector by componentwise application.

- For a vector x, $\mathrm{supp}^+(x) = \{i \mid x_i > 0\}$ and $\mathrm{supp}^-(x) = \{i \mid x_i < 0\}$ denote the positive and negative supports of x, respectively.

- The ℓ_∞-norm of a vector x is denoted as $\|x\|_\infty$, i.e.,

  $$\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|).$$

  We also use the following variants:

  $$\|x\|_\infty^+ = \max(0, x_1, x_2, \ldots, x_n)$$

  and

  $$\|x\|_\infty^- = \max(0, -x_1, -x_2, \ldots, -x_n).$$

- For two vectors $p = (p_1, p_2, \ldots, p_n)$ and $x = (x_1, x_2, \ldots, x_n)$, their inner product is denoted by $\langle p, x \rangle$, i.e., $\langle p, x \rangle = p^\top x = \sum_{i=1}^n p_i x_i$, where $p^\top$ is the transpose of p viewed as a column vector.

- For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ or $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$,

$$\begin{aligned} \mathrm{dom}\, f &= \{x \mid -\infty < f(x) < +\infty\}, \\ \arg\min f &= \{x \mid f(x) \leq f(y) \ \text{ for all } y\}, \\ \arg\max f &= \{x \mid f(x) \geq f(y) \ \text{ for all } y\}. \end{aligned}$$

  These notations are used also for $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ or $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$. We sometimes use $\mathrm{dom}_{\mathbb{R}} f$ and $\mathrm{dom}_{\mathbb{Z}} f$ to emphasize that $\mathrm{dom}\, f \subseteq \mathbb{R}^n$ and $\mathrm{dom}\, f \subseteq \mathbb{Z}^n$.

- For a set function $f : 2^N \to \mathbb{R} \cup \{+\infty\}$ or $f : 2^N \to \mathbb{R} \cup \{-\infty\}$,

$$\begin{aligned} \mathrm{dom}\, f &= \{X \subseteq N \mid -\infty < f(X) < +\infty\}, \\ \arg\min f &= \{X \subseteq N \mid f(X) \leq f(Y) \ \text{ for all } Y \subseteq N\}, \\ \arg\max f &= \{X \subseteq N \mid f(X) \geq f(Y) \ \text{ for all } Y \subseteq N\}. \end{aligned}$$

- For a function f and a vector p, $f[-p]$ means the function defined by

$$f[-p](x) = f(x) - p^{\top} x = f(x) - \langle p, x \rangle.$$

  If f is a set function, $f[-p]$ is the set function defined by $f[-p](X) = f(X) - p(X)$.

- For a function f, four variants of the conjugate function of f are denoted as

$$\begin{aligned} f^{\bullet}(p) &= \sup\{\langle p, x \rangle - f(x)\}, & f^{\circ}(p) &= \inf\{\langle p, x \rangle - f(x)\}, \\ f^{\triangledown}(p) &= \sup\{f(x) - \langle p, x \rangle\}, & f^{\vartriangle}(p) &= \inf\{f(x) + \langle p, x \rangle\}. \end{aligned}$$

- The convex closure of a function f is denoted by $\overline{f}$. The convex hull of a set S is denoted by $\overline{S}$.

- $D(p; f)$ denotes the demand correspondence for a price vector p and a valuation function f, defined in (3.16) and (4.21).

- $C(\cdot)$ denotes a choice function. $C(\cdot\,; f)$ denotes the choice function determined by a valuation function f, defined in (3.17) and (4.23).

- $\mathrm{tw}(\cdot)$ denotes the twisting of a set or a vector, defined in (3.18) and (4.25), respectively.

- For an arc a in a directed graph, $\partial^+ a$ denotes the initial (tail) vertex of a, and $\partial^- a$ the terminal (head) vertex of a. That is, $\partial^+ a = u$ and $\partial^- a = v$ if $a = (u, v)$.

- For a flow ξ in a network, $\partial \xi$ denotes the boundary vector on the vertex set, defined in (4.36). For a matching M, ∂M denotes the set of the vertices incident to some edge in M.

- For a potential p defined on the vertex set of a network, δp denotes the coboundary of p, the vector on the arc set defined in (12.20).

3. M$^\natural$-CONCAVE SET FUNCTION

First we introduce M$^\natural$-concavity for set functions. Let N be a finite set, say, $N = \{1, 2, \ldots, n\}$, $\mathscr{F}$ be a nonempty family of subsets of N, and $f : \mathscr{F} \to \mathbb{R}$ be a real-valued function on $\mathscr{F}$. In economic applications, we may think of f as a single-unit valuation (binary valuation) over combinations of indivisible commodities N, where $\mathscr{F}$ represents the set of feasible combinations.

3.1. Exchange property

Let $\mathscr{F}$ be a nonempty family of subsets of a finite set $N = \{1, 2, \ldots, n\}$. We say that a function $f : \mathscr{F} \to \mathbb{R}$ is *M$^\natural$-concave*, if, for any $X, Y \in \mathscr{F}$ and $i \in X \setminus Y$, we have (i) $X - i \in \mathscr{F}$, $Y + i \in \mathscr{F}$ and

$$f(X) + f(Y) \leq f(X - i) + f(Y + i), \tag{3.1}$$

or (ii) there exists some $j \in Y \setminus X$ such that $X - i + j \in \mathscr{F}$, $Y + i - j \in \mathscr{F}$ and

$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j). \tag{3.2}$$

Here we use short-hand notations $X - i = X \setminus \{i\}$, $Y + i = Y \cup \{i\}$, $X - i + j = (X \setminus \{i\}) \cup \{j\}$, and $Y + i - j = (Y \cup \{i\}) \setminus \{j\}$. This property is referred to as the *exchange property*.

A more compact way of defining M$^\natural$-concavity, free from explicit reference to the domain $\mathscr{F}$, is to define a function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ to be M$^\natural$-concave if it has the following property:

(M$^\natural$-EXC) For any $X, Y \subseteq N$ and $i \in X \setminus Y$, we have

$$f(X)+f(Y) \leq \begin{array}{l}\max(f(X-i)+f(Y+i), \\ \max_{j \in Y \setminus X}\{f(X-i+j)+f(Y+i-j)\}),\end{array} \tag{3.3}$$

where $(-\infty)+a = a+(-\infty) = (-\infty)+(-\infty) = -\infty$ for $a \in \mathbb{R}$, $-\infty \leq -\infty$, and a maximum taken over an empty set is defined to be $-\infty$. The family of subsets X for which $f(X)$ is finite is called the *effective domain* of f, and denoted as $\mathrm{dom}\, f$, i.e., $\mathrm{dom}\, f = \{X \mid f(X) > -\infty\}$. When f is regarded as a function on $\mathscr{F} = \mathrm{dom}\, f$, it is an M$^\natural$-concave function in the original sense.

As a (seemingly) stronger condition than (M$^\natural$-EXC) we may also conceive the *multiple exchange property*:

(M$^\natural$-EXC$_\mathbf{m}$) For any $X, Y \subseteq N$ and $I \subseteq X \setminus Y$, there exists $J \subseteq Y \setminus X$ such that $f(X)+f(Y) \leq f((X \setminus I) \cup J) + f((Y \setminus J) \cup I)$, i.e.,

$$f(X)+f(Y) \leq \max_{J \subseteq Y \setminus X}\{f((X \setminus I) \cup J) + f((Y \setminus J) \cup I)\}. \tag{3.4}$$

Recently it has been shown (Murota, 2016) that (M$^\natural$-EXC$_\mathrm{m}$) is equivalent to (M$^\natural$-EXC).

Theorem 3.1. *A function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ satisfies* (M$^\natural$-EXC) *if and only if it satisfies* (M$^\natural$-EXC$_\mathrm{m}$)*. Hence, every M$^\natural$-concave function has the multiple exchange property* (M$^\natural$-EXC$_\mathrm{m}$).

Remark 3.1. The multiple exchange property (M$^\natural$-EXC$_\mathrm{m}$) here is the same as the "strong no complementarities property (SNC)" introduced by Gul & Stacchetti (1999) where it is shown that (SNC) implies the gross substitutes property (GS). On the other hand, (GS) is known (Fujishige & Yang, 2003) to be equivalent to (M$^\natural$-EXC) (see Theorem 3.7). Therefore, Theorem 3.1 above reveals that (SNC) is equivalent to (GS). This settles the question since 1999: Is (SNC) strictly stronger than (GS) or not? We now know that (SNC) is equivalent to (GS). See Murota (2016) for details. ∎

It follows from the definition of an M$^\natural$-concave function that the (effective) domain $\mathscr{F}$ of an M$^\natural$-concave function has the following exchange property:

(B$^\natural$-EXC) For any $X, Y \in \mathscr{F}$ and $i \in X \setminus Y$, we have (i) $X-i \in \mathscr{F}$, $Y+i \in \mathscr{F}$ or

(ii) there exists some $j \in Y \setminus X$ such that $X-i+j \in \mathscr{F}$, $Y+i-j \in \mathscr{F}$.

This means that $\mathscr{F}$ forms a matroid-like structure,[5] called a *generalized matroid* (*g-matroid*), or an *M*$^\natural$*-convex family*.[6] An M$^\natural$-convex family $\mathscr{F}$ containing the empty set forms the family of independent sets of a matroid. For example, for integers a, b with $0 \leq a \leq b \leq n$, $\mathscr{F}_{ab} = \{X \mid a \leq |X| \leq b\}$ is an M$^\natural$-convex family, and $\mathscr{F}_{0b}$ (with $a = 0$) forms the family of independent sets of a matroid.

Remark 3.2. It follows from Theorem 3.1 that a nonempty family $\mathscr{F} \subseteq 2^N$ satisfies (B$^\natural$-EXC) if and only if it satisfies the multiple exchange axiom:

(B$^\natural$-EXC$_\mathbf{m}$) For any $X, Y \in \mathscr{F}$ and $I \subseteq X \setminus Y$, there exists $J \subseteq Y \setminus X$ such that $(X \setminus I) \cup J \in \mathscr{F}$ and $(Y \setminus J) \cup I \in \mathscr{F}$. ■

M$^\natural$-concavity can be characterized by a local exchange property under the assumption that function f is (effectively) defined on an M$^\natural$-convex family of sets (Murota, 1996a, 2003; Murota & Shioura, 1999). The conditions (3.5)–(3.7) below are "local" in the sense that they require the exchangeability of the form of (3.3) only for (X, Y) with $\max(|X \setminus Y|, |Y \setminus X|) \leq 2$.

Theorem 3.2. *A set function* $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ *is M*$^\natural$*-concave if and only if* $\mathrm{dom}\, f$ *is an M*$^\natural$*-convex family and the following three conditions hold:*

$$f(X + i + j) + f(X) \leq f(X + i) + f(X + j) \tag{3.5}$$

for all $X \subseteq N$ *and for all* $i, j \in N \setminus X$ *with* $i \neq j$*; and*

$$f(X + i + j) + f(X + k) \leq \max\{f(X + i + k) + f(X + j),\; f(X + j + k) + f(X + i)\} \tag{3.6}$$

for all $X \subseteq N$ *and for all distinct* $i, j, k \in N \setminus X$*; and*

$$f(X + i + j) + f(X + k + l) \leq \max\{f(X + i + k) + f(X + j + l),\; f(X + j + k) + f(X + i + l)\} \tag{3.7}$$

for all $X \subseteq N$ *and for all distinct* $i, j, k, l \in N \setminus X$.

5 See, e.g., Murota (2000b), Oxley (2011), and Schrijver (2003) for matroids.

6 A subset of N can be identified with a 0-1 vector (characteristic vector in (2.1)), and accordingly, a family of subsets can be identified with a set of 0-1 vectors. We call a family of subsets an *M*$^\natural$*-convex family* if the corresponding set of 0-1 vectors is an M$^\natural$-convex set as a subset of $\mathbb{Z}^N$.

When the effective domain $\mathrm{dom}\, f$ contains the emptyset, the local exchange condition for M$^\natural$-concavity takes a simpler form without involving (3.7); see Reijnierse et al. (2002, Theorem 10), Müller (2006, Theorem 13.5), Shioura & Tamura (2015, Theorem 6.5).

Theorem 3.3. *Let $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ be a set function such that* $\mathrm{dom}\, f$ *is an M$^\natural$-convex family containing $\emptyset$ (the empty set). Then f is M$^\natural$-concave if and only if* (3.5) *and* (3.6) *hold.*

It is known (Murota, 2003, Theorem 6.19) that an M$^\natural$-concave function is *submodular*, i.e.,

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \qquad (X, Y \subseteq N). \tag{3.8}$$

More precisely, the condition (3.5) above is equivalent to the submodularity (3.8) as long as $\mathrm{dom}\, f$ is M$^\natural$-convex (Shioura & Tamura, 2015, Proposition 6.1).

Because of the additional condition (3.6) for M$^\natural$-concavity, not every submodular set function is M$^\natural$-concave. Thus, M$^\natural$-concave set functions form a proper subclass of submodular set functions.

Remark 3.3. It follows from (M$^\natural$-EXC) that M$^\natural$-concave set functions enjoy the following exchange properties under cardinality constraints (Murota & Shioura, 1999, Lemmas 4.3 and 4.6):

• For any $X, Y \subseteq N$ with $|X| < |Y|$,

$$f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X + j) + f(Y - j)\}. \tag{3.9}$$

• For any $X, Y \subseteq N$ with $|X| = |Y|$ and $i \in X \setminus Y$,

$$f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X - i + j) + f(Y + i - j)\}. \tag{3.10}$$

The former property, in particular, implies the cardinal-monotonicity of the induced choice function; see Theorem 3.10 and its proof. ■

Remark 3.4. For a set family $\mathscr{F}$ consisting of equi-cardinal sets (i.e., $|X| = |Y|$ for all $X, Y \in \mathscr{F}$) the exchange property (B$^\natural$-EXC) takes a simpler form: For any $X, Y \in \mathscr{F}$ and $i \in X \setminus Y$, there exists some $j \in Y \setminus X$ such that $X - i + j \in \mathscr{F}$, $Y + i - j \in \mathscr{F}$. This means that $\mathscr{F}$ forms the family of bases of a matroid. An

M$^\natural$-concave function defined on matroid bases is called a *valuated matroid* in Dress & Wenzel (1990, 1992) and Murota (2000b, Chapter 5) or an *M-concave set function* in Murota (1996a, 2003). The exchange property for M-concavity reads: A set function f is M-concave if and only if (3.10) holds for any $X, Y \subseteq N$ and $i \in X \setminus Y$. A corollary of Theorem 3.1: Every M-concave function (valuated matroid) f has the multiple exchange property (M$^\natural$-EXC$_\mathrm{m}$) with $|J| = |I|$. A further corollary of this fact is a classical result in matroid theory: The base family of a matroid has the multiple exchange property (B$^\natural$-EXC$_\mathrm{m}$) with $|J| = |I|$; see, e.g., Schrijver (2003, Section 39.9a). ∎

3.2. Maximization and single improvement property

For an M$^\natural$-concave function, the maximality of a function value is characterized by a local condition (Murota, 2003, Theorem 6.26).

Theorem 3.4. *Let $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and $X \in \mathrm{dom}\, f$. Then X is a maximizer of f if and only if*

$$f(X) \geq f(X - i + j) \quad (\forall i \in X,\ \forall j \in N \setminus X), \tag{3.11}$$
$$f(X) \geq f(X - i) \quad (\forall i \in X), \tag{3.12}$$
$$f(X) \geq f(X + j) \quad (\forall j \in N \setminus X). \tag{3.13}$$

As a discrete analogue of the subgradient inequality for convex functions, we have the inequality (3.14) in the following theorem.[7]

Theorem 3.5. *Let $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and $X, Y \in \mathrm{dom}\, f$. Then*

$$f(Y) - f(X) \leq \hat{f}(X, Y), \tag{3.14}$$

where $\hat{f}(X, Y)$ is defined as follows:

- *When $|X| = |Y|$,*

$$\hat{f}(X, Y) = \max_{\sigma} \left(\sum_{i \in X \setminus Y} [f(X - i + \sigma(i)) - f(X)] \right),$$

  where the maximum is taken over all one-to-one correspondences $\sigma : X \setminus Y \to Y \setminus X$.

[7] This is a reformulation of the "upper-bound lemma" (Murota, 2000b, Lemma 5.2.29) for valuated matroids to M$^\natural$-concave functions. See also Murota (2003, Proposition 6.25).

- *When* $|X| < |Y|$,

$$\hat{f}(X,Y) = \max_{\sigma} \Bigg(\sum_{i \in X \setminus Y} [f(X - i + \sigma(i)) - f(X)] + \sum_{j \in Y \setminus (X \cup \sigma(X))} [f(X + j) - f(X)] \Bigg),$$

where the maximum is taken over all injections $\sigma : X \setminus Y \to Y \setminus X$.

- *When* $|X| > |Y|$,

$$\hat{f}(X,Y) = \max_{\tau} \Bigg(\sum_{j \in Y \setminus X} [f(X - \tau(j) + j) - f(X)] + \sum_{i \in X \setminus (Y \cup \tau(Y))} [f(X - i) - f(X)] \Bigg),$$

where the maximum is taken over all injections $\tau : Y \setminus X \to X \setminus Y$.

For a vector $p = (p_i \mid i \in N) \in \mathbb{R}^N$ we use the notation $f[-p]$ to mean the function $f(X) - p(X)$, where $X \subseteq N$ and $p(X) = \sum_{i \in X} p_i$. That is,

$$f[-p](X) = f(X) - p(X) \qquad (X \subseteq N). \tag{3.15}$$

Note that $f[-p]$ is M$^\natural$-concave if and only if f is M$^\natural$-concave.

The "if" part of Theorem 3.4, which is the content of the theorem, can be restated as follows: If X is not a maximizer of f, there exists $Y \subseteq N$ such that $|X \setminus Y| \leq 1$, $|Y \setminus X| \leq 1$, and $f(X) < f(Y)$. By considering this property for $f[-p]$ with varying p, we are naturally led to the *single improvement property* of Gul & Stacchetti (1999):

(SI) For any $p \in \mathbb{R}^N$, if X is not a maximizer of $f[-p]$, there exists $Y \subseteq N$ such that $|X \setminus Y| \leq 1$, $|Y \setminus X| \leq 1$, and $f[-p](X) < f[-p](Y)$.

The above argument shows that (SI) is true for M$^\natural$-concave functions. In fact, (SI) is equivalent to M$^\natural$-concavity (Fujishige & Yang, 2003).

3.3. Maximizers and gross substitutability

For a vector $p = (p_i \mid i \in N) \in \mathbb{R}^N$ we consider the maximizers of the function $f[-p](X) = f(X) - p(X)$, where $p(X) = \sum_{i \in X} p_i$ for $X \subseteq N$. We denote the set of these maximizers by

$$D(p; f) = \arg\max_X \{f(X) - p(X) \mid X \subseteq N\}. \tag{3.16}$$

In economic applications, p is a price vector and $D(p) = D(p; f)$ represents the demand correspondence.

It is one of the most fundamental facts in discrete convex analysis that the M$^\natural$-concavity of a function is characterized in terms of the M$^\natural$-convexity of its maximizers; see Murota (1996a), Murota (2003, Theorem 6.30), and Murota & Shioura (1999).

Theorem 3.6. *A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is M$^\natural$-concave if and only if, for every vector $p \in \mathbb{R}^N$, $D(p; f)$ is an M$^\natural$-convex family. That is, f satisfies* (M$^\natural$-EXC) *if and only if, for every $p \in \mathbb{R}^N$, $D(p; f)$ satisfies* (B$^\natural$-EXC).

The following are two versions of the multiple exchange property of $D(p; f)$:

(NC) For any $p \in \mathbb{R}^N$, if $X, Y \in D(p; f)$ and $I \subseteq X \setminus Y$, there exists $J \subseteq Y \setminus X$ such that $(X \setminus I) \cup J \in D(p; f)$,

(NCsim) For any $p \in \mathbb{R}^N$, if $X, Y \in D(p; f)$ and $I \subseteq X \setminus Y$, there exists $J \subseteq Y \setminus X$ such that $(X \setminus I) \cup J \in D(p; f)$ and $(Y \setminus J) \cup I \in D(p; f)$.

The condition (NC), introduced by Gul & Stacchetti (1999) is called "no complementarities property" and (NCsim) is a simultaneous (or symmetric) version of (NC) introduced by Murota (2016). These conditions, (NC) and (NCsim), are equivalent to each other, and are equivalent to the M$^\natural$-concavity of f; see Remark 3.1 as well as Murota (2016) for details.

In the above we have looked at the family $D(p; f)$ of the maximizers for each $p \in \mathbb{R}^N$. We now investigate how $D(p; f)$ changes with the variation of p.

A set function (single-unit valuation function) $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is said to have the *gross substitutes property* if [8]

[8] To be precise, Kelso & Crawford (1982) and also Gul & Stacchetti (1999) treat the case of $f : 2^N \to \mathbb{R}$.

(GS) For any $p, q \in \mathbb{R}^N$ with $p \leq q$ and $X \in D(p; f)$, there exists $Y \in D(q; f)$ such that $\{i \in X \mid p_i = q_i\} \subseteq Y$.

The concept of gross substitutes property, introduced by Kelso & Crawford (1982), has turned out to be crucial in economics; see, e.g., Roth & Sotomayor (1990), Bikhchandani & Mamer (1997), Gul & Stacchetti (1999), Ausubel & Milgrom (2002), Milgrom (2004), Hatfield & Milgrom (2005), Ausubel (2006), Sun & Yang (2006), Milgrom & Strulovici (2009), and Hatfield et al. (2016).

The following theorem, due to Fujishige & Yang (2003), plays the key role in connecting discrete convex analysis and economics.

Theorem 3.7. *A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ has the gross substitutes property* (GS) *if and only if it is $\mathrm{M}^\natural$-concave.*

It is known (Hatfield & Milgrom, 2005; Milgrom & Strulovici, 2009) that the gross substitutes property, and hence $\mathrm{M}^\natural$-concavity, implies the law of aggregate demand in the following form:

(LAD) For any $p, q \in \mathbb{R}^N$ with $p \leq q$ and $X \in D(p; f)$, there exists $Y \in D(q; f)$ such that $|X| \geq |Y|$.

Gross substitutes properties for multi-unit valuations are treated in Section 4.3.

3.4. Choice function

A function $C : 2^N \to 2^N$ is called a *choice function* if $C(Z) \subseteq Z$ for all $Z \subseteq N$. We have $C(\emptyset) = \emptyset$ and, possibly, $C(Z) = \emptyset$ for some nonempty subsets Z. A choice function C is said to be *consistent* if $C(X) \subseteq Y \subseteq X$ implies $C(Y) = C(X)$. Here we discuss two other properties of choice functions, substitutability and cardinal monotonicity, which are closely related to $\mathrm{M}^\natural$-concavity.

The *substitutability* of a choice function C means the following property (Roth, 1984; Roth & Sotomayor, 1990)

($\mathrm{SC_{ch}}$) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \supseteq Z_2$ it holds that $Z_2 \cap C(Z_1) \subseteq C(Z_2)$.

Several apparently different formulations of substitutability, each equivalent to ($\mathrm{SC_{ch}}$), are found in the literature:

- For any $Z_1, Z_2 \subseteq N$ with $Z_1 \supseteq Z_2$ it holds that $Z_1 \setminus C(Z_1) \supseteq Z_2 \setminus C(Z_2)$.
- $i \in C(X)$ implies $i \in C(Y \cup \{i\})$ for $Y \subseteq X$.
- For any $X \subseteq N$ and any distinct $i, j \in X$, $i \in C(X)$ implies $i \in C(X \setminus \{j\})$.

A choice function C is said to be *cardinal-monotone* if $|C(Y)| \leq |C(X)|$ for all $Y \subseteq X$ (Alkan, 2002). This property is called *increasing property* by Fleiner (2003) and *law of aggregate demand* by Hatfield & Milgrom (2005).

Remark 3.5. As is well known, consistency and substitutability together are equivalent to *path independence* of Plott (1973) which is characterized by the condition: $C(C(X) \cup Y) = C(X \cup Y)$ for all $X, Y \subseteq N$. This condition is equivalent to: $C(C(X) \cup C(Y)) = C(X \cup Y)$ for all $X, Y \subseteq N$. ■

Remark 3.6. The above-mentioned properties of choice functions are well-known key properties in economics and game theory. In the stable matching problem, for example, consistency and substitutability (i.e., path independence) guarantee, roughly, the existence of a stable matching. If, in addition, the choice functions are cardinal-monotone, then the stable matchings form a nice lattice (with simple lattice operations, being distributive, etc.). To quote Alkan (2002, Theorem 10): "The set of stable matchings in any two-sided market with path-independent cardinal-monotone choice functions is a distributive lattice under the common preferences of all agents on one side of the market. The supremum (infimum) operation of the lattice for each side consists componentwise of the join (meet) operation in the revealed preference ordering of associated agents. The lattice has the polarity, unicardinality and complementarity properties." ■

Remark 3.7. A function $C : 2^N \to 2^N$ is called *comonotone* if there exists a monotone function $g : 2^N \to 2^N$ such that $C(X) = X \setminus g(X)$ for all $X \subseteq N$ (Fleiner, 2003). A function $C : 2^N \to 2^N$ is comonotone if and only if C is a choice function with substitutability. The fixed point approach to stable matchings of Fleiner (2003) is based on the observation that stable matchings correspond to fixed points of a certain monotone function associated with the choice functions and the deferred acceptance algorithm of Gale & Shapley (1962) can be regarded as an iteration of this function. See also Farooq et al. (2012). ■

A *choice correspondence* means a function $C : 2^N \to 2^{2^N}$ such that $\emptyset \neq C(Z) \subseteq 2^Z$ for all $Z \subseteq N$. It should be clear that the value $C(Z)$ is not a subset of N but a family of subsets of N. If $C(Z)$ consists of a single subset for each $Z \subseteq N$, then C can be identified with a choice function $C : 2^N \to 2^N$.

The *substitutability* of a choice correspondence C is formulated as follows (Sotomayor, 1999, Definition 4):

($\mathbf{SC}^1_{\mathrm{ch}}$) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \supseteq Z_2$ and any $X_1 \in C(Z_1)$, there exists $X_2 \in C(Z_2)$ such that $Z_2 \cap X_1 \subseteq X_2$.

($\mathbf{SC}^2_{\mathrm{ch}}$) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \supseteq Z_2$ and any $X_2 \in C(Z_2)$, there exists $X_1 \in C(Z_1)$ such that $Z_2 \cap X_1 \subseteq X_2$.

For a choice function $C : 2^N \to 2^N$, ($\mathrm{SC}^1_{\mathrm{ch}}$) and ($\mathrm{SC}^2_{\mathrm{ch}}$) are each equivalent to ($\mathrm{SC}_{\mathrm{ch}}$).

Choice function induced from a valuation function: A valuation function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\emptyset \in \mathrm{dom}\, f$ induces a choice correspondence $C : 2^N \to 2^{2^N}$ by

$$C(Z) = C(Z; f) = \arg\max\{f(Y) \mid Y \subseteq Z\}. \tag{3.17}$$

The assumption "$\emptyset \in \mathrm{dom}\, f$" ensures that $C(Z; f) \neq \emptyset$ for every $Z \subseteq N$. In general, the maximizer is not unique, and accordingly, C is a choice correspondence (i.e., $C(Z; f)$ is a family of subsets of N).

While ($\mathrm{SC}^1_{\mathrm{ch}}$) and ($\mathrm{SC}^2_{\mathrm{ch}}$) above formulate the substitutability for a choice correspondence, (SC^1) and (SC^2) below are the corresponding conditions for a valuation function f. That is, a valuation function f satisfies (SC^1) if and only if the induced choice correspondence $C(\,\cdot\,; f)$ satisfies ($\mathrm{SC}^1_{\mathrm{ch}}$), and similarly for ($\mathrm{SC}^2$) and ($\mathrm{SC}^2_{\mathrm{ch}}$).

($\mathbf{SC}^1$) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \supseteq Z_2$ and any $X_1 \in C(Z_1; f)$, there exists $X_2 \in C(Z_2; f)$ such that $Z_2 \cap X_1 \subseteq X_2$.

($\mathbf{SC}^2$) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \supseteq Z_2$ and any $X_2 \in C(Z_2; f)$, there exists $X_1 \in C(Z_1; f)$ such that $Z_2 \cap X_1 \subseteq X_2$.

These two conditions are independent of each other; see Examples 3.1 and 3.2 in Farooq & Tamura (2004).

A connection to M$^\natural$-concavity is pointed out by Eguchi et al. (2003) (see also Fujishige & Tamura, 2006). This is another important finding, on top of

Theorem 3.7 (equivalence of M$^\natural$-concavity to (GS)), which has reinforced the connection between discrete convex analysis and economics.

Theorem 3.8. *Every M$^\natural$-concave function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\emptyset \in \mathrm{dom} f$ satisfies* (SC^1) *and* (SC^2). *That is, the choice correspondence induced from an M$^\natural$-concave set function has the substitutability properties* (SC$^1_{\mathrm{ch}}$) *and* (SC$^2_{\mathrm{ch}}$).

Proof. Assume $Z_1 \supseteq Z_2$.

Proof of (SC^1): Let $X_1 \in C(Z_1; f)$ and take $X_2 \in C(Z_2; f)$ with minimum $|(Z_2 \cap X_1) \setminus X_2|$. To prove by contradiction, suppose that there exists $i \in (Z_2 \cap X_1) \setminus X_2$. Since $i \in X_1 \setminus X_2$, (M$^\natural$-EXC) implies (i) $f(X_1) + f(X_2) \leq f(X_1 - i) + f(X_2 + i)$ or (ii) there exists $j \in X_2 \setminus X_1$ such that $f(X_1) + f(X_2) \leq f(X_1 - i + j) + f(X_2 + i - j)$. In case (i) we note $X_1 - i \subseteq Z_1$ and $X_2 + i \subseteq Z_2$, from which follow $f(X_1 - i) \leq f(X_1)$ and $f(X_2 + i) \leq f(X_2)$. Therefore, the inequalities are in fact equalities, and $X_1 - i \in C(Z_1; f)$ and $X_2 + i \in C(Z_2; f)$. But we have $|(Z_2 \cap X_1) \setminus (X_2 + i)| = |(Z_2 \cap X_1) \setminus X_2| - 1$, which contradicts the choice of X_2. In case (ii) we note $X_1 - i + j \subseteq Z_1$ and $X_2 + i - j \subseteq Z_2$, from which follow $f(X_1 - i + j) \leq f(X_1)$ and $f(X_2 + i - j) \leq f(X_2)$. Therefore, the inequalities are in fact equalities, and $X_1 - i + j \in C(Z_1; f)$ and $X_2 + i - j \in C(Z_2; f)$. But we have $|(Z_2 \cap X_1) \setminus (X_2 + i - j)| = |(Z_2 \cap X_1) \setminus X_2| - 1$, which contradicts the choice of X_2.

Proof of (SC^2): Let $X_2 \in C(Z_2; f)$ and take $X_1 \in C(Z_1; f)$ with minimum $|(Z_2 \cap X_1) \setminus X_2|$. By the same argument as above we obtain (i) $X_1 - i \in C(Z_1; f)$ with $|(Z_2 \cap (X_1 - i)) \setminus X_2| = |(Z_2 \cap X_1) \setminus X_2| - 1$, or (ii) $X_1 - i + j \in C(Z_1; f)$ with $|(Z_2 \cap (X_1 - i + j)) \setminus X_2| = |(Z_2 \cap X_1) \setminus X_2| - 1$. This is a contradiction to the choice of X_1. $\square$

When the maximizer is unique in (3.17) for every Z, we say that f is *unique-selecting*. In this case, C in (3.17) is a choice function (i.e., $C(Z; f)$ is a subset of N for every Z), and (SC^1) and (SC^2) both reduce to the following condition:

(SC) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \supseteq Z_2$ it holds that $Z_2 \cap C(Z_1; f) \subseteq C(Z_2; f)$.

Theorem 3.8 yields, as a corollary, the following result of Eguchi & Fujishige (2002).

Theorem 3.9. *Every unique-selecting M$^\natural$-concave function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\emptyset \in \mathrm{dom} f$ satisfies* (SC). *That is, the choice function induced from a unique-selecting M$^\natural$-concave set function has the substitutability property* (SC$_{\mathrm{ch}}$).

Unique-selecting $\mathrm{M}^\natural$-concave functions are well-behaved also with respect to cardinal monotonicity. The following is a special case of Murota & Yokoi (2015, Lemma 4.5).

Theorem 3.10. *Every unique-selecting $M^\natural$-concave function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\emptyset \in \mathrm{dom}\, f$ induces a choice function with cardinal monotonicity.*

Proof. The proof is based on the exchange property (3.9) in Remark 3.3. To prove by contradiction, suppose that there exist X and Y such that $X \supseteq Y$ and $|C(X)| < |C(Y)|$. Set $X^* = C(X)$ and $Y^* = C(Y)$. Then $|X^*| < |Y^*|$. By the exchange property (3.9) there exists $j \in Y^* \setminus X^*$ such that $f(X^*) + f(Y^*) \leq f(X^* + j) + f(Y^* - j)$. Here we have $f(X^* + j) < f(X^*)$ since $X^* + j \subseteq X$ and X^* is the unique maximizer, and also $f(Y^* - j) < f(Y^*)$ since $Y^* - j \subseteq Y$ and Y^* is the unique maximizer. This is a contradiction. $\square$

Thus, $\mathrm{M}^\natural$-concave valuation functions entail the three desirable properties. Recall Remark 3.6 for the implications of this fact.

Theorem 3.11. *The choice function induced from a unique-selecting $M^\natural$-concave set function f with $\emptyset \in \mathrm{dom}\, f$ has consistency, substitutability, and cardinal monotonicity.*

Finally, we mention a theorem that characterizes $\mathrm{M}^\natural$-concavity in terms of a parametrized version of (SC^1) and (SC^2). Recall from (3.15) the notation $f[-p](X) = f(X) - p(X)$ for $p \in \mathbb{R}^N$ and $X \subseteq N$. If f is an $\mathrm{M}^\natural$-concave function (not assumed to be unique-selecting), $f[-p]$ is also $\mathrm{M}^\natural$-concave, and hence is equipped with the properties (SC^1) and (SC^2) by Theorem 3.8. In other words, an $\mathrm{M}^\natural$-concave function f has the following properties.

($\mathbf{SC^1_G}$) For any $p \in \mathbb{R}^N$, $f[-p]$ satisfies (SC^1).

($\mathbf{SC^2_G}$) For any $p \in \mathbb{R}^N$, $f[-p]$ satisfies (SC^2).

The following theorem, due to Farooq & Tamura (2004), states that these two conditions are equivalent, and each of them characterizes $\mathrm{M}^\natural$-concavity.

Theorem 3.12. *For a set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, we have the equivalence: f is $M^\natural$-concave $\iff$ ($\mathrm{SC^1_G}$) $\iff$ ($\mathrm{SC^2_G}$).*

3.5. Twisted M$^\natural$-concavity

Let W be a subset of N. For any subset X of N we define

$$\mathrm{tw}(X) = (X \setminus W) \cup (W \setminus X). \tag{3.18}$$

A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is said to be a *twisted M$^\natural$-concave function* with respect to W, if the function $\tilde{f} : 2^N \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\tilde{f}(X) = f(\mathrm{tw}(X)) \qquad (X \subseteq N) \tag{3.19}$$

is an M$^\natural$-concave function (Ikebe & Tamura, 2015). The same concept was introduced earlier by Sun & Yang (2006, 2009) under the name of *GM-concave functions*. Note that f is twisted M$^\natural$-concave with respect to W if and only if it is twisted M$^\natural$-concave with respect to $U = N \setminus W$.

Mathematically, twisted M$^\natural$-concavity is equivalent to the original M$^\natural$-concavity through twisting, and all the properties and theorems about M$^\natural$-concave functions can be translated into those about twisted M$^\natural$-concave functions. However, twisted M$^\natural$-concave functions are convenient sometimes in the modeling in economics.

For example, as pointed out by Ikebe & Tamura (2015), twisted M$^\natural$-concavity implies the same-side substitutability (SSS) and the cross-side complementarity (CSC) proposed by Ostrovsky (2008) in discussing supply chain networks. For a choice function $C : 2^N \to 2^N$ the *same-side substitutability* (SSS) with respect to the bipartition (U,W) of N means the following property:

(SSS) (i) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \cap U \supseteq Z_2 \cap U$ and $Z_1 \cap W = Z_2 \cap W$, we have $Z_2 \cap C(Z_1) \cap U \subseteq C(Z_2) \cap U$, and (ii) the same statement with U and W interchanged,

and the *cross-side complementarity* (CSC) means

(CSC) (i) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \cap U \supseteq Z_2 \cap U$ and $Z_1 \cap W = Z_2 \cap W$, we have $C(Z_1) \cap W \supseteq C(Z_2) \cap W$, and (ii) the same statement with U and W interchanged.

For our exposition it is convenient to combine these two into a single property:

(SSS-CSC) (i) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \cap U \supseteq Z_2 \cap U$ and $Z_1 \cap W = Z_2 \cap W$, we have $Z_2 \cap C(Z_1) \cap U \subseteq C(Z_2) \cap U$ and $C(Z_1) \cap W \supseteq C(Z_2) \cap W$, and (ii) the same statement with U and W interchanged.

The connection to twisted M$^\natural$-concavity is given in the following theorem,[9] to be ascribed to Ikebe & Tamura (2015). Recall from (3.17) the definition of the choice function induced from a valuation function: $C(Z) = C(Z; f) = \arg\max\{f(Y) \mid Y \subseteq Z\}$.

Theorem 3.13. *The choice function induced from a unique-selecting twisted M$^\natural$-concave set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\emptyset \in \mathrm{dom}\, f$ has the property* (SSS-CSC).

For choice correspondences we need to consider the following pair of conditions.

(SSS-CSC^1) (i) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \cap U \supseteq Z_2 \cap U$ and $Z_1 \cap W = Z_2 \cap W$ and any $X_1 \in C(Z_1)$, there exists $X_2 \in C(Z_2)$ such that $Z_2 \cap X_1 \cap U \subseteq X_2 \cap U$ and $X_1 \cap W \supseteq X_2 \cap W$, and (ii) the same statement with U and W interchanged.

(SSS-CSC^2) (i) For any $Z_1, Z_2 \subseteq N$ with $Z_1 \cap U \supseteq Z_2 \cap U$ and $Z_1 \cap W = Z_2 \cap W$ and any $X_2 \in C(Z_2)$, there exists $X_1 \in C(Z_1)$ such that $Z_2 \cap X_1 \cap U \subseteq X_2 \cap U$ and $X_1 \cap W \supseteq X_2 \cap W$, and (ii) the same statement with U and W interchanged.

The following theorem (Ikebe & Tamura, 2015) states that these two properties are implied by twisted M$^\natural$-concavity.

Theorem 3.14. *The choice correspondence induced from a twisted M$^\natural$-concave set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\emptyset \in \mathrm{dom}\, f$ has the properties* (SSS-CSC^1) *and* (SSS-CSC^2).

Proof. We prove (SSS-CSC^1)-(i) and (SSS-CSC^2)-(i); the proofs of (SSS-CSC^1)-(ii) and (SSS-CSC^2)-(ii) are obtained by interchanging U and W. Assume $Z_1 \cap U \supseteq Z_2 \cap U$ and $Z_1 \cap W = Z_2 \cap W$, and let $\tilde{f}$ be the M$^\natural$-concave function in (3.19) associated with f. For $X_1 \subseteq Z_1$ and $X_2 \subseteq Z_2$ define

$$\Phi(X_1, X_2) = |(Z_2 \cap X_1 \cap U) \setminus (X_2 \cap U)| + |(X_2 \cap W) \setminus (X_1 \cap W)|.$$

[9] Theorem 3.13 can be understood as a twisted version of Theorem 3.9, though a straightforward translation of Theorem 3.9 via twisting does not seem to yield Theorem 3.13. Theorem 3.13 can be proved as a special case of Theorem 3.14 below, for which a direct proof is given.

Proof of (SSS-CSC^1)-(i): Let $X_1 \in C(Z_1; f)$ and take $X_2 \in C(Z_2; f)$ with $\Phi(X_1, X_2)$ minimum. To prove by contradiction, suppose that there exists $i \in \big((Z_2 \cap X_1 \cap U) \setminus (X_2 \cap U)\big) \cup \big((X_2 \cap W) \setminus (X_1 \cap W)\big)$. Since $i \in \mathrm{tw}(X_1) \setminus \mathrm{tw}(X_2)$, (M$^\natural$-EXC) for $\tilde{f}$ implies

(i) $\tilde{f}(\mathrm{tw}(X_1)) + \tilde{f}(\mathrm{tw}(X_2)) \leq \tilde{f}(\mathrm{tw}(X_1) - i) + \tilde{f}(\mathrm{tw}(X_2) + i)$ or

(ii) there exists $j \in \mathrm{tw}(X_2) \setminus \mathrm{tw}(X_1)$ such that $\tilde{f}(\mathrm{tw}(X_1)) + \tilde{f}(\mathrm{tw}(X_2)) \leq \tilde{f}(\mathrm{tw}(X_1) - i + j) + \tilde{f}(\mathrm{tw}(X_2) + i - j)$.

Letting

$$\hat{X}_1 = \begin{cases} \mathrm{tw}(\mathrm{tw}(X_1) - i) & \text{(in (i))}, \\ \mathrm{tw}(\mathrm{tw}(X_1) - i + j)) & \text{(in (ii))}, \end{cases} \quad \hat{X}_2 = \begin{cases} \mathrm{tw}(\mathrm{tw}(X_2) + i) & \text{(in (i))}, \\ \mathrm{tw}(\mathrm{tw}(X_2) + i - j)) & \text{(in (ii))}, \end{cases}$$

we can express the above inequalities in (i) and (ii) as

$$f(X_1) + f(X_2) \leq f(\hat{X}_1) + f(\hat{X}_2).$$

As can be verified easily, we have $\hat{X}_1 \subseteq Z_1$ and $\hat{X}_2 \subseteq Z_2$, from which follow $f(\hat{X}_1) \leq f(X_1)$ and $f(\hat{X}_2) \leq f(X_2)$ since $X_1 \in C(Z_1; f)$ and $X_2 \in C(Z_2; f)$. Therefore, the inequalities are in fact equalities, and $\hat{X}_1 \in C(Z_1; f)$ and $\hat{X}_2 \in C(Z_2; f)$. But we have $\Phi(X_1, \hat{X}_2) = \Phi(X_1, X_2) - 1$, which contradicts the choice of X_2.

Proof of (SSS-CSC^2)-(i): Let $X_2 \in C(Z_2; f)$ and take $X_1 \in C(Z_1; f)$ with $\Phi(X_1, X_2)$ minimum. By the same argument as above we obtain $\hat{X}_1 \in C(Z_1; f)$ with $\Phi(\hat{X}_1, X_2) = \Phi(X_1, X_2) - 1$. This is a contradiction to the choice of X_1. □

The concept of twisted M$^\natural$-concavity can also be defined for functions on integer vectors $\mathbb{Z}^N$ to be used for multi-unit models. See Section 4.5.

3.6. Examples

Here are some examples of M$^\natural$-concave set functions.

1. For real numbers a_i indexed by $i \in N$, the *additive valuation*

$$f(X) = \sum_{i \in X} a_i \qquad (X \subseteq N) \tag{3.20}$$

is an M$^\natural$-concave function.

2. For a set of nonnegative numbers a_i indexed by $i \in N$, the *maximum-value function* (*unit-demand utility*)

$$f(X) = \max_{i \in X} a_i \qquad (X \subseteq N) \tag{3.21}$$

with $f(\emptyset) = 0$ is an M$^\natural$-concave function.

3. For a univariate concave function $\varphi : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ (i.e., if $\varphi(t-1) + \varphi(t+1) \leq 2\varphi(t)$ for all integers t), the function f defined by

$$f(X) = \varphi(|X|) \qquad (X \subseteq N) \tag{3.22}$$

is M$^\natural$-concave. Such f is called a *symmetric concave valuation*.

4. For a family of univariate concave functions $\{\varphi_A \mid A \in \mathscr{T}\}$ indexed by a family $\mathscr{T}$ of subsets of N, the function

$$f(X) = \sum_{A \in \mathscr{T}} \varphi_A(|A \cap X|) \qquad (X \subseteq N) \tag{3.23}$$

is submodular. A function f of the form (3.23) is called *laminar concave*, if $\mathscr{T}$ is a laminar family, i.e., if $[A, B \in \mathscr{T} \Rightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B]$. A laminar concave function is M$^\natural$-concave. See Murota (2003, Note 6.11) for a proof. A special case of (3.23) with $\mathscr{T} = \{N\}$ reduces to (3.22).

5. Given a matroid[10] on N in terms of the family $\mathscr{I}$ of independent sets, the *rank function* f is defined by

$$f(X) = \max\{|I| \mid I \in \mathscr{I},\ I \subseteq X\} \qquad (X \subseteq N), \tag{3.24}$$

which denotes the maximum size of an independent set contained in X. A matroid rank function (3.24) is M$^\natural$-concave. A *weighted matroid rank function* (or *weighted matroid valuation*) is a function represented as

$$f(X) = \max\{w(I) \mid I \in \mathscr{I},\ I \subseteq X\} \qquad (X \subseteq N) \tag{3.25}$$

with some weight $w \in \mathbb{R}^N$, where $w(I) = \sum_{i \in I} w_i$. A weighted matroid rank function (3.25) is M$^\natural$-concave (Shioura, 2012). See Murota (2010) for an elementary proof for the M$^\natural$-concavity of (3.25) as well as (3.24).

[10] For matroids, see, e.g., Murota (2000b), Oxley (2011), and Schrijver (2003).

6. Let $G = (S, T; E)$ be a bipartite graph with vertex bipartition (S, T) and edge set E, and suppose that each edge $e \in E$ is associated with weight $w_e \in \mathbb{R}$. For $M \subseteq E$, we denote by ∂M the set of the vertices incident to some edge in M, and call M a *matching* if $|S \cap \partial M| = |M| = |T \cap \partial M|$. For $X \subseteq T$ denote by $f(X)$ the maximum weight of a matching that precisely matches X in T, i.e.,

$$f(X) = \max\{w(M) \mid M \text{ is a matching}, T \cap \partial M = X\} \tag{3.26}$$

with $w(M) = \sum_{e \in M} w_e$, where $f(X) = -\infty$ if no such M exists for X. Then $f : 2^T \to \mathbb{R} \cup \{-\infty\}$ is an M$^\natural$-concave function. See Murota (1996c, Example 3.3) or Murota (2000b, Example 5.2.4) for proofs. Such function is called an *assignment valuation* by Hatfield & Milgrom (2005). Assignment valuations cover a fairly large class of M$^\natural$-concave functions, but not every M$^\natural$-concave function can be represented in the form of (3.26), as shown by Ostrovsky & Paes Leme (2015).

7. Let $G = (S, T; E)$ be a bipartite graph with vertex bipartition (S, T) and edge set E, with weight $w_e \in \mathbb{R}$ associated with each edge $e \in E$. Furthermore, suppose that a matroid on S is given in terms of the family $\mathscr{I}$ of independent sets (see Fig. 1). For $X \subseteq T$ denote by $f(X)$ the maximum weight of a matching such that the end-vertices in S form an independent set and the end-vertices in T are equal to X, i.e.,

$$f(X) = \max\{w(M) \mid \begin{array}{l} M \text{ is a matching}, \\ S \cap \partial M \in \mathscr{I},\ T \cap \partial M = X\}, \end{array} \tag{3.27}$$

where $f(X) = -\infty$ if no such M exists for X. We call such f an *independent assignment valuation*. It is known that an independent assignment valuation is M$^\natural$-concave. For proofs, see Murota (2000b, Example 5.2.18), Murota (2003, Section 9.6.2), and Kobayashi et al. (2007). If the given matroid is a free matroid with $\mathscr{I} = 2^S$, (3.27) reduces to (3.26).

3.7. Concluding remarks of section 3

We collect here the conditions that characterize M$^\natural$-concave set functions:

– Exchange property (M$^\natural$-EXC) (Section 3.1)
– Multiple exchange property (M$^\natural$-EXC$_\mathrm{m}$)

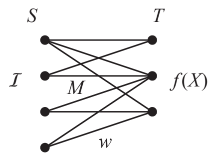

Figure 1: Independent assignment valuation

= Strong no complementarities property (SNC) (Section 3.1)
– Local exchange property (Theorems 3.2 and 3.3) (Section 3.1)
– Single improvement property (SI) (Section 3.2)
– Exchange property ($B^{\natural}$-EXC) for the maximizers $D(p;f)$ (Section 3.3)
– Multiple (one-sided) exchange property for the maximizers $D(p;f)$
= No complementarities property (NC) (Section 3.3)
– Multiple exchange property (NCsim) for the maximizers $D(p;f)$ (Section 3.3)
– Gross substitutability (GS) (Section 3.3)
– Parametrized substitutability (SC^1_G) (Section 3.4)
– Parametrized substitutability (SC^2_G) (Section 3.4)

4. $M^{\natural}$-CONCAVE FUNCTION ON $\mathbb{Z}^N$

In Section 3 we have considered $M^{\natural}$-concave set functions, which correspond to single-unit valuations with substitutability. In this section we deal with $M^{\natural}$-concave functions defined on integer vectors, $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$, which correspond to multi-unit valuations with substitutability.

4.1. Exchange property

Let N be a finite set, say, $N = \{1,2,\ldots,n\}$ for $n \geq 1$. For a vector $z \in \mathbb{R}^N$ in general, define the *positive* and *negative supports* of z as

$$\mathrm{supp}^+(z) = \{i \mid z_i > 0\}, \qquad \mathrm{supp}^-(z) = \{j \mid z_j < 0\}. \tag{4.1}$$

Recall that, for $i \in N$, the ith unit vector is denoted by χ_i.

We say that a function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is *$M^{\natural}$-concave*, if, for any $x, y \in \mathbb{Z}^N$ and $i \in \mathrm{supp}^+(x-y)$, we have (i)

$$f(x) + f(y) \leq f(x - \chi_i) + f(y + \chi_i) \tag{4.2}$$

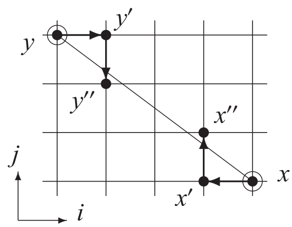

Figure 2: Nearer pair in the definition of M$^\natural$-concave functions

or (ii) there exists some $j \in \mathrm{supp}^-(x-y)$ such that

$$f(x)+f(y) \leq f(x-\chi_i+\chi_j)+f(y+\chi_i-\chi_j). \tag{4.3}$$

This property is referred to as the *exchange property*. See Fig. 2, in which $(x',y') = (x-\chi_i, y+\chi_i)$ and $(x'',y'') = (x-\chi_i+\chi_j, y+\chi_i-\chi_j)$.

A more compact expression of the exchange property is as follows:

(M$^\natural$-EXC[$\mathbb{Z}$]) For any $x,y \in \mathbb{Z}^N$ and $i \in \mathrm{supp}^+(x-y)$, we have

$$f(x)+f(y) \leq \max_{j \in \mathrm{supp}^-(x-y) \cup \{0\}} \{f(x-\chi_i+\chi_j)+f(y+\chi_i-\chi_j)\}, \tag{4.4}$$

where $\chi_0 = \mathbf{0}$ (zero vector). In the above statement we may change "For any $x,y \in \mathbb{Z}^N$" to "For any $x,y \in \mathrm{dom}\, f$" since if $x \notin \mathrm{dom}\, f$ or $y \notin \mathrm{dom}\, f$, (4.4) trivially holds with $f(x)+f(y) = -\infty$. An M$^\natural$-concave function f with $\mathrm{dom}\, f \subseteq \{0,1\}^N$ can be identified with an M$^\natural$-concave set function introduced in Section 3.1. A function f is called *M$^\natural$-convex* if $-f$ is M$^\natural$-concave.

It follows from (M$^\natural$-EXC[$\mathbb{Z}$]) that the effective domain $B = \mathrm{dom}\, f$ of an M$^\natural$-concave function f has the following exchange property:

(B$^\natural$-EXC[$\mathbb{Z}$]) For any $x,y \in B$ and $i \in \mathrm{supp}^+(x-y)$, we have (i) $x-\chi_i \in B$, $y+\chi_i \in B$ or
(ii) there exists some $j \in \mathrm{supp}^-(x-y)$ such that $x-\chi_i+\chi_j \in B$, $y+\chi_i-\chi_j \in B$.

A set $B \subseteq \mathbb{Z}^N$ having this property is called an *M$^\natural$-convex set* (or *integral generalized polymatroid*, *integral g-polymatroid*). An M$^\natural$-convex set contained in the unit cube $\{0,1\}^N$ can be identified with an M$^\natural$-convex family of subsets (Section 3.1).

M$^\natural$-concavity can be characterized by a local exchange property under the assumption that function f is (effectively) defined on an M$^\natural$-convex set (Murota, 1996a, 2003; Murota & Shioura, 1999). The conditions (4.5)–(4.9) below are "local" in the sense that they require the exchangeability of the form of (4.4) only for some (x,y) with $\|x-y\|_1 \leq 4$.

Theorem 4.1. *A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is M$^\natural$-concave if and only if* $\mathrm{dom}\, f$ *is an M$^\natural$-convex set and the following conditions hold:*

$$f(x+2\chi_i) + f(x) \leq 2f(x+\chi_i) \tag{4.5}$$

for all $x \in \mathbb{Z}^N$ and for all $i \in N$; and

$$f(x+\chi_i+\chi_j) + f(x) \leq f(x+\chi_i) + f(x+\chi_j) \tag{4.6}$$

for all $x \in \mathbb{Z}^N$ and for all distinct $i, j \in N$; and

$$f(x+2\chi_i) + f(x+\chi_k) \leq f(x+\chi_i+\chi_k) + f(x+\chi_i) \tag{4.7}$$

for all $x \in \mathbb{Z}^N$ and for all distinct $i,k \in N$; and

$$f(x+\chi_i+\chi_j) + f(x+\chi_k) \leq \max \begin{array}{l} \{f(x+\chi_i+\chi_k) + f(x+\chi_j), \\ f(x+\chi_j+\chi_k) + f(x+\chi_i)\} \end{array} \tag{4.8}$$

for all $x \in \mathbb{Z}^N$ and for all distinct $i, j, k \in N$; and

$$\begin{array}{ll} f(x+\chi_i+\chi_j) + & f(x+\chi_k+\chi_l) \leq \\ & \max\{f(x+\chi_i+\chi_k) + f(x+\chi_j+\chi_l), \\ & f(x+\chi_j+\chi_k) + f(x+\chi_i+\chi_l)\} \end{array} \tag{4.9}$$

for all $x \in \mathbb{Z}^N$ and for all $i, j, k, l \in N$ with $\{i, j\} \cap \{k, l\} = \emptyset$. Here we allow the possibility of $i = j$ or $k = l$.

When the effective domain $\mathrm{dom}\, f$ is an M$^\natural$-convex set such that $\mathbf{0} \in \mathrm{dom}\, f \subseteq \mathbb{Z}^N_+$, the local exchange condition above takes a simpler form that does not involve (4.9) (Shioura & Tamura, 2015, Theorem 6.8). To cover the case of $\mathrm{dom}\, f = \mathbb{Z}^N$ we weaken the assumption on $\mathrm{dom}\, f$ to:

$$x, y \in \mathrm{dom}\, f \Longrightarrow x \wedge y \in \mathrm{dom}\, f. \tag{4.10}$$

Theorem 4.2. *Let $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be a function such that* $\mathrm{dom}\, f$ *is an M$^\natural$-convex set satisfying* (4.10). *Then f is M$^\natural$-concave if and only if* (4.5), (4.6), (4.7) *and* (4.8) *hold.*

Proof. The proof of Shioura & Tamura (2015, Theorem 6.8) works under the weaker condition (4.10). □

The local exchange property above admits a natural reformulation in terms of the discrete Hessian matrix when $\mathrm{dom}\, f = \mathbb{Z}^N$. For $x \in \mathbb{Z}^N$ and $i, j \in N$ define

$$H_{ij}(x) = f(x + \chi_i + \chi_j) - f(x + \chi_i) - f(x + \chi_j) + f(x), \tag{4.11}$$

and let $H_f(x) = (H_{ij}(x) \mid i, j \in N)$ be the matrix consisting of those components. This matrix $H_f(x)$ is called the *discrete Hessian matrix* of f at x. The following theorem, due to Hirai & Murota (2004) and Murota (2007) can be derived from Theorem 4.2.

Theorem 4.3. *A function $f : \mathbb{Z}^N \to \mathbb{R}$ is M$^\natural$-concave if and only if the discrete Hessian matrix $H_f(x) = (H_{ij}(x))$ satisfies the following conditions for each $x \in \mathbb{Z}^N$:*

$$H_{ij}(x) \leq 0 \qquad \textit{for any } (i, j), \tag{4.12}$$

$$H_{ij}(x) \leq \max(H_{ik}(x), H_{jk}(x)) \qquad \textit{if } \{i, j\} \cap \{k\} = \emptyset. \tag{4.13}$$

Proof. The correspondence between the conditions in Theorems 4.2 and 4.3 is quite straightforward. With the use of (4.11) we can easily verify: (4.5) $\Leftrightarrow$ $H_{ii}(x) \leq 0$, (4.6) $\Leftrightarrow$ $H_{ij}(x) \leq 0$ $(i \neq j)$, (4.7) $\Leftrightarrow$ $H_{ii}(x) \leq H_{ik}(x)$ $(i \neq k)$, and (4.8) $\Leftrightarrow$ $H_{ij}(x) \leq \max(H_{ik}(x), H_{jk}(x))$ (i, j, k: distinct). □

It is known (Murota, 2003, Theorem 6.19) that an M$^\natural$-concave function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is *submodular* on the integer lattice, i.e.,

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \qquad (x, y \in \mathbb{Z}^N). \tag{4.14}$$

More precisely, the condition (4.6) above is equivalent to the submodularity (4.14) as long as $\mathrm{dom}\, f$ is M$^\natural$-convex (Shioura & Tamura, 2015, Proposition 6.1). Because of the additional conditions for M$^\natural$-concavity, not every submodular function is M$^\natural$-concave. Thus, M$^\natural$-concave functions form a proper subclass of submodular functions on $\mathbb{Z}^N$.

It is also known in Murota (1996a, Theorem 4.6) and Murota (2003, Theorem 6.42) that an M$^{\natural}$-concave function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is *concave-extensible*, i.e., there exists a concave function $\overline{f} : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ such that $\overline{f}(x) = f(x)$ for all $x \in \mathbb{Z}^N$.

Remark 4.1. It follows from (M$^{\natural}$-EXC[$\mathbb{Z}$]) that M$^{\natural}$-concave functions enjoy the following exchange properties under size constraints (Murota & Shioura, 1999, Lemmas 4.3 and 4.6):
- For any $x, y \in \mathbb{Z}^N$ with $x(N) < y(N)$,

$$f(x) + f(y) \leq \max_{j \in \mathrm{supp}^-(x-y)} \{f(x + \chi_j) + f(y - \chi_j)\}. \tag{4.15}$$

- For any $x, y \in \mathbb{Z}^N$ with $x(N) = y(N)$ and $i \in \mathrm{supp}^+(x-y)$,

$$f(x) + f(y) \leq \max_{j \in \mathrm{supp}^-(x-y)} \{f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)\}. \tag{4.16}$$

The former property, in particular, implies the size-monotonicity of the induced choice function; see Theorem 4.9 and its proof. ∎

Remark 4.2. If $B \subseteq \mathbb{Z}^N$ lies in a hyperplane with a constant component sum (i.e., $x(N) = y(N)$ for all $x, y \in B$), the exchange property (B$^{\natural}$-EXC[$\mathbb{Z}$]) takes a simpler form (without the possibility of $j = 0$): For any $x, y \in B$ and $i \in \mathrm{supp}^+(x-y)$, there exists some $j \in \mathrm{supp}^-(x-y)$ such that $x - \chi_i + \chi_j \in B$, $y + \chi_i - \chi_j \in B$. A set $B \subseteq \mathbb{Z}^N$ having this exchange property is called an *M-convex set* (or *integral base polyhedron*). An M$^{\natural}$-concave function defined on an M-convex set is called an *M-concave function* (Murota, 1996a, 2003). The exchange property for M-concavity reads: A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is M-concave if and only if, for any $x, y \in \mathbb{Z}^N$ and $i \in \mathrm{supp}^+(x-y)$, it holds that

$$f(x) + f(y) \leq \max_{j \in \mathrm{supp}^-(x-y)} \{f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)\}. \tag{4.17}$$

M-concave functions and M$^{\natural}$-concave functions are equivalent concepts, in that M$^{\natural}$-concave functions in n variables can be obtained as projections of M-concave functions in $n+1$ variables. More formally, let "0" denote a new element not in N and $\tilde{N} = \{0\} \cup N$. A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is M$^{\natural}$-concave if and only if the function $\tilde{f} : \mathbb{Z}^{\tilde{N}} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 = -x(N) \\ -\infty & \text{otherwise} \end{cases} \qquad (x_0 \in \mathbb{Z}, x \in \mathbb{Z}^N) \tag{4.18}$$

is an M-concave function. A function f is called *M-convex* if $-f$ is M-concave. ∎

4.2. Maximization and single improvement property

For an M$^\natural$-concave function, the maximality of a function value is characterized by a local condition as follows, where $\chi_0 = \mathbf{0}$ (Murota, 2003, Proposition 6.23, Theorem 6.26).

Theorem 4.4. *Let $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and $x \in$* $\mathrm{dom}\, f$.
(1) *If $f(x) < f(y)$ for $y \in \mathrm{dom}\, f$, then $f(x) < f(x - \chi_i + \chi_j)$ for some $i \in$* $\mathrm{supp}^+(x-y) \cup \{0\}$ *and* $j \in \mathrm{supp}^-(x-y) \cup \{0\}$.
(2) *x is a maximizer of f if and only if*

$$f(x) \geq f(x - \chi_i + \chi_j) \quad (\forall i, j \in N \cup \{0\}). \tag{4.19}$$

For a vector $p = (p_i \mid i \in N) \in \mathbb{R}^N$ we use the notation $f[-p]$ to mean the function $f(x) - p^\top x$, where $p^\top$ means the transpose of p. That is,

$$f[-p](x) = f(x) - p^\top x \qquad (x \in \mathbb{Z}^N). \tag{4.20}$$

By considering the properties of (1) and (2) in Theorem 4.4 for $f[-p]$ with varying p, we are naturally led to (SSI[$\mathbb{Z}$]) and (SI[$\mathbb{Z}$]) below:[11]

(SSI[$\mathbb{Z}$]) For any $p \in \mathbb{R}^N$ and $x, y \in \mathrm{dom}\, f$ with $f[-p](x) < f[-p](y)$, there exists $i \in \mathrm{supp}^+(x-y) \cup \{0\}$ and $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that $f[-p](x) < f[-p](x - \chi_i + \chi_j)$.

(SI[$\mathbb{Z}$]) For any $p \in \mathbb{R}^N$, if $x \in \mathrm{dom}\, f$ is not a maximizer of $f[-p]$, there exists $i \in N \cup \{0\}$ and $j \in N \cup \{0\}$ such that $f[-p](x) < f[-p](x - \chi_i + \chi_j)$.

The stronger version (SSI[$\mathbb{Z}$]) is shown to be equivalent to M$^\natural$-concavity (Murota & Tamura, 2003b, Theorem 7). This property is named the *strong single improvement property* in Shioura & Tamura (2015). The latter (SI[$\mathbb{Z}$]) is the vector version of single improvement property (Section 3.2), called the *multi-unit single improvement property* by Milgrom & Strulovici (2009). We can see from Milgrom & Strulovici (2009, Theorem 13) that (SI[$\mathbb{Z}$]) is equivalent to M$^\natural$-concavity under the assumption of concave-extensibility of f and boundedness of $\mathrm{dom}\, f$.

[11] (SSI[$\mathbb{Z}$]) here is denoted as (M$^\natural$-SI[$\mathbb{Z}$]) in Murota (2003).

4.3. Maximizers and gross substitutability

For a vector $p = (p_i \mid i \in N) \in \mathbb{R}^N$ we consider the maximizers of the function $f[-p](x) = f(x) - p^\top x$. We denote the set of these maximizers by

$$D(p; f) = \arg\max_x \{f(x) - p^\top x\}. \tag{4.21}$$

In economic applications, p is a price vector and $D(p) = D(p; f)$ represents the demand correspondence.

It is one of the most fundamental facts in discrete convex analysis that the M$^\natural$-concavity of a function is characterized in terms of the M$^\natural$-convexity of its maximizers; see Murota (1996a), Murota (2003, Theorem 6.30), and Murota & Shioura (1999).

Theorem 4.5. *Let $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be a function with a bounded effective domain. Then f is M$^\natural$-concave if and only if, for every vector $p \in \mathbb{R}^N$, $D(p; f)$ is an M$^\natural$-convex set. That is, f satisfies* (M$^\natural$-EXC[$\mathbb{Z}$]) *if and only if, for every $p \in \mathbb{R}^N$, $D(p; f)$ satisfies* (B$^\natural$-EXC[$\mathbb{Z}$]).

As a straightforward extension of the gross substitutes condition from single-unit valuations (Section 3.3) to multi-unit valuations it seems natural to conceive the following condition:

(GS[$\mathbb{Z}$]) For any $p, q \in \mathbb{R}^N$ with $p \leq q$ and $x \in D(p; f)$, there exists $y \in D(q; f)$ such that $x_i \leq y_i$ for all $i \in N$ with $p_i = q_i$.

It turns out, however, that this condition alone is too weak to be fruitful, mathematically and economically. Subsequently, several different strengthened forms of (GS[$\mathbb{Z}$]) are proposed in the literature, including Danilov et al. (2003); Murota & Tamura (2003b); Milgrom & Strulovici (2009); Shioura & Tamura (2015).

Among others we start with the *projected gross substitutes condition*[12] (PRJ-GS[$\mathbb{Z}$]) of Murota & Tamura (2003b):

(PRJ-GS[$\mathbb{Z}$]) For any $p, q \in \mathbb{R}^N$ with $p \leq q$, any $p_0, q_0 \in \mathbb{R}$ with $p_0 \leq q_0$ and $x \in D(p - p_0\mathbf{1}; f)$, there exists $y \in D(q - q_0\mathbf{1}; f)$ such that (i) $x_i \leq y_i$ for all $i \in N$ with $p_i = q_i$ and (ii) $x(N) \geq y(N)$ if $p_0 = q_0$,

[12] (PRJ-GS[$\mathbb{Z}$]) is denoted as (M$^\natural$-GS[$\mathbb{Z}$]) in Murota (2003, §6.8).

where $x(N) = \sum_{i \in N} x_i$ and $y(N) = \sum_{i \in N} y_i$. By fixing $p_0 = q_0 = 0$ in (PRJ-GS[$\mathbb{Z}$]) we obtain the following condition:

(GS&LAD[$\mathbb{Z}$]) For any $p, q \in \mathbb{R}^N$ with $p \leq q$ and $x \in D(p; f)$, there exists $y \in D(q; f)$ such that (i) $x_i \leq y_i$ for all $i \in N$ with $p_i = q_i$ and (ii) $x(N) \geq y(N)$.

As the acronym (GS&LAD[$\mathbb{Z}$]) shows, this condition is a combination of (GS[$\mathbb{Z}$]) above and the law of aggregate demand:

(LAD[$\mathbb{Z}$]) For any $p, q \in \mathbb{R}^N$ with $p \leq q$ and $x \in D(p; f)$, there exists $y \in D(q; f)$ such that $x(N) \geq y(N)$.

This condition is studied by Hatfield & Milgrom (2005) and Milgrom & Strulovici (2009). Note, however, that imposing (GS&LAD[$\mathbb{Z}$]) on f is not the same as imposing (GS[$\mathbb{Z}$]) and (LAD[$\mathbb{Z}$]) on f, since in (GS&LAD[$\mathbb{Z}$]) both (i) and (ii) must be satisfied by the same vector y. Obviously, (GS&LAD[$\mathbb{Z}$]) implies (GS[$\mathbb{Z}$]) and (LAD[$\mathbb{Z}$]). The amalgamated form (GS&LAD[$\mathbb{Z}$]) is given in Murota et al. (2013a), whereas the juxtaposition of (GS[$\mathbb{Z}$]) and (LAD[$\mathbb{Z}$]) is in Milgrom & Strulovici (2009, Theorem 13 (iv)). We may also consider the following variant (Shioura & Tamura, 2015; Shioura & Yang, 2015) of (GS&LAD[$\mathbb{Z}$]), where the vector q takes a special form[13] $p + \delta \chi_k$ with $k \in N$ and $\delta > 0$:

(GS&LAD$'$[$\mathbb{Z}$]) For any $p \in \mathbb{R}^N$, $k \in N$, $\delta > 0$ and $x \in D(p; f)$, there exists $y \in D(p + \delta \chi_k; f)$ such that (i) $x_i \leq y_i$ for all $i \in N \setminus \{k\}$ and (ii) $x(N) \geq y(N)$.

M$^\natural$-concavity can be characterized by these properties as the following theorem indicates; see Murota & Tamura (2003b), Danilov et al. (2003), Milgrom & Strulovici (2009, Theorem 13), Shioura & Tamura (2015, Theorem 4.1), and Murota (2003, Theorems 6.34, 6.36). It refers to two other conditions (SWGS[$\mathbb{Z}$]) and (SS[$\mathbb{Z}$]), which are explained in Remark 4.3 below.

Theorem 4.6. *Let $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be a concave-extensible function with a bounded effective domain. Then we have the following equivalence:* (M$^\natural$-EXC[$\mathbb{Z}$]) $\Longleftrightarrow$ (PRJ-GS[$\mathbb{Z}$]) $\Longleftrightarrow$ (GS&LAD[$\mathbb{Z}$]) $\Longleftrightarrow$ (GS[$\mathbb{Z}$]) & (LAD[$\mathbb{Z}$]) $\Longleftrightarrow$ (GS&LAD$'$[$\mathbb{Z}$]) $\Longleftrightarrow$ (SWGS[$\mathbb{Z}$]). *If* $\mathrm{dom}\, f$ *is contained in* $\mathbb{Z}_+^N$, *each of these conditions is equivalent to* (SS[$\mathbb{Z}$]).

[13] Recall that χ_k denotes the kth unit vector.

Remark 4.3. The *step-wise gross substitutes condition* (Danilov et al., 2003) means:

(SWGS[$\mathbb{Z}$]) For any $p \in \mathbb{R}^N$, $k \in N$ and $x \in D(p; f)$, at least one of (i) and (ii) holds true:[14]
(i) $x \in D(p + \delta \chi_k; f)$ for all $\delta \geq 0$,
(ii) there exists $\delta \geq 0$ and $y \in D(p + \delta \chi_k; f)$ such that $y_k = x_k - 1$ and $y_i \geq x_i$ for all $i \in N \setminus \{k\}$.

The *strong substitute condition* (Milgrom & Strulovici, 2009) for a multi-unit valuation f means the condition (GS[$\mathbb{Z}$]) for the single-unit valuation f^{B} corresponding to f:

(SS[$\mathbb{Z}$]) The function f^{B} associated with f satisfies the condition (GS[$\mathbb{Z}$]).

More specifically, the function f^{B} is defined as follows. Let $u \in \mathbb{Z}_+^N$ be a vector such that $\mathrm{dom}\, f \subseteq [\mathbf{0}, u]_{\mathbb{Z}}$. Consider a set $N^{\mathrm{B}} = \{(i, \beta) \mid i \in N,\ \beta \in \mathbb{Z},\ 1 \leq \beta \leq u_i\}$ and define $f^{\mathrm{B}} : \mathbb{Z}^{N^{\mathrm{B}}} \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f^{\mathrm{B}} \subseteq \{0, 1\}^{N^{\mathrm{B}}}$ by

$$f^{\mathrm{B}}(x^{\mathrm{B}}) = f(x), \quad x^{\mathrm{B}} \in \{0, 1\}^{N^{\mathrm{B}}}, \quad x_i = \sum_{\beta=1}^{u_i} x^{\mathrm{B}}_{(i,\beta)} \quad (i \in N). \tag{4.22}$$

∎

4.4. Choice function

Let $b \in \mathbb{Z}_+^N$ be an upper bound vector and $\mathscr{B} = \{x \in \mathbb{Z}_+^N \mid x \leq b\}$ be the set of feasible vectors. A function $C : \mathscr{B} \to \mathscr{B}$ is called a *choice function* if $C(x) \leq x$ for all $x \in \mathscr{B}$. Three important properties are identified in the literature (Alkan & Gale, 2003):

- C is called *consistent* if $C(x) \leq y \leq x$ implies $C(y) = C(x)$,
- C is called *persistent* if $x \geq y$ implies $y \wedge C(x) \leq C(y)$,
- C is called *size-monotone* if $x \geq y$ implies $|C(x)| \geq |C(y)|$, where $|C(x)| = \sum_{i \in N} C(x)_i$.

[14] Recall that χ_k denotes the kth unit vector.

Remark 4.4. Alkan & Gale (2003) consider a stable allocation model that extends the stable matching model of Alkan (2002). If the choice functions are consistent and persistent, the set of stable allocations is nonempty and forms a lattice. Moreover, if the choice functions are also size-monotone, the lattice of stable allocations is distributive and has several significant properties, called polarity, complementarity, and uni-size property. ∎

For a given function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ we define

$$C(z) = C(z; f) = \arg\max\{f(y) \mid y \leq z\}. \tag{4.23}$$

In general, the maximizer may not be unique, and hence $C(z; f) \subseteq \mathbb{Z}^N$. We also have the possibility of $C(z; f) = \emptyset$ to express the nonexistence of a maximizer.

An important property of M$^\natural$-concave functions, closely related to persistence, can be found in Eguchi et al. (2003, Lemma 1) and also in Fujishige & Tamura (2006, Lemma 5.2).

Theorem 4.7. *Let $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function. Then the following hold.*

(SC^1[$\mathbb{Z}$]) *For any $z_1, z_2 \in \mathbb{Z}^N$ with $z_1 \geq z_2$ and $C(z_2; f) \neq \emptyset$ and for any $x_1 \in C(z_1; f)$, there exists $x_2 \in C(z_2; f)$ such that $z_2 \wedge x_1 \leq x_2$.*

(SC^2[$\mathbb{Z}$]) *For any $z_1, z_2 \in \mathbb{Z}^N$ with $z_1 \geq z_2$ and $C(z_1; f) \neq \emptyset$ and for any $x_2 \in C(z_2; f)$, there exists $x_1 \in C(z_1; f)$ such that $z_2 \wedge x_1 \leq x_2$.*

Proof. Assume $z_1 \geq z_2$. For $x_1 \leq z_1$ and $x_2 \leq z_2$ define

$$\Phi(x_1, x_2) = \sum\{(x_1)_i - (x_2)_i \mid i \in \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\}.$$

Proof of (SC^1[$\mathbb{Z}$]): Let $x_1 \in C(z_1; f)$ and take $x_2 \in C(z_2; f)$ with minimum $\Phi(x_1, x_2)$. To prove by contradiction, suppose that there exists $i \in \mathrm{supp}^+((z_2 \wedge x_1) - x_2)$. Since $i \in \mathrm{supp}^+(x_1 - x_2)$, (M$^\natural$-EXC[$\mathbb{Z}$]) implies there exists $j \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ such that

$$f(x_1) + f(x_2) \leq f(x_1 - \chi_i + \chi_j) + f(x_2 + \chi_i - \chi_j).$$

Here we have $x_1 - \chi_i + \chi_j \leq z_1$ and $x_2 + \chi_i - \chi_j \leq z_2$; the former is obvious if $j = 0$ and otherwise, it follows from $(x_1)_j < (x_2)_j \leq (z_2)_j \leq (z_1)_j$, and the latter follows from $(x_2)_i < (z_2)_i$. This implies that $f(x_1 - \chi_i + \chi_j) \leq f(x_1)$ and

$f(x_2+\chi_i-\chi_j) \leq f(x_2)$ since $x_1 \in C(z_1;f)$ and $x_2 \in C(z_2;f)$. Therefore, the inequalities are in fact equalities, and $x_1-\chi_i+\chi_j \in C(z_1;f)$ and $x_2+\chi_i-\chi_j \in C(z_2;f)$. But we have $\Phi(x_1,x_2+\chi_i-\chi_j) = \Phi(x_1,x_2)-1$, which contradicts the choice of x_2.

Proof of (SC$^2[\mathbb{Z}]$): Let $x_2 \in C(z_2;f)$ and take $x_1 \in C(z_1;f)$ with minimum $\Phi(x_1,x_2)$. By the same argument as above we obtain $x_1-\chi_i+\chi_j \in C(z_1;f)$ with $\Phi(x_1-\chi_i+\chi_j,x_2) = \Phi(x_1,x_2)-1$. This is a contradiction to the choice of x_1. □

When the maximizer is unique in (4.23) for every z, we say that f is *unique-selecting*. In the following we assume that f is unique-selecting and

$$\mathbf{0} \in \operatorname{dom} f \subseteq \mathbb{Z}_+^N. \tag{4.24}$$

Then C in (4.23) can be regarded as a choice function $C : \mathscr{B} \to \mathscr{B}$.

The induced choice function C is obviously consistent for any valuation function f. For persistence, M$^\natural$-concavity plays an essential role. The following theorem of Eguchi et al. (2003) can be obtained as a corollary of Theorem 4.7, since for unique-selecting valuation functions, (SC$^1[\mathbb{Z}]$) and (SC$^2[\mathbb{Z}]$) are equivalent and both coincide with persistence.

Theorem 4.8. *Every unique-selecting M$^\natural$-concave function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ with (4.24) induces a persistent choice function.*

The size-monotonicity is also implied by M$^\natural$-concavity (Murota & Yokoi, 2015).

Theorem 4.9. *Every unique-selecting M$^\natural$-concave function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ with (4.24) induces a size-monotone choice function.*

Proof. The proof is based on the exchange property (4.15) in Remark 4.1. To prove by contradiction, suppose that there exist $x,y \in \mathbb{Z}^N$ such that $x \geq y$ and $|C(x;f)| < |C(y;f)|$. Set $x^* = C(x;f)$ and $y^* = C(y;f)$. Then $|x^*| < |y^*|$. By the exchange property (4.15) there exists $j \in \operatorname{supp}^-(x^*-y^*)$ such that $f(x^*)+f(y^*) \leq f(x^*+\chi_j)+f(y^*-\chi_j)$. Here we have $f(x^*+\chi_j) < f(x^*)$ since $x^*+\chi_j \leq x$ by $x^*_j < y^*_j \leq y_j \leq x_j$ and x^* is the unique maximizer. We also have $f(y^*-\chi_j) < f(y^*)$ since $y^*-\chi_j \leq y^* \leq y$ and y^* is the unique maximizer. This is a contradiction. □

Thus, M$^\natural$-concave valuation functions entail the three desired properties, consistency, persistence, and size-monotonicity.[15] Recall Remark 4.4 for the implications of this fact.

Theorem 4.10. *For a unique-selecting M$^\natural$-concave value function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ with (4.24), the choice function C induced from f is consistent, persistent, and size-monotone.*

Finally, we mention a theorem that characterizes M$^\natural$-concavity in terms of a parametrized version of (SC$^1[\mathbb{Z}]$) and (SC$^2[\mathbb{Z}]$). Recall from (4.20) the notation $f[-p](x) = f(x) - p^\top x$ for $p \in \mathbb{R}^N$ and $x \in \mathbb{Z}^N$. If f is an M$^\natural$-concave function (not assumed to be unique-selecting), $f[-p]$ is also M$^\natural$-concave, and hence is equipped with the properties (SC$^1[\mathbb{Z}]$) and (SC$^2[\mathbb{Z}]$) by Theorem 4.7. In other words, an M$^\natural$-concave function f has the following properties.

(SC$^1_{\mathrm{G}}[\mathbb{Z}]$) For any $p \in \mathbb{R}^N$, $f[-p]$ satisfies (SC$^1[\mathbb{Z}]$).

(SC$^2_{\mathrm{G}}[\mathbb{Z}]$) For any $p \in \mathbb{R}^N$, $f[-p]$ satisfies (SC$^2[\mathbb{Z}]$).

The following theorem, due to Farooq & Shioura (2005), states that each of these conditions characterizes M$^\natural$-concavity.

Theorem 4.11. *For a function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ with a bounded nonempty effective domain, we have the equivalence: f is M$^\natural$-concave $\iff$* (SC$^1_{\mathrm{G}}[\mathbb{Z}]$) $\iff$ (SC$^2_{\mathrm{G}}[\mathbb{Z}]$).

4.5. Twisted M$^\natural$-concavity

Let W be a subset of N. For any vector $x \in \mathbb{Z}^N$ we define $\mathrm{tw}(x) \in \mathbb{Z}^N$ by specifying its ith component $\mathrm{tw}(x)_i$ as

$$\mathrm{tw}(x)_i = \begin{cases} x_i & (i \in N \setminus W), \\ -x_i & (i \in W). \end{cases} \tag{4.25}$$

A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is said to be a *twisted M$^\natural$-concave function* with respect to W, if the function $\tilde{f} : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\tilde{f}(x) = f(\mathrm{tw}(x)) \qquad (x \in \mathbb{Z}^N) \tag{4.26}$$

[15] Theorem 4.10 can be extended to quasi M$^\natural$-concave value functions; see Murota & Yokoi (2015).

is an M$^\natural$-concave function (Ikebe & Tamura, 2015). The same concept has been introduced by Shioura & Yang (2015), almost at the same time and independently, under the name of *GM-concave functions*; see also Sun & Yang (2008) and Section 14.5. Note that f is twisted M$^\natural$-concave with respect to W if and only if it is twisted M$^\natural$-concave with respect to $U = N \setminus W$.

Mathematically, twisted M$^\natural$-concavity is equivalent to the original M$^\natural$-concavity through twisting, and all the properties and theorems about M$^\natural$-concave functions can be translated into those about twisted M$^\natural$-concave functions. In such translations it is often adequate to define the *twisted demand correspondence* as[16]

$$\tilde{D}(p;f) = \arg\max_{x} \{f(x) - \mathrm{tw}(p)^\top x\}. \tag{4.27}$$

A twisted version of (GS&LAD$'[\mathbb{Z}]$) is introduced by Ikebe et al. (2015) as the *generalized full substitutes* (GFS[$\mathbb{Z}$]) condition:

(GFS[$\mathbb{Z}$]) (i) For any $p \in \mathbb{R}^N$, $\tilde{D}(p;f)$ is a discrete convex set.[17]

(ii) For any $p \in \mathbb{R}^N$, $k \in U$, $\delta > 0$, and $x \in \tilde{D}(p;f)$, there exists $y \in \tilde{D}(p + \delta\chi_k; f)$ such that

$$\begin{array}{rcl} x_i & \leq & y_i \text{ for all } i \in U \setminus \{k\}, \\ x_i & \geq & y_i \text{ for all } i \in W, \\ x(U) - x(W) & \geq & y(U) - y(W). \end{array} \tag{4.28}$$

(iii) For any $p \in \mathbb{R}^N$, $k \in W$, $\delta > 0$, and $x \in \tilde{D}(p;f)$, there exists $y \in \tilde{D}(p - \delta\chi_k; f)$ such that

$$\begin{array}{rcl} x_i & \leq & y_i \text{ for all } i \in W \setminus \{k\}, \\ x_i & \geq & y_i \text{ for all } i \in U, \\ x(W) - x(U) & \geq & y(W) - y(U). \end{array} \tag{4.29}$$

The following theorem[18] (Ikebe et al., 2015) characterizes twisted M$^\natural$-concavity in terms of this condition.

[16] Note: $x \in \tilde{D}(p;f) \iff \mathrm{tw}(x) \in D(p;\tilde{f})$.

[17] That is, $\tilde{D}(p;f)$ should coincide with the integer points contained in the convex hull of $\tilde{D}(p;f)$.

[18] Theorem 4.12 can be understood as a twisted version of the equivalence "(GS&LAD$'[\mathbb{Z}]$) $\Leftrightarrow$ (M$^\natural$-EXC[$\mathbb{Z}$])" in Theorem 4.6.

Theorem 4.12. *Let $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be a concave-extensible*[19] *function with a bounded effective domain. Then f satisfies* (GFS[$\mathbb{Z}$]) *if and only if it is a twisted $M^\natural$-concave function with respect to W.*

In the modeling of a trading network (supply chain network), where an agent is identified with a vertex (node) of the network, each vertex (agent) is associated with a valuation function f defined on the set of arcs incident to the vertex. Denoting the set of in-coming arcs to the vertex by U and the set of out-going arcs from the vertex by W, the function f is defined on $U \cup W$. Twisted M$^\natural$-concave functions are used effectively in this context (Ikebe & Tamura, 2015; Ikebe et al., 2015; Candogan et al., 2016). See Section 14.2.

With the use of the ordinary (un-twisted) demand correspondence

$$D(p; f) = \arg\max_{x} \{f(x) - p^\top x\}, \tag{4.30}$$

a similar condition was formulated by Shioura & Yang (2015), independently of Ikebe et al. (2015), to deal with economies with two classes of indivisible goods such that goods in the same class are substitutable and goods across two classes are complementary. The condition, called the *generalized gross substitutes and complements* (GGSC[$\mathbb{Z}$]) condition, reads as follows:

(GGSC[$\mathbb{Z}$]) (i) For any $p \in \mathbb{R}^N$, $D(p; f)$ is a discrete convex set.
(ii) For any $p \in \mathbb{R}^N$, $k \in U$, $\delta > 0$, and $x \in D(p; f)$, there exists $y \in D(p + \delta\chi_k; f)$ that satisfies (4.28).
(iii) For any $p \in \mathbb{R}^N$, $k \in W$, $\delta > 0$, and $x \in D(p; f)$, there exists $y \in D(p + \delta\chi_k; f)$ that satisfies (4.29).

This condition also characterizes twisted M$^\natural$-concavity (Shioura & Yang, 2015).

Theorem 4.13. *Let $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be a concave-extensible function with a bounded effective domain. Then f satisfies* (GGSC[$\mathbb{Z}$]) *if and only if it is a twisted $M^\natural$-concave function with respect to W.*

[19] The concave-extensibility of f is assumed here for the consistency with the statement of Theorem 4.6. Mathematically, this assumption can be omitted, since the condition (i) in (GFS[$\mathbb{Z}$]) is equivalent to the concave-extensibility of f and twisted M$^\natural$-concave functions are concave-extensible. Similarly in Theorem 4.13.

Although Theorems 4.12 and 4.13 have significances in different contexts, they are in fact two variants of the same mathematical statement. Note that (GSF[$\mathbb{Z}$]) and (GGSC[$\mathbb{Z}$]) are equivalent, since

$$D(p;f) = \tilde{D}(\mathrm{tw}(p);f), \qquad \mathrm{tw}(p+\delta\chi_k) = \begin{cases} \mathrm{tw}(p)+\delta\chi_k & (k \in U), \\ \mathrm{tw}(p)-\delta\chi_k & (k \in W). \end{cases}$$

The multi-unit (or vector) version of the same-side substitutability (SSS) and the cross-side complementarity (CSC) of Ostrovsky (2008) can be formulated for a correspondence $C : \mathbb{Z}^N \to 2^{\mathbb{Z}^N}$ as follows, where, for any $z \in \mathbb{Z}^N$, the subvector of z on U is denoted by $z^U \in \mathbb{Z}^U$ and similarly the subvector on W by $z^W \in \mathbb{Z}^W$.

(SSS-CSC^1[$\mathbb{Z}$]) (i) For any $z_1, z_2 \in \mathbb{Z}^N$ with $z_1^U \geq z_2^U$, $z_1^W = z_2^W$ and $C(z_2) \neq \emptyset$ and for any $x_1 \in C(z_1)$, there exists $x_2 \in C(z_2)$ such that $z_2^U \wedge x_1^U \leq x_2^U$ and $x_1^W \geq x_2^W$, and (ii) the same statement with U and W interchanged.

(SSS-CSC^2[$\mathbb{Z}$]) (i) For any $z_1, z_2 \in \mathbb{Z}^N$ with $z_1^U \geq z_2^U$, $z_1^W = z_2^W$ and $C(z_1) \neq \emptyset$ and for any $x_2 \in C(z_2)$ there exists $x_1 \in C(z_1)$, such that $z_2^U \wedge x_1^U \leq x_2^U$ and $x_1^W \geq x_2^W$, and (ii) the same statement with U and W interchanged.

The following theorem (Ikebe & Tamura, 2015) states that these two properties are implied by twisted M$^\natural$-concavity. Recall from (4.23) that a valuation function f induces the correspondence[20] $C(z) = C(z;f) = \arg\max\{f(y) \mid y \leq z\}$ $(z \in \mathbb{Z}^N)$.

Theorem 4.14. *For any twisted M$^\natural$-concave function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$, the induced correspondence C has the properties* (SSS-CSC^1[$\mathbb{Z}$]) *and* (SSS-CSC^2[$\mathbb{Z}$]).

Proof. We prove (SSS-CSC^1[$\mathbb{Z}$])-(i) and (SSS-CSC^2[$\mathbb{Z}$])-(i); the proofs of (SSS-CSC^1[$\mathbb{Z}$])-(ii) and (SSS-CSC^2[$\mathbb{Z}$])-(ii) are obtained by interchanging U and W. Assume $z_1^U \geq z_2^U$, $z_1^W = z_2^W$ and $C(z_1;f) \neq \emptyset$, and let $\tilde{f}$ be the M$^\natural$-concave function in (4.26) associated with f. For $x_1 \leq z_1$ and $x_2 \leq z_2$ define

$$\begin{aligned}\Phi(x_1,x_2) = &\sum\{(x_1)_i - (x_2)_i \mid i \in U \cap \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\} \\ &+ \sum\{(x_2)_i - (x_1)_i \mid i \in W \cap \mathrm{supp}^+(x_2 - x_1)\}.\end{aligned}$$

[20] It may be that $C(z) = \emptyset$ if $\mathrm{dom}\, f$ is unbounded below or $\{y \mid y \leq z\} \cap \mathrm{dom}\, f = \emptyset$. The condition "$C(z_2) \neq \emptyset$" in (SSS-CSC^1[$\mathbb{Z}$]), for example, takes care of this possibility.

Proof of (SSS-CSC^1[$\mathbb{Z}$])-(i): Let $x_1 \in C(z_1;f)$ and take $x_2 \in C(z_2;f)$ with $\Phi(x_1,x_2)$ minimum. To prove by contradiction, suppose that there exists

$$i \in \big(U \cap \mathrm{supp}^+((z_2 \wedge x_1) - x_2)\big) \cup \big(W \cap \mathrm{supp}^+(x_2 - x_1)\big).$$

Then $i \in \mathrm{supp}^+(\mathrm{tw}(x_1) - \mathrm{tw}(x_2))$, and (M$^\natural$-EXC[$\mathbb{Z}$]) for $\tilde{f}$ implies that there exists $j \in \mathrm{supp}^-(\mathrm{tw}(x_1) - \mathrm{tw}(x_2)) \cup \{0\}$ such that

$$\tilde{f}(\mathrm{tw}(x_1)) + \tilde{f}(\mathrm{tw}(x_2)) \leq \tilde{f}(\mathrm{tw}(x_1) - \chi_i + \chi_j) + \tilde{f}(\mathrm{tw}(x_2) + \chi_i - \chi_j).$$

Letting $\hat{x}_1 = \mathrm{tw}(\mathrm{tw}(x_1) - \chi_i + \chi_j))$ and $\hat{x}_2 = \mathrm{tw}(\mathrm{tw}(x_2) + \chi_i - \chi_j))$ we can express the above inequality as

$$f(x_1) + f(x_2) \leq f(\hat{x}_1) + f(\hat{x}_2).$$

By considering all possibilities ($i \in U$ or $i \in W$, and $j \in U$ or $j \in W$ or $j = 0$), we can verify that $\hat{x}_1 \leq z_1$ and $\hat{x}_2 \leq z_2$, from which follow $f(\hat{x}_1) \leq f(x_1)$ and $f(\hat{x}_2) \leq f(x_2)$ since $x_1 \in C(z_1;f)$ and $x_2 \in C(z_2;f)$. Therefore, the inequalities are in fact equalities, and $\hat{x}_1 \in C(z_1;f)$ and $\hat{x}_2 \in C(z_2;f)$. But we have $\Phi(x_1,\hat{x}_2) = \Phi(x_1,x_2) - 1$, which contradicts the choice of x_2.

Proof of (SSS-CSC^2[$\mathbb{Z}$])-(i): Let $x_2 \in C(z_2;f)$ and take $x_1 \in C(z_1;f)$ with minimum $\Phi(x_1,x_2)$. By the same argument as above we obtain $\hat{x}_1 \in C(z_1;f)$ with $\Phi(\hat{x}_1,x_2) = \Phi(x_1,x_2) - 1$. This is a contradiction to the choice of x_1. □

4.6. Examples

Here are some examples of M$^\natural$-concave functions in integer variables.

1. A linear (or affine) function

$$f(x) = \alpha + \langle p,x \rangle \tag{4.31}$$

   with $p \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ is M$^\natural$-concave if $\mathrm{dom}\, f$ is an M$^\natural$-convex set.

2. A quadratic function $f : \mathbb{Z}^N \to \mathbb{R}$ defined by

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \tag{4.32}$$

with $a_{ij} = a_{ji} \in \mathbb{R}$ $(i, j = 1, \ldots, n)$ is M$^\natural$-concave if and only if

$$\begin{array}{lll} a_{ij} & \leq & 0 \ \text{ for all } (i,j), \ \text{ and} \\ a_{ij} & \leq & \max(a_{ik}, a_{jk}) \ \text{ when } \{i,j\} \cap \{k\} = \emptyset. \end{array} \tag{4.33}$$

The Hessian matrix $H_f(x) = (H_{ij}(x))$ defined in (4.11) is given by $H_{ij}(x) = 2a_{ij}$, and (4.33) above is consistent with (4.12), (4.13) in Theorem 4.3.

3. A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is called *separable concave* if it can be represented as
$$f(x) = \sum_{i \in N} \varphi_i(x_i) \qquad (x \in \mathbb{Z}^N) \tag{4.34}$$
for univariate concave functions[21] $\varphi_i : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ $(i \in N)$. A separable concave function is M$^\natural$-concave. In (4.4) for (M$^\natural$-EXC[$\mathbb{Z}$]) we can always take $j = 0$, i.e., (4.2).

4. A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is called *laminar concave* if it can be represented as
$$f(x) = \sum_{A \in \mathscr{T}} \varphi_A(x(A)) \qquad (x \in \mathbb{Z}^N) \tag{4.35}$$
for a laminar family $\mathscr{T} \subseteq 2^N$ and a family of univariate concave functions $\varphi_A : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ indexed by $A \in \mathscr{T}$, where $x(A) = \sum_{i \in A} x_i$. A laminar concave function is M$^\natural$-concave; see Murota (2003, Note 6.11) for a proof. A special case of (4.35) with $\mathscr{T} = \{\{1\}, \{2\}, \ldots, \{n\}\}$ reduces to the separable convex function (4.34).

5. M$^\natural$-concave functions arise from the maximum weight of nonlinear network flows. Let $G = (V, A)$ be a directed graph with two disjoint vertex subsets $S \subseteq V$ and $T \subseteq V$ specified as the entrance and the exit. Suppose that, for each arc $a \in A$, we are given a univariate concave function $\varphi_a : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ representing the weight of flow on the arc a. Let $\xi \in \mathbb{Z}^A$ be a vector representing an integer flow, and $\partial\xi \in \mathbb{Z}^V$ be the boundary of flow ξ defined for every $v \in V$ by
$$\begin{array}{rl} \partial\xi(v) = & \sum\{\xi(a) \mid \text{ arc } a \text{ leaves } v\ \} \\ & -\sum\{\xi(a) \mid \text{ arc } a \text{ enters } v\ \}. \end{array} \tag{4.36}$$

[21] Recall that $\varphi : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ is called concave if $\varphi(t-1) + \varphi(t+1) \leq 2\varphi(t)$ for all integers t.

Then, the maximum weight of a flow that realizes a supply/demand specification on the exit T in terms of $x \in \mathbb{Z}^T$ is expressed by

$$f(x) = \sup_{\xi} \quad \begin{array}{l} \{\sum_{a \in A} \varphi_a(\xi(a)) \mid (\partial \xi)(v) = -x(v) \ (v \in T), \\ (\partial \xi)(v) = 0 \ (v \in V \setminus (S \cup T))\} \end{array} \tag{4.37}$$

where no constraint is imposed on $(\partial \xi)(v)$ for entrance vertices $v \in S$. This function is M$^\natural$-concave, provided that f does not take the value $+\infty$ and $\mathrm{dom}\, f$ is nonempty. If $S = \emptyset$, the function f is M-concave, since $\sum_{v \in T} x(v) = -\sum_{v \in T} (\partial \xi)(v) = \sum_{v \in V \setminus T} (\partial \xi)(v) = 0$ in this case. See Murota (1998, Example 2.3) and Murota (2003, Section 2.2.2) for details. The maximum weight of a matching in (3.26) can be understood as a special case of (4.37).

4.7. Concluding remarks of section 4

The concept of M-convex functions is formulated by Murota (1996a) as a generalization of valuated matroids of Dress & Wenzel (1990, 1992). Then M$^\natural$-convex functions are introduced by Murota & Shioura (1999) as a variant of M-convex functions. Quasi M-convex functions are introduced by Murota & Shioura (2003). The concept of M-convex functions is extended to functions on jump systems by Murota (2006); see also Kobayashi et al. (2007).

Unimodularity is closely related to discrete convexity. For a fixed unimodular matrix U we may consider a change of variables $x \mapsto Ux$ for $x \in \mathbb{Z}^n$ to define a class of functions $\{f(Ux) \mid f : \text{M}^\natural\text{-concave}\}$ as a variant of M$^\natural$-concave functions. Twisted M$^\natural$-concave functions (Section 4.5) are a typical example of this construction with $U = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$; see Sun & Yang (2008) and Section 14.5 for further discussion in this direction.

5. M$^\natural$-CONCAVE FUNCTION ON $\mathbb{R}^N$

In Sections 3 and 4, we have considered M$^\natural$-concave functions on 2^N and $\mathbb{Z}^N$, which correspond to valuations for indivisible goods with substitutability. In this section we deal with M$^\natural$-concave functions in real vectors, $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$, which correspond to valuations for divisible goods with substitutability. M$^\natural$-concave functions in real variables are investigated by Murota & Shioura (2000, 2004a,b).

5.1. Exchange property

We say that a function $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ is *M$^\natural$-concave* if it is a concave function (in the ordinary sense) that satisfies

(M$^\natural$-EXC[$\mathbb{R}$]) For any $x, y \in \mathbb{R}^N$ and $i \in \mathrm{supp}^+(x-y)$, there exist $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ and a positive number $\alpha_0 \in \mathbb{R}_{++}$ such that

$$f(x) + f(y) \leq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \tag{5.1}$$

for all $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq \alpha_0$.

In the following we restrict ourselves to closed proper[22] M$^\natural$-concave functions, for which the closure of the effective domain $\mathrm{dom}\, f$ is a well-behaved polyhedron (g-polymatroid, or M$^\natural$-convex polyhedron[23]); see Murota & Shioura (2008, Theorem 3.2). Often we are interested in polyhedral M$^\natural$-concave functions.

Remark 5.1. It follows from (M$^\natural$-EXC[$\mathbb{R}$]) that M$^\natural$-concave functions enjoy the following exchange properties under size constraints:

• For any $x, y \in \mathbb{R}^N$ with $x(N) < y(N)$, there exists $\alpha_0 \in \mathbb{R}_{++}$ such that

$$f(x) + f(y) \leq \max_{j \in \mathrm{supp}^-(x-y)} \{f(x + \alpha\chi_j) + f(y - \alpha\chi_j)\} \tag{5.2}$$

for all $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq \alpha_0$.

• For any $x, y \in \mathbb{R}^N$ with $x(N) = y(N)$ and $i \in \mathrm{supp}^+(x-y)$, there exists $\alpha_0 \in \mathbb{R}_{++}$ such that

$$f(x) + f(y) \leq \max_{j \in \mathrm{supp}^-(x-y)} \{f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j))\} \tag{5.3}$$

for all $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq \alpha_0$. ∎

Remark 5.2. If $\mathrm{dom}\, f \subseteq \mathbb{R}^N$ lies in a hyperplane with a constant component sum (i.e., $x(N) = y(N)$ for all $x, y \in \mathrm{dom}\, f$), the exchange property (M$^\natural$-EXC[$\mathbb{R}$]) takes a simpler form excluding the possibility of $j = 0$. A function $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ having this exchange property is called an *M-concave function*. That is, a concave function f is M-concave if and only if (5.3) holds. ∎

[22] A concave function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be *proper* if $\mathrm{dom}\, f$ is nonempty, and *closed* if the hypograph $\{(x, \beta) \in \mathbb{R}^{n+1} \mid \beta \leq f(x)\}$ is a closed subset of $\mathbb{R}^{n+1}$.

[23] A polyhedron P is called an *M$^\natural$-convex polyhedron* if its (concave) indicator function f is M$^\natural$-concave, where $f(x) = 0$ for $x \in P$ and $= -\infty$ for $x \notin P$. See Murota (2003, Section 4.8) for details.

5.2. Maximizers and gross substitutability

For $p \in \mathbb{R}^N$ we denote the set of the maximizers of $f[-p](x) = f(x) - p^\top x$ by $D(p;f) \subseteq \mathbb{R}^N$ (cf. (4.21)). M$^\natural$-concavity of a function f is characterized by the M$^\natural$-convexity of $D(p;f)$ (Murota & Shioura, 2000, Theorem 5.2).

Theorem 5.1. *A polyhedral concave function $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ is M$^\natural$-concave if and only if, for every vector $p \in \mathbb{R}^N$, $D(p;f)$ is an M$^\natural$-convex polyhedron.*[24]

(GS[$\mathbb{R}$]) For any $p, q \in \mathbb{R}^N$ with $p \leq q$ and $x \in D(p;f)$, there exists $y \in D(q;f)$ such that $x_i \leq y_i$ for all $i \in N$ with $p_i = q_i$.

The following theorem is given by Danilov et al. (2003).

Theorem 5.2. *A polyhedral M$^\natural$-concave function $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ with a bounded effective domain satisfies* (GS[$\mathbb{R}$]).

Proof. This follows from Theorem 7.5 (2) and Theorem 7.7 in Section 7.2.1. □

Example 5.1. Here is an example to show that (GS[$\mathbb{R}$]) does not imply M$^\natural$-concavity. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1, x_2) = \min(2, x_1 + 2x_2)$ on $\mathrm{dom}\, f = \mathbb{R}^2$. This function is not M$^\natural$-concave because (M$^\natural$-EXC[$\mathbb{R}$]) fails for $x = (2,0)$, $y = (0,1)$ and $i = 1$. However, it satisfies (GS[$\mathbb{R}$]), which can be verified easily. Thus the converse of Theorem 5.2 does not hold. ∎

5.3. Choice function

In Theorem 4.10 in Section 4.4 we have seen, for the multi-unit indivisible goods, the choice function induced from a unique-selecting M$^\natural$-concave value function is consistent, persistent, and size-monotone in the sense of Alkan & Gale (2003). In this section we point out that this is also the case with divisible goods; recall Remark 4.4 in Section 4.4 for the implications of this fact.

For a choice function $C : \mathscr{B} \to \mathscr{B}$ with $\mathscr{B} = \{x \in \mathbb{R}^N_+ \mid x \leq b\}$ for some $b \in \mathbb{R}^N_+$, consistency means $[\, C(x) \leq y \leq x \Rightarrow C(y) = C(x) \,]$, persistence means $[\, x \geq y \Rightarrow y \wedge C(x) \leq C(y) \,]$, and size-monotonicity means $[\, x \geq y \Rightarrow |C(x)| \geq |C(y)| \,]$, where $|C(x)| = \sum_{i \in N} C(x)_i$ (sum of the components).

[24] See the footnote 23.

Theorem 5.3. *For a unique-selecting M$^\natural$-concave value function* $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ *with* $\mathbf{0} \in \mathrm{dom}\, f \subseteq \mathbb{R}^N_+$, *the induced choice function* $C(x; f) = \arg\max\{f(y) \mid y \leq x\}$ *is consistent, persistent, and size-monotone.*[25]

Proof. The consistency is obvious from the definition of $C(x; f)$.

To prove persistence[26] by contradiction, suppose that $y \wedge C(x; f) \leq C(y; f)$ fails for some $x, y \in \mathbb{R}^N$ with $x \geq y$. Set $x^* = C(x; f)$, $y^* = C(y; f)$. Since $y \wedge x^* \leq y^*$ fails, there exists some $i \in N$ such that $y_i \wedge x^*_i > y^*_i$. Then $i \in \mathrm{supp}^+(x^* - y^*)$. We apply (M$^\natural$-EXC[$\mathbb{R}$]) to x^*, y^* and i, to obtain $j \in \mathrm{supp}^-(x^* - y^*) \cup \{0\}$ and $\alpha_0 > 0$ such that

$$f(x^*) + f(y^*) \leq f(x^* - \alpha(\chi_i - \chi_j)) + f(y^* + \alpha(\chi_i - \chi_j)) \qquad (5.4)$$

for all α with $0 < \alpha \leq \alpha_0$. For sufficiently small $\alpha > 0$ we also have $x^* - \alpha(\chi_i - \chi_j) \leq x$ and $y^* + \alpha(\chi_i - \chi_j) \leq y$; the former follows from $x^*_j < y^*_j \leq y_j \leq x_j$ for $j \in \mathrm{supp}^-(x^* - y^*)$, and the latter from $y^*_i < y_i \wedge x^*_i \leq y_i$. On the right-hand side of (5.4), we have $f(x^* - \alpha(\chi_i - \chi_j)) < f(x^*)$ since $x^* - \alpha(\chi_i - \chi_j) \leq x$ and $x^* = C(x; f)$ is the unique maximizer of f in $\{z \in \mathbb{R}^N \mid z \leq x\}$, and similarly, $f(y^* + \alpha(\chi_i - \chi_j)) < f(y^*)$. This is a contradiction, proving persistence.

To prove size-monotonicity by contradiction, suppose that there exist $x, y \in \mathbb{R}^N$ such that $x \geq y$ and $|C(x; f)| < |C(y; f)|$. Set $x^* = C(x; f)$ and $y^* = C(y; f)$. Then $|x^*| < |y^*|$. By the exchange property (5.2) in Remark 5.1, there exists $j \in \mathrm{supp}^-(x^* - y^*)$ such that $f(x^*) + f(y^*) \leq f(x^* + \alpha\chi_j) + f(y^* - \alpha\chi_j)$ for sufficiently small $\alpha > 0$. Here we have $f(x^* + \alpha\chi_j) < f(x^*)$ since $x^* + \alpha\chi_j \leq x$ by $x^*_j < y^*_j \leq y_j \leq x_j$ and x^* is the unique maximizer. We also have $f(y^* - \alpha\chi_j) < f(y^*)$ since $y^* - \alpha\chi_j \leq y^* \leq y$ and y^* is the unique maximizer. This is a contradiction, proving size-monotonicity. □

5.4. Examples

Here are some examples of M$^\natural$-concave functions in real variables.

1. A function $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ is called *laminar concave* if it can be represented as

$$f(x) = \sum_{A \in \mathcal{T}} \varphi_A(x(A)) \qquad (x \in \mathbb{R}^N) \qquad (5.5)$$

[25] As in Section 4.4, f is said to be unique-selecting if $C(x; f)$ consists of a single element for every x.

[26] This proof for persistence is an adaptation of the one in Murota & Yokoi (2015, Lemma 3.3).

for a laminar family $\mathscr{T} \subseteq 2^N$ and a family of univariate (closed proper) concave functions $\varphi_A : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ indexed by $A \in \mathscr{T}$, where $x(A) = \sum_{i \in A} x_i$. A laminar concave function is M$^\natural$-concave.

2. M$^\natural$-concave functions arise from the maximum weight of nonlinear network flows. Let $G = (V, A)$ be a directed graph with two disjoint vertex subsets $S \subseteq V$ and $T \subseteq V$ specified as the entrance and the exit. Suppose that, for each arc $a \in A$, we are given a univariate (closed proper) concave function $\varphi_a : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ representing the weight of flow on the arc a. Let $\xi \in \mathbb{R}^A$ be a vector representing a flow, and $\partial\xi \in \mathbb{R}^V$ be the boundary of flow ξ defined by (4.36). Then, the maximum weight of a flow that realizes a supply/demand specification on the exit T in terms of $x \in \mathbb{R}^T$ is expressed by a function $f : \mathbb{R}^T \to \mathbb{R} \cup \{-\infty\}$ defined as (4.37). This function is M$^\natural$-concave, provided that f does not take the value $+\infty$ and $\operatorname{dom} f$ is nonempty. If $S = \emptyset$, the function f is M-concave. See Murota (2003, Section 2.2.1) and Murota & Shioura (2004a, Theorem 2.10) for details.

5.5. Concluding remarks of section 5

The concept of M-concave functions in continuous variables is introduced for polyhedral concave functions by Murota & Shioura (2000) and for general concave functions by Murota & Shioura (2004a). This is partly motivated by a phenomenon inherent in the network flow/tension problem described in Section 5.4.

6. OPERATIONS FOR M$^\natural$-CONCAVE FUNCTIONS

6.1. Basic operations

Basic operations on M$^\natural$-concave functions on $\mathbb{Z}^n$ are presented here, whereas the most powerful operation, transformation by networks, is treated in Section 6.2.

M$^\natural$-concave functions admit the following operations.

Theorem 6.1. *Let $f, f_1, f_2 : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ be M$^\natural$-concave functions.*
(1) *For nonnegative $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, $\alpha f(x) + \beta$ is M$^\natural$-concave in x.*
(2) *For $a \in \mathbb{Z}^N$, $f(a - x)$ and $f(a + x)$ are M$^\natural$-concave in x.*

(3) *For* $p \in \mathbb{R}^N$, $f[-p]$ *is* $M^\natural$*-concave, where* $f[-p]$ *is defined by* (4.20).
(4) *For univariate concave functions* $\varphi_i : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ *indexed by* $i \in N$,

$$\tilde{f}(x) = f(x) + \sum_{i \in N} \varphi_i(x_i) \qquad (x \in \mathbb{Z}^N) \tag{6.1}$$

is $M^\natural$*-concave, provided* $\mathrm{dom}\,\tilde{f} \neq \emptyset$.
(5) *For* $a \in (\mathbb{Z} \cup \{-\infty\})^N$ *and* $b \in (\mathbb{Z} \cup \{+\infty\})^N$, *the restriction of* f *to the integer interval* $[a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z}^N \mid a \leq x \leq b\}$ *defined by*

$$f_{[a,b]_{\mathbb{Z}}}(x) = \begin{cases} f(x) & (x \in [a,b]_{\mathbb{Z}}), \\ -\infty & (x \notin [a,b]_{\mathbb{Z}}) \end{cases} \tag{6.2}$$

is $M^\natural$*-concave, provided* $\mathrm{dom}\, f_{[a,b]_{\mathbb{Z}}} \neq \emptyset$.
(6) *For* $U \subseteq N$, *the restriction of* f *to* U *defined by*

$$f_U(y) \;=\; f(y, \mathbf{0}_{N \setminus U}) \qquad (y \in \mathbb{Z}^U) \tag{6.3}$$

is $M^\natural$*-concave, provided* $\mathrm{dom}\, f_U \neq \emptyset$, *where* $\mathbf{0}_{N\setminus U}$ *means the zero vector in* $\mathbb{Z}^{N\setminus U}$.
(7) *For* $U \subseteq N$, *the projection of* f *to* U *defined by*

$$f^U(y) \;=\; \sup\{f(y,z) \mid z \in \mathbb{Z}^{N\setminus U}\} \qquad (y \in \mathbb{Z}^U) \tag{6.4}$$

is $M^\natural$*-concave, provided* $f^U < +\infty$.
(8) *For* $U \subseteq N$, *the function* $\tilde{f}$ *defined at* $y \in \mathbb{Z}^U$ *and* $w \in \mathbb{Z}$ *by*

$$\tilde{f}(y,w) \;=\; \sup\{f(y,z) \mid z(N \setminus U) = w, z \in \mathbb{Z}^{N\setminus U}\} \tag{6.5}$$

is $M^\natural$*-concave, provided* $\tilde{f} < +\infty$.
(9) *Integer (supremal) convolution* $f_1 \Box f_2 : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty, +\infty\}$ *defined at* $x \in \mathbb{Z}^N$ *by*

$$(f_1 \Box f_2)(x) = \sup\{f_1(x_1) + f_2(x_2) \mid x = x_1 + x_2, x_1, x_2 \in \mathbb{Z}^N\} \tag{6.6}$$

is $M^\natural$*-concave, provided* $(f_1 \Box f_2) < +\infty$.

Proof. See Murota (2003, Theorem 6.15) for the proofs of (1) to (8). In view of the importance of convolution operations we give a straightforward alternative proof of (9) in Remark 6.2. □

Remark 6.1. Theorem 6.1 (9) for M$^\natural$-concavity of convolutions has an implication of great economic significance. Suppose that $U_1, U_2, \ldots, U_k$ represent utility functions. Then the aggregated utility is given by their convolution $U_1 \Box U_2 \Box \cdots \Box U_k$. Theorem 6.1 (9) means that substitutability is preserved in this aggregation operation. ∎

Remark 6.2. A proof for M$^\natural$-concavity of the convolution (6.6) is given here.[27] Let f_1 and f_2 be M$^\natural$-concave functions, and $f = f_1 \Box f_2$. First we treat the case where $\mathrm{dom}\, f_1$ and $\mathrm{dom}\, f_2$ are bounded. Then $\mathrm{dom}\, f = \mathrm{dom}\, f_1 + \mathrm{dom}\, f_2$ (Minkowski sum) is bounded. For each $p \in \mathbb{R}^N$ we have $f[-p] = (f_1[-p]) \Box (f_2[-p])$, from which follows

$$\arg\max(f[-p]) = \arg\max(f_1[-p]) + \arg\max(f_2[-p]).$$

In this expression, both $\arg\max(f_1[-p])$ and $\arg\max(f_2[-p])$ are M$^\natural$-convex sets by Theorem 4.5 (only if part), and therefore, their Minkowski sum (the right-hand side) is M$^\natural$-convex (Murota, 2003, Theorem 4.23). This means that $\arg\max(f[-p])$ is M$^\natural$-convex for each $p \in \mathbb{R}^N$, which implies the M$^\natural$-concavity of f by Theorem 4.5 (if part).

The general case without the boundedness assumption on effective domains can be treated via limiting procedure as follows. For $l = 1,2$ and $k = 1,2,\ldots$, define $f_l^{(k)} : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ by

$$f_l^{(k)}(x) = \begin{cases} f_l(x) & \text{if } \|x\|_\infty \le k \\ -\infty & \text{otherwise,} \end{cases}$$

which is an M$^\natural$-concave function with a bounded effective domain, provided that k is large enough to ensure $\mathrm{dom}\, f_l^{(k)} \neq \emptyset$. For each k, the convolution $f^{(k)} = f_1^{(k)} \Box f_2^{(k)}$ is M$^\natural$-concave by the above argument, and moreover, $\lim_{k\to\infty} f^{(k)}(x) = f(x)$ for each x. It remains to demonstrate the property (M$^\natural$-EXC[$\mathbb{Z}$]) for f. Take $x, y \in \mathrm{dom}\, f$ and $i \in \mathrm{supp}^+(x-y)$. There exists $k_0 = k_0(x,y)$, depending on x and y, such that $x, y \in \mathrm{dom}\, f^{(k)}$ for every $k \ge k_0$. Since $f^{(k)}$ is M$^\natural$-concave, there exists $j_k \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that

$$f^{(k)}(x) + f^{(k)}(y) \le f^{(k)}(x - \chi_i + \chi_{j_k}) + f^{(k)}(y + \chi_i - \chi_{j_k}).$$

[27] This proof is an adaptation of the proof (Murota, 2004a) for M-convex functions to M$^\natural$-concave functions. See Murota (2003, Note 9.30) for another proof using a network transformation.

Since $\mathrm{supp}^-(x-y)\cup\{0\}$ is a finite set, at least one element of $\mathrm{supp}^-(x-y)\cup\{0\}$ appears infinitely many times in the sequence $\{j_k\}$. More precisely, there exists $j\in\mathrm{supp}^-(x-y)\cup\{0\}$ and an increasing subsequence $k(1)<k(2)<\cdots$ such that $j_{k(t)}=j$ for $t=1,2,\ldots$. By letting $k\to\infty$ along this subsequence in the above inequality we obtain

$$f(x)+f(y)\le f(x-\chi_i+\chi_j)+f(y+\chi_i-\chi_j).$$

Thus $f=f_1\Box f_2$ satisfies (M$^\natural$-EXC[$\mathbb{Z}$]), which proves Theorem 6.1 (9). ∎

Remark 6.3. A sum of M$^\natural$-concave functions is not necessarily M$^\natural$-concave. This implies, in particular, that an M$^\natural$-concave function does not necessarily remain M$^\natural$-concave when its effective domain is restricted to an M$^\natural$-convex set. For example,[28] let $S_1=S_0\cup\{(0,1,1)\}$ and $S_2=S_0\cup\{(1,1,0)\}$ with $S_0=\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,0,1)\}$, and let $f_i:\mathbb{Z}^3\to\mathbb{R}\cup\{-\infty\}$ be the (concave) indicator function[29] of S_i for $i=1,2$. Then f_1+f_2 is the indicator function of $S_1\cap S_2=S_0$. Here S_1 and S_2 are M$^\natural$-convex sets, whereas S_0 is not.[30] Accordingly, f_1 and f_2 are M$^\natural$-concave functions, but their sum f_1+f_2 is not M$^\natural$-concave. Functions represented as a sum of two M$^\natural$-concave functions are an intriguing mathematical object, investigated under the name of *$M_2^\natural$-concave function* in Murota (2003, Section 8.3). ∎

Remark 6.4. For a function $f:\mathbb{Z}^n\to\mathbb{R}\cup\{-\infty\}$ and a positive integer α, the function $f^\alpha:\mathbb{Z}^n\to\mathbb{R}\cup\{-\infty\}$ defined by $f^\alpha(x)=f(\alpha x)$ $(x\in\mathbb{Z}^n)$ is called a *domain scaling* of f. If $\alpha=2$, for instance, this amounts to considering the function values only on vectors of even integers. Scaling is one of the common techniques used in designing efficient algorithms and this is particularly true of network flow algorithms. Unfortunately, M$^\natural$-concavity is not preserved under scaling. For example,[31] let f be the indicator function of a set $S=\{c_1(1,0,-1)+c_2(1,0,0)+c_3(0,1,-1)+c_4(0,1,0)\mid c_i\in\{0,1\}\}\subseteq\mathbb{Z}^3$. This f is an M$^\natural$-concave function, but f^2 (=f^α with $\alpha=2$), being the indicator function of $\{(0,0,0),(1,1,-1)\}$, is not M$^\natural$-concave. Nevertheless, scaling of an M$^\natural$-concave function is useful in designing efficient algorithms (Murota,

[28] This example is a reformulation of Murota (2003, Note 4.25) for M-convex functions to M$^\natural$-concave functions.

[29] $f_i(x)=0$ for $x\in S_i$ and $=-\infty$ for $x\notin S_i$.

[30] (B$^\natural$-EXC[$\mathbb{Z}$]) fails for S_0 with $x=(1,0,1)$, $y=(0,1,0)$, and $i=1$.

[31] This example is a reformulation of Murota (2003, Note 6.18) for M-convex functions to M$^\natural$-concave functions.

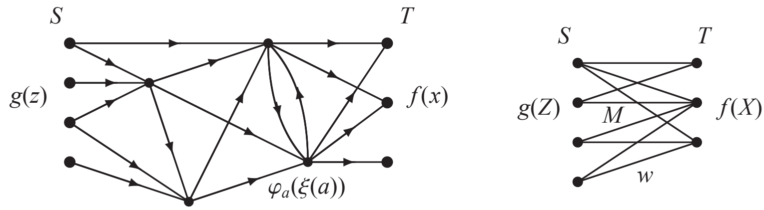

Figure 3: Transformation by a network and a bipartite graph

2003, Section 10.1). It is worth mentioning that some subclasses of M$^\natural$-concave functions are closed under scaling operation; linear, quadratic, separable, and laminar M$^\natural$-concave functions, respectively, form such subclasses. ■

Remark 6.5. A class of set functions, named matroid-based valuations, is defined by Ostrovsky & Paes Leme (2015) with the use of the convolution operation as well as the contraction operation. For set functions $f_1, f_2 : 2^N \to \mathbb{R} \cup \{-\infty\}$, the convolution of f_1 and f_2 is defined by $(f_1 \Box f_2)(X) = \max_{Y \subseteq X}(f_1(Y) + f_2(X \setminus Y))$ for $X \subseteq N$. For a set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ and a subset T of N, the *contraction* of T is defined as $f_T(X) = f(X \cup T) - f(T)$ for $X \subseteq N \setminus T$. A set function f is said to be a *matroid-based valuation*, if it can be constructed by repeated application of convolution and contraction to weighted matroid valuations (3.25). By Theorem 6.1, matroid-based valuations are M$^\natural$-concave functions. It is conjectured in Ostrovsky & Paes Leme (2015) that every M$^\natural$-concave function is a matroid-based valuation. ■

6.2. Transformation by networks

M$^\natural$-concave functions can be transformed through networks. Let $G = (V, A)$ be a directed graph with two disjoint vertex subsets $S \subseteq V$ and $T \subseteq V$ specified as the entrance and the exit (Fig. 3, left). Suppose that, for each arc $a \in A$, we are given a univariate concave function $\varphi_a : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ representing the weight of flow on the arc a. Let $\xi \in \mathbb{Z}^A$ be a vector representing a flow, and $\partial\xi \in \mathbb{Z}^V$ be the boundary of flow ξ defined by (4.36).

Given a function $g : \mathbb{Z}^S \to \mathbb{R} \cup \{-\infty\}$ on the entrance set S, we define a function $f : \mathbb{Z}^T \to \mathbb{R} \cup \{-\infty, +\infty\}$ on the exit set T by

$$f(x) = \sup_{\xi, z}\{g(z) + \sum_{a \in A} \varphi_a(\xi(a)) \mid \xi \in \mathbb{Z}^A, \partial\xi = (z, -x, \mathbf{0}) \in \mathbb{Z}^S \times \mathbb{Z}^T \times \mathbb{Z}^{V \setminus (S \cup T)}\}. \tag{6.7}$$

This function $f(x)$ represents the maximum weight to meet the demand specification x at the exit, subject to the flow conservation at the vertices not in $S \cup T$. The weight consists of two parts, the weight $g(z)$ of supply z at the entrance S and the weight $\sum_{a \in A} \varphi_a(\xi(a))$ in the arcs.

We can regard (6.7) as a transformation of g to f by the network. If the given function g is M$^\natural$-concave, the resultant function f is also M$^\natural$-concave, provided that f does not take the value $+\infty$ and $\mathrm{dom}\, f$ is nonempty. In other words, the transformation (6.7) by a network preserves M$^\natural$-concavity. See Murota (2003, Section 9.6) for a proof. An alternative proof is given by Kobayashi et al. (2007).

In particular, an M$^\natural$-concave set function is transformed to another M$^\natural$-concave set function through a bipartite graph (Fig. 3, right). Let $G = (S, T; E)$ be a bipartite graph with vertex bipartition (S, T) and edge set E, with weight $w_e \in \mathbb{R}$ associated with each edge $e \in E$. Given an M$^\natural$-concave set function $g : 2^S \to \mathbb{R} \cup \{-\infty\}$ on S, define a set function f on T by

$$f(X) = \max\{g(Z) + w(M) \mid \begin{array}{l} M \text{ is a matching,} \\ S \cap \partial M = Z, T \cap \partial M = X\} \end{array} \tag{6.8}$$

where $f(X) = -\infty$ if no such M exists for X. If g is M$^\natural$-concave, then f is also M$^\natural$-concave, as long as $\mathrm{dom}\, f$ is nonempty. A proof tailored to set functions is given in the proof of Murota (2000b, Theorem 5.2.18).

6.3. Concluding remarks of section 6

Efficient algorithms are available for the operations listed in Theorem 6.1. In particular, the convolution (6.6), corresponding to the aggregation of utility functions, can be computed efficiently (Murota & Tamura, 2003a). The transformation by networks is also accompanied by efficient algorithms. For M$^\natural$-concave function maximization algorithms, see Murota (2003, Chapter 10), and more recent papers, e.g., Shioura (2004), Tamura (2005), Murota (2010), Moriguchi et al. (2011), Fujishige et al. (2015), and Shioura (2015).

7. CONJUGACY AND L$^\natural$-CONVEXITY

Conjugacy under the Legendre transformation is one of the most appealing facts in convex analysis. This is also the case in discrete convex analysis. The conjugacy theorem in discrete convex analysis says that the Legendre transformation gives a one-to-one correspondence between M$^\natural$-concave functions and L$^\natural$-convex functions. Since M$^\natural$-concavity expresses substitutability of valuation or utility functions, L$^\natural$-convexity characterizes substitutability in terms of indirect utility functions. This fact has a significant application to auction theory, to be expounded in Section 8.

7.1. L$^\natural$-convex function

The concept of L$^\natural$-convexity is defined for functions in discrete (integer) variables and for those in continuous (real) variables. We start with discrete variables.

7.1.1. L$^\natural$-convex function on $\mathbb{Z}^n$

First recall that a function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called *submodular* if

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (p, q \in \mathbb{Z}^n), \tag{7.1}$$

where $p \vee q$ and $p \wedge q$ mean the vectors of componentwise maximum and minimum of p and q, respectively. To define L$^\natural$-convexity of g, we consider a function $\tilde{g}$ in $n+1$ variables $(p_0, p) = (p_0, p_1, \ldots, p_n)$ defined as

$$\tilde{g}(p_0, p) = g(p - p_0\mathbf{1}) \qquad (p_0 \in \mathbb{Z}, p \in \mathbb{Z}^n), \tag{7.2}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$. Then we say that $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is *L$^\natural$-convex* if the associated function $\tilde{g} : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ is a submodular function in (p_0, p), i.e., if for all $p_0, q_0 \in \mathbb{Z}$ and for all $p, q \in \mathbb{Z}^n$ it holds

$$\begin{aligned} g(p - p_0\mathbf{1}) + g(q - q_0\mathbf{1}) \geq \; & g((p \vee q) - (p_0 \vee q_0)\mathbf{1}) \\ & + g((p \wedge q) - (p_0 \wedge q_0)\mathbf{1}). \end{aligned} \tag{7.3}$$

Remark 7.1. The significance of the extra variable p_0 in the definition of L$^\natural$-convexity is most transparent when $n = 1$. When $n = 1$ we have $(p \vee q, p \wedge q) = (p, q)$ or (q, p), according to whether $p \geq q$ or $p \leq q$. Hence the submodular inequality (7.1) is always satisfied, and every function $g : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ is submodular. On the other hand, the inequality (7.3) for $(p_0, p) = (1, t)$ and $(q_0, q) = (0, t+1)$ yields $g(t-1) + g(t+1) \geq 2g(t)$ for $t \in \mathbb{Z}$, which shows the convexity of g on $\mathbb{Z}$. The converse is also true. Therefore, a function $g : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ is L$^\natural$-convex if and only if $g(t-1) + g(t+1) \geq 2g(t)$ for all $t \in \mathbb{Z}$. ∎

Remark 7.2. For a set function $\mu : 2^N \to \mathbb{R} \cup \{+\infty\}$, L$^\natural$-convexity is equivalent to submodularity (3.8). Recall the notation χ_X for the characteristic vector of a subset X; see (2.1). A set function μ can be identified with a function $g : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, g \subseteq \{0,1\}^N$ by $\mu(X) = g(\chi_X)$ for $X \subseteq N$, and μ is submodular if and only if the corresponding g is L$^\natural$-convex. ∎

Remark 7.3. Matroid rank functions have a dual character of being both L$^\natural$-convex and M$^\natural$-concave. It is L$^\natural$-convex as it is submodular, and M$^\natural$-concave as already mentioned in Section 3.6. ∎

L$^\natural$-convexity can be characterized by a number of equivalent conditions (Favati & Tardella, 1990; Fujishige & Murota, 2000; Murota, 2003).

Theorem 7.1. *For $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ the following conditions,* (a) *to* (d)*, are equivalent:*

(a) *L$^\natural$-convexity, i.e.,* (7.3).

(b) *Translation-submodularity:*[32] *for all $p, q \in \mathbb{Z}^n$ and for all $\alpha \in \mathbb{Z}_+$*

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})). \tag{7.4}$$

(c) *Discrete midpoint convexity: for all $p, q \in \mathbb{Z}^n$*

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) \tag{7.5}$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the integer vectors obtained by componentwise rounding-up and rounding-down to the nearest integers, respectively.

(d) *For any $p, q \in \mathbb{Z}^n$ with $\mathrm{supp}^+(p - q) \neq \emptyset$, it holds that*[33]

$$g(p) + g(q) \geq g(p - \chi_A) + g(q + \chi_A), \tag{7.6}$$

where $A = \arg\max_i \{p_i - q_i\}$.

It is known (Murota, 2003, Theorem 7.20) that an L$^\natural$-convex function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is *convex-extensible*, i.e., there exists a convex function $\overline{g} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $\overline{g}(p) = g(p)$ for all $p \in \mathbb{Z}^n$. Moreover, the convex extension $\overline{g}$ can be constructed by a simple procedure; see Murota (2003, Theorem 7.19).

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[32] This condition is labeled as (SBF$^\natural[\mathbb{Z}]$) in Murota (2003, Section 7.1). Note that α is restricted to be nonnegative, and the inequality (7.4) for $\alpha = 0$ coincides with submodularity (7.1).

[33] This condition is labeled as (L$^\natural$-APR[$\mathbb{Z}$]) in Murota (2003, Section 7.2). Recall the notation χ_A for the characteristic vector of A, as defined in (2.1).

Remark 7.4. A nonempty set $P \subseteq \mathbb{Z}^n$ is called an *L$^\natural$-convex set* if its indicator function[34] is an L$^\natural$-convex function. In other words, $P \neq \emptyset$ is an L$^\natural$-convex set if it satisfies one of the following equivalent conditions, where $p, q \in \mathbb{Z}^n$ and $p_0, q_0 \in \mathbb{Z}$:

(a) $p - p_0\mathbf{1},\ q - q_0\mathbf{1}\ \in P \implies (p \vee q) - (p_0 \vee q_0)\mathbf{1},\ (p \wedge q) - (p_0 \wedge q_0)\mathbf{1}\ \in P$.

(b) $p, q \in P,\ \alpha \in \mathbb{Z}_+ \implies (p - \alpha\mathbf{1}) \vee q,\ p \wedge (q + \alpha\mathbf{1})\ \in P$.

(c) $p, q \in P \implies \left\lceil \frac{p+q}{2} \right\rceil,\ \left\lfloor \frac{p+q}{2} \right\rfloor\ \in P$.

(d) $p, q \in P,\quad \mathrm{supp}^+(p - q) \neq \emptyset \implies p - \chi_A,\ q + \chi_A\ \in P$ with $A = \arg\max_i \{p_i - q_i\}$.

For an L$^\natural$-convex function g, the effective domain $\mathrm{dom}\, g$ and the set of minimizers $\arg\min g$ are L$^\natural$-convex sets. See Murota (2003, Section 5.5) for more about L$^\natural$-convex sets. ∎

Remark 7.5. A function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called an *L-convex function* if it is an L$^\natural$-convex function such that there exists $r \in \mathbb{R}$ for which $g(p + \mathbf{1}) = g(p) + r$ for all $p \in \mathbb{Z}^n$. L-convex functions and L$^\natural$-convex functions are equivalent concepts, in that L$^\natural$-convex functions in n variables can be identified, up to the constant r, with L-convex functions in $n+1$ variables. Indeed, a function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is L$^\natural$-convex if and only if the function $\tilde{g} : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ in (7.2) is an L-convex function (with $r = 0$). ∎

7.1.2. L$^\natural$-convex function on $\mathbb{R}^n$

We turn to continuous variables. A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *L$^\natural$-convex* if it is a convex function (in the ordinary sense) such that $\tilde{g}(p_0, p) = g(p - p_0\mathbf{1})$ $(p_0 \in \mathbb{R}, p \in \mathbb{R}^n)$ is a submodular function in $n+1$ variables, i.e., for all $p_0, q_0 \in \mathbb{R}$ and for all $p, q \in \mathbb{R}^n$

$$\begin{aligned} g(p - p_0\mathbf{1}) + g(q - q_0\mathbf{1}) \geq\ & g((p \vee q) - (p_0 \vee q_0)\mathbf{1}) \\ & + g((p \wedge q) - (p_0 \wedge q_0)\mathbf{1}). \end{aligned} \tag{7.7}$$

[34] $g(p) = 0$ for $p \in P$ and $= +\infty$ for $p \notin P$.

In the following we restrict ourselves to closed proper $\mathrm{L}^\natural$-convex functions,[35] for which the closure of the effective domain $\mathrm{dom}\, g$ is a well-behaved polyhedron ($\mathrm{L}^\natural$-convex polyhedron[36]); see Murota & Shioura (2008, Theorem 3.3). For a closed proper convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the condition (7.7) for $\mathrm{L}^\natural$-convexity is equivalent to translation-submodularity: for all $p, q \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{R}_+$

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})). \tag{7.8}$$

Often we are interested in polyhedral $\mathrm{L}^\natural$-convex functions.

$\mathrm{L}^\natural$-convex functions in real variables are investigated by Murota & Shioura (2000, 2004a,b, 2008).

7.2. Conjugacy

7.2.1. *Functions in continuous variables*

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (not necessarily convex) with $\mathrm{dom}\, f \neq \emptyset$, the *convex conjugate* $f^\bullet : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{R}^n\} \qquad (p \in \mathbb{R}^n), \tag{7.9}$$

where $\langle p, x\rangle = \sum_{i=1}^n p_i x_i$ is the inner product of $p = (p_i) \in \mathbb{R}^n$ and $x = (x_i) \in \mathbb{R}^n$. The function $f^\bullet$ is also referred to as the (convex) *Legendre(–Fenchel) transform* of f, and the mapping $f \mapsto f^\bullet$ as the (convex) *Legendre(–Fenchel) transformation*. A fundamental theorem in convex analysis states that the Legendre transformation gives a symmetric one-to-one correspondence in the class of all closed proper convex functions. That is, for a closed proper convex function f, the conjugate function $f^\bullet$ is a closed proper convex function and the *biconjugacy* $(f^\bullet)^\bullet = f$ holds.

To formulate the correspondence between concave functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ and convex functions $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ and $\mathrm{dom}\, g \neq \emptyset$, we introduce the following variants of the transformation (7.9):

$$f^\triangledown(p) = \sup\{f(x) - \langle p, x\rangle \mid x \in \mathbb{R}^n\} \qquad (p \in \mathbb{R}^n), \tag{7.10}$$

$$g^\vartriangle(x) = \inf\{g(p) + \langle p, x\rangle \mid p \in \mathbb{R}^n\} \qquad (x \in \mathbb{R}^n), \tag{7.11}$$

[35] A convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *proper* if $\mathrm{dom}\, g$ is nonempty, and *closed* if the epigraph $\{(p, \alpha) \in \mathbb{R}^{n+1} \mid \alpha \geq g(p)\}$ is a closed subset of $\mathbb{R}^{n+1}$.

[36] A polyhedron is called an *$L^\natural$-convex polyhedron* if its (convex) indicator function is $\mathrm{L}^\natural$-convex. See Murota (2003, Section 5.6) for details.

where $f^{\triangledown} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g^{\triangle} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$. The biconjugacy is expressed as $(f^{\triangledown})^{\triangle} = f$, $(g^{\triangle})^{\triangledown} = g$ for closed proper concave functions f and closed proper convex functions g.

Theorem 7.2.
(1) *The transformations* (7.10) *and* (7.11) *give a one-to-one correspondence between the classes of all closed proper concave functions f and closed proper convex functions g.*
(2) *For a closed proper concave function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, the conjugate function $f^{\triangledown} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a closed proper convex function and $(f^{\triangledown})^{\triangle} = f$.*
(3) *For a closed proper convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the conjugate function $g^{\triangle} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is a closed proper concave function and $(g^{\triangle})^{\triangledown} = g$.*

Addition of combinatorial ingredients to the above theorem yields the conjugacy theorem between M$^\natural$-concave and L$^\natural$-convex functions (Murota & Shioura, 2004a).

Theorem 7.3.
(1) *The transformations* (7.10) *and* (7.11) *give a one-to-one correspondence between the classes of all closed proper M$^\natural$-concave functions f and closed proper L$^\natural$-convex functions g.*
(2) *For a closed proper M$^\natural$-concave function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, the conjugate function $f^{\triangledown} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a closed proper L$^\natural$-convex function and $(f^{\triangledown})^{\triangle} = f$.*
(3) *For a closed proper L$^\natural$-convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the conjugate function $g^{\triangle} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is a closed proper M$^\natural$-concave function and $(g^{\triangle})^{\triangledown} = g$.*

The M$^\natural$/L$^\natural$-conjugacy is also valid for polyhedral concave/convex functions; see Murota & Shioura (2000) and Murota (2003, Theorem 8.4).

Theorem 7.4.
(1) *The transformations* (7.10) *and* (7.11) *give a one-to-one correspondence between the classes of all polyhedral M$^\natural$-concave functions f and polyhedral L$^\natural$-convex functions g.*
(2) *For a polyhedral M$^\natural$-concave function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, the conjugate function $f^{\triangledown} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a polyhedral L$^\natural$-convex function and $(f^{\triangledown})^{\triangle} = f$.*

(3) *For a polyhedral* $\mathrm{L}^\natural$*-convex function* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$*, the conjugate function* $g^\triangle : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ *is a polyhedral* $\mathrm{M}^\natural$*-concave function and* $(g^\triangle)^\triangledown = g$.

As corollaries of the conjugacy theorems, the following characterizations of $\mathrm{M}^\natural$-concavity and $\mathrm{L}^\natural$-convexity in terms of the conjugate functions are obtained.

Theorem 7.5.
(1) *A function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ *is closed proper* $\mathrm{M}^\natural$*-concave if and only if the conjugate function* $f^\triangledown : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *by* (7.10) *is closed proper* $\mathrm{L}^\natural$*-convex.*
(2) *A function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ *is polyhedral* $\mathrm{M}^\natural$*-concave if and only if the conjugate function* $f^\triangledown : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *by* (7.10) *is polyhedral* $\mathrm{L}^\natural$*-convex.*

Theorem 7.6.
(1) *A function* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is closed proper* $\mathrm{L}^\natural$*-convex if and only if the conjugate function* $g^\triangle : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ *by* (7.11) *is closed proper* $\mathrm{M}^\natural$*-concave.*
(2) *A function* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is polyhedral* $\mathrm{L}^\natural$*-convex if and only if the conjugate function* $g^\triangle : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ *by* (7.11) *is polyhedral* $\mathrm{M}^\natural$*-concave.*

$\mathrm{L}^\natural$-convexity, being equivalent to translation-submodularity, is a stronger property than mere submodularity. When we replace $\mathrm{L}^\natural$-convexity of $f^\triangledown$ in Theorem 7.5 (2) with submodularity, we obtain a larger class of polyhedral concave functions f than $\mathrm{M}^\natural$-concave functions. The following theorem is ascribed to Danilov & Lang (2001) in Danilov et al. (2003); see also Shioura & Tamura (2015, Appendix) for technical supplements.

Theorem 7.7. *Let* $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ *be a polyhedral concave function with a bounded effective domain. Then the following conditions are equivalent:*[37]
(a) f *satisfies* (GS[$\mathbb{R}$]).
(b) *For every* $p \in \mathbb{R}^N$*, each edge (one-dimensional face) of* $D(p; f)$ *is parallel to a vector* d *with* $|\mathrm{supp}^+(d)| \leq 1$ *and* $|\mathrm{supp}^-(d)| \leq 1$.
(c) $f^\triangledown : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ *by* (7.10) *is a submodular function.*

[37] Recall the definition of (GS[$\mathbb{R}$]) from Section 5.2. Also recall from Theorem 5.2 that polyhedral $\mathrm{M}^\natural$-concave functions satisfy (GS[$\mathbb{R}$]).

Remark 7.6. In Danilov et al. (2003) a bounded polyhedron P is called a *quasi-polymatroid* if each edge (one-dimensional face) is parallel to a vector d with $|\mathrm{supp}^+(d)| \leq 1$ and $|\mathrm{supp}^-(d)| \leq 1$. It follows from Fujishige et al. (2004, Theorem 3.1) that every face of a quasi-polymatroid whose normal vector has the full support N is obtained from an M-convex polyhedron (base polyhedron) by a scaling along axes. We mention in passing that a pointed convex polyhedron is called *polybasic* if each edge is parallel to a vector d with $|\mathrm{supp}^+(d)| + |\mathrm{supp}^-(d)| \leq 2$ (Fujishige et al., 2004). ∎

Remark 7.7. In the canonical situation, where $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly concave smooth function, the equivalence between (GS[$\mathbb{R}$]) of f and the submodularity of $g = f^{\triangledown}$ is easily derived by simple calculus. Let $x(p)$ be the unique maximizer of $f(x) - \langle p, x\rangle$. We have $p_i = \partial f / \partial x_i$ for $i = 1, \ldots, n$, and $g(p) = f(x(p)) - \langle p, x(p)\rangle$. This implies $\partial g / \partial p_i = -x_i$ $(i = 1, \ldots, n)$, and hence $\partial^2 g / \partial p_i \partial p_j = -\partial x_i / \partial p_j$ $(i, j = 1, \ldots, n)$. On the other hand, the submodularity of g is equivalent to $\partial^2 g / \partial p_i \partial p_j \leq 0$ $(i \neq j)$, and (GS[$\mathbb{R}$]) of f is represented as $\partial x_i / \partial p_j \geq 0$ $(i \neq j)$. ∎

7.2.2. *Functions in discrete variables*

We turn to functions defined on integer vectors. For functions $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ and $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ and $\mathrm{dom}\, g \neq \emptyset$, the transformations (7.10) and (7.11) are modified to

$$f^{\triangledown}(p) = \sup\{f(x) - \langle p, x\rangle \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{R}^n), \tag{7.12}$$

$$g^{\vartriangle}(x) = \inf\{g(p) + \langle p, x\rangle \mid p \in \mathbb{Z}^n\} \qquad (x \in \mathbb{R}^n), \tag{7.13}$$

where $f^{\triangledown} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g^{\vartriangle} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$.

The conjugacy between M$^\natural$-concavity and L$^\natural$-convexity in this case reads as follows.[38]

Theorem 7.8.
(1) *For an M$^\natural$-concave function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$, the conjugate function $f^{\triangledown} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a (locally polyhedral) L$^\natural$-convex function, and $(f^{\triangledown})^{\vartriangle}(x) = f(x)$ for $x \in \mathbb{Z}^n$.*

[38] In Theorem 7.8 (1), $^{\triangledown}$ is defined by (7.12) and $^{\vartriangle}$ by (7.11). In (2), $^{\vartriangle}$ is defined by (7.13) and $^{\triangledown}$ by (7.10).

(2) *For an* $L^\natural$*-convex function* $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$*, the conjugate function* $g^\triangle : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ *is a (locally polyhedral)* $M^\natural$*-concave function, and* $(g^\triangle)^\triangledown(p) = g(p)$ *for* $p \in \mathbb{Z}^n$.

For integer-valued functions f and g, $f^\triangledown(p)$ and $g^\triangle(x)$ are integers for integer vectors p and x. Hence (7.12) with $p \in \mathbb{Z}^n$ and (7.13) with $x \in \mathbb{Z}^n$, i.e.,

$$f^\triangledown(p) = \sup\{f(x) - \langle p,x\rangle \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{Z}^n), \tag{7.14}$$

$$g^\triangle(x) = \inf\{g(p) + \langle p,x\rangle \mid p \in \mathbb{Z}^n\} \qquad (x \in \mathbb{Z}^n), \tag{7.15}$$

define transformations of $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$ to $f^\triangledown : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ and $g : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ to $g^\triangle : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$, respectively.

The conjugacy theorem for integer-valued discrete-variable $\mathrm{M}^\natural$-concave and $\mathrm{L}^\natural$-convex functions reads as follows; see Murota (1998) and Murota (2003, Theorem 8.12).

Theorem 7.9.
(1) *The transformations* (7.14) *and* (7.15) *give a one-to-one correspondence between the classes of all integer-valued* $M^\natural$*-concave functions* f *and integer-valued* $L^\natural$*-convex functions* g.
(2) *For an integer-valued* $M^\natural$*-concave function* $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$*, the conjugate function* $f^\triangledown : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ *is an integer-valued* $L^\natural$*-convex function and* $(f^\triangledown)^\triangle = f$.
(3) *For an integer-valued* $L^\natural$*-convex function* $g : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$*, the conjugate function* $g^\triangle : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$ *is an integer-valued* $M^\natural$*-concave function and* $(g^\triangle)^\triangledown = g$.

As corollaries of the conjugacy theorems, the following characterizations of $\mathrm{M}^\natural$-concavity and $\mathrm{L}^\natural$-convexity in terms of the conjugate functions are obtained.

Theorem 7.10.
(1) *A function* $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ *is* $M^\natural$*-concave if and only if the conjugate function* $f^\triangledown : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *by* (7.12) *is (locally polyhedral)* $L^\natural$*-convex.*
(2) *A function* $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$ *is* $M^\natural$*-concave if and only if the conjugate function* $f^\triangledown : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ *by* (7.14) *is* $L^\natural$*-convex.*

Theorem 7.11.
(1) *A function* $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ *is* $L^\natural$*-convex if and only if the conjugate function* $g^\triangle : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ *by* (7.13) *is (locally polyhedral)* $M^\natural$*-concave.*

(2) *A function* $g : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ *is* $L^\natural$*-convex if and only if the conjugate function* $g^\triangle : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$ *by* (7.15) *is* $M^\natural$*-concave.*

$\mathrm{L}^\natural$-convexity, being equivalent to translation-submodularity, is a stronger property than mere submodularity. Naturally, we may wonder if $\mathrm{L}^\natural$-convexity of $f^\triangledown$ in Theorem 7.10 can be replaced by submodularity. However, the following example denies this possibility.[39]

Example 7.1. Here is an example of a function f such that the conjugate function $f^\triangledown$ is submodular, but f is not $\mathrm{M}^\natural$-concave. Let $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$ be defined by $f(x_1,x_2) = \min(2, x_1 + 2x_2)$ on $\mathrm{dom}\, f = \{(x_1,x_2) \in \mathbb{Z}^2 \mid 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 1\}$, whose numerical values are

$$f(0,0) = 0,\ f(1,0) = 1,\ f(2,0) = 2; \quad f(0,1) = f(1,1) = f(2,1) = 2.$$

This function is not $\mathrm{M}^\natural$-concave because ($\mathrm{M}^\natural$-EXC[$\mathbb{Z}$]) fails for $x = (2,0)$, $y = (0,1)$ and $i = 1$. The conjugate function $f^\triangledown : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ of (7.12) is given by

$$\begin{aligned} f^\triangledown(p_1,p_2) &= \max(0, 2-2p_1, 2-p_2, 2-2p_1-p_2) \\ &= \begin{cases} 0 & (p_1 \geq 1, p_2 \geq 2), \\ 2-2p_1 & (2p_1 \leq \min(2,p_2), p_2 \geq 0), \\ 2-p_2 & (p_2 \leq \min(2,2p_1), p_1 \geq 0), \\ 2-2p_1-p_2 & (p_1 \leq 0, p_2 \leq 0). \end{cases} \end{aligned}$$

The function $f^\triangledown$ is submodular, as is easily verified, but it is not $\mathrm{L}^\natural$-convex since the translation-submodularity (7.8) fails for $g = f^\triangledown$, $p = (1,2)$, $q = (0,0)$ and $\alpha = 1$ with $g(p) + g(q) = 0 + 2 = 2$ and $g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})) = g(0,1) + g(1,1) = 2 + 1 = 3$. It is also noted that $f^\triangledown(p_1/2, p_2)$ is $\mathrm{L}^\natural$-convex in (p_1,p_2). ∎

In spite of the above example, $\mathrm{M}^\natural$-concavity of a set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ can be characterized by submodularity of the conjugate function $f^\triangledown$, which is defined by

$$f^\triangledown(p) = \max\{f(X) - p(X) \mid X \subseteq N\} \qquad (p \in \mathbb{R}^n) \tag{7.16}$$

as an adaptation of (7.12).

[39] Shioura & Tamura (2015, Example 7.4) also shows this. See Theorem 7.7 for the continuous case.

Theorem 7.12. *A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is M$^\natural$-concave if and only if the conjugate function $f^\triangledown : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ in (7.16) is submodular.*

This theorem can be derived from a combination of Ausubel & Milgrom (2002, Theorem 10) with Theorem 3.7 in Section 3.3; see also Shioura & Tamura (2015, Section 7.2.2) for an alternative proof.

7.3. Minimization of L$^\natural$-convex functions

The set of the minimizers of an L$^\natural$-convex function on $\mathbb{Z}^n$ forms a well-behaved "discrete convex" subset of $\mathbb{Z}^n$. Recall from Remark 7.4 that a nonempty set $P \subseteq \mathbb{Z}^n$ is called an L$^\natural$-convex set if

$$p, q \in P \implies (p - \alpha \mathbf{1}) \vee q,\ p \wedge (q + \alpha \mathbf{1}) \in P \qquad (\forall \alpha \in \mathbb{Z}_+). \tag{7.17}$$

This condition with $\alpha = 0$ gives

$$p, q \in P \implies p \vee q,\ p \wedge q \in P, \tag{7.18}$$

which shows that an L$^\natural$-convex set forms a sublattice of $\mathbb{Z}^n$. A bounded L$^\natural$-convex set has the (uniquely determined) maximal element and the (uniquely determined) minimal element.

Theorem 7.13. *Let $g : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ be an L$^\natural$-convex function and assume* $\arg\min g \neq \emptyset$. *Then the set of the minimizers* $\arg\min g$ *is an L$^\natural$-convex set. If* $\arg\min g$ *is bounded, there exist the maximal and the minimal minimizer of g.*

Proof. This follows easily from the translation-submodularity in Theorem 7.1 (b). □

For an L$^\natural$-convex function, the minimality of a function value is characterized by a local condition as follows (Murota, 2003, Theorem 7.14). Recall the notation χ_Y for the characteristic vector of a subset Y; see (2.1).

Theorem 7.14. *Let $g : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ be an L$^\natural$-convex function and $p \in$* $\mathrm{dom}\, g$.
(1) *If $g(p) > g(q)$ for $q \in \mathrm{dom}\, g$, then $g(p) > g(p + \chi_Y)$ for some $Y \subseteq \mathrm{supp}^+(q - p)$ or $g(p) > g(p - \chi_Z)$ for some $Z \subseteq \mathrm{supp}^-(q - p)$.*
(2) *p is a minimizer of g if and only if*

$$g(p) \leq g(p + \chi_Y) \quad (\forall Y \subseteq N), \qquad g(p) \leq g(p - \chi_Z) \quad (\forall Z \subseteq N). \tag{7.19}$$

Proof. (1) This follows from Theorem 7.15 below. If $g(q) < g(p)$ in (7.20), $g(p+\chi_{Y_k}) - g(p) < 0$ for some k or $g(p-\chi_{Z_j}) - g(p) < 0$ for some j. (2) This is immediate from (1). □

Theorem 7.15. *Let $g : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ be an $L^\natural$-convex function. For $p, q \in$ $\mathrm{dom}\, g$ we have*

$$g(q) \geq g(p) + \sum_{k=1}^{m} [g(p+\chi_{Y_k}) - g(p)] + \sum_{j=1}^{l} [g(p-\chi_{Z_j}) - g(p)], \tag{7.20}$$

where[40] $\emptyset \neq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_m = \mathrm{supp}^+(q-p)$, $\emptyset \neq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_l =$ $\mathrm{supp}^-(q-p)$, *and*

$$q - p = \sum_{k=1}^{m} \chi_{Y_k} - \sum_{j=1}^{l} \chi_{Z_j}. \tag{7.21}$$

Proof. (1) If $\mathrm{supp}^+(q-p)$ is nonempty, (7.6) for (q,p) implies

$$g(q) \geq g(p+\chi_{Y_1}) + g(q-\chi_{Y_1}) - g(p) = [g(p+\chi_{Y_1}) - g(p)] + g(q_2),$$

where $q_2 = q - \chi_{Y_1}$. If $\mathrm{supp}^+(q_2 - p)$ is nonempty, (7.6) for (q_2, p) implies

$$g(q_2) \geq g(p+\chi_{Y_2}) + g(q_2-\chi_{Y_2}) - g(p) = [g(p+\chi_{Y_2}) - g(p)] + g(q_3),$$

where $q_3 = q_2 - \chi_{Y_2} = q - \chi_{Y_1} - \chi_{Y_2}$. Repeating this, we obtain $q' = q - \sum_{k=1}^{m} \chi_{Y_k} = p \wedge q$ and

$$g(q) \geq g(q') + \sum_{k=1}^{m} [g(p+\chi_{Y_k}) - g(p)]. \tag{7.22}$$

By the similar procedure starting with (p, q') we obtain $p = q' + \sum_{j=1}^{l} \chi_{Z_j}$ and

$$g(q') \geq g(p) + \sum_{j=1}^{l} [g(p-\chi_{Z_j}) - g(p)]. \tag{7.23}$$

Adding (7.22) and (7.23) we obtain (7.20). □

[40] The decomposition (7.21) is uniquely determined: $m = \max(0, q_1 - p_1, \ldots, q_n - p_n)$, $Y_k = \{i \mid q_i - p_i \geq m+1-k\}$ $(k = 1, \ldots, m)$; $l = \max(0, p_1 - q_1, \ldots, p_n - q_n)$, $Z_j = \{i \mid p_i - q_i \geq l+1-j\}$ $(j = 1, \ldots, l)$.

7.3.1. Algorithms for L$^\natural$-convex minimization

Algorithms for L$^\natural$-convex function minimization are considered by Murota (2000a), Kolmogorov & Shioura (2009), Murota & Shioura (2014, 2016), Murota et al. (2016), and Shioura (2017); see also Murota (2003, Section 10.3). Among others we present here the following two algorithms.[41]

Algorithm GREEDY
Step 0: Find a vector $p^\circ \in \mathrm{dom}\, g$ and set $p := p^\circ$.
Step 1: Find $\varepsilon \in \{+1, -1\}$ and $X \subseteq N$ that minimize $g(p + \varepsilon\chi_X)$.
Step 2: If $g(p) \leq g(p + \varepsilon\chi_X)$, then output p and stop.
Step 3: Set $p := p + \varepsilon\chi_X$ and go to Step 1.

Algorithm GREEDYUPMINIMAL
Step 0: Find a vector $p^\circ \in \mathrm{dom}\, g$ such that $\{q \mid q \geq p^\circ\} \cap \arg\min g \neq \emptyset$ and set $p := p^\circ$.
Step 1: Find the minimal minimizer $X \subseteq N$ of $g(p + \chi_X)$.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p + \chi_X$ and go to Step 1.

The algorithm GREEDY can start with an arbitrary initial vector p° in the effective domain, and the vector p may increase or decrease depending on $\varepsilon = +1$ or -1. The output of the algorithm GREEDY is not uniquely determined, varying with the choice of ε and X in case of ties in minimizing $g(p + \varepsilon\chi_X)$ in Step 1. Step 1 amounts to minimizing two set functions $\rho_+(X) = g(p + \chi_X) - g(p)$ and $\rho_-(X) = g(p - \chi_X) - g(p)$ over all subsets X of N. As a consequence of submodularity of g, both ρ_+ and ρ_- are submodular set functions and they can be minimized efficiently (i.e., in strongly polynomial time). The second algorithm, GREEDYUPMINIMAL, keeps increasing the vector p, until it reaches the smallest minimizer of g that is greater than or equal to p°. Accordingly, the initial vector p° must be small enough to ensure $\{q \mid q \geq p^\circ\} \cap \arg\min g \neq \emptyset$. If g has the minimal minimizer $p^*_{\min}$ and $p^\circ \leq p^*_{\min}$, then the algorithm GREEDYUPMINIMAL outputs $p^*_{\min}$.

The correctness of the algorithms, at their termination, is guaranteed by Theorem 7.14, whereas the following exact bounds for the number of updates of p are established recently in Murota & Shioura (2014).

[41] Algorithm GREEDY is called “steepest descent algorithm” in Murota (2003, Section 10.3.1).

Theorem 7.16.
(1) *The number of updates of p in the algorithm* GREEDY *is exactly equal to*

$$\mu(p^\circ) = \min\{\|p^\circ - p^*\|_\infty^+ + \|p^\circ - p^*\|_\infty^- \mid p^* \in \arg\min g\} \tag{7.24}$$

under the assumption of $\arg\min g \neq \emptyset$, *where* $\|q\|_\infty^+ = \max(0, q_1, q_2, \ldots, q_n)$ *and* $\|q\|_\infty^- = \max(0, -q_1, -q_2, \ldots, -q_n)$.
(2) *The number of updates of p in the algorithm* GREEDYUPMINIMAL *is exactly equal to*[42]

$$\hat{\mu}(p^\circ) = \min\{\|p^\circ - p^*\|_\infty \mid p^* \in \arg\min g,\ p^\circ \leq p^*\} \tag{7.25}$$

under the assumption of $\{q \mid q \geq p^\circ\} \cap \arg\min g \neq \emptyset$. *If the minimal minimizer* $p^*_{\min}$ *exists and* $p^\circ \leq p^*_{\min}$, *then* $\hat{\mu}(p^\circ) = \|p^\circ - p^*_{\min}\|_\infty$.

We can conceive variants of GREEDYUPMINIMAL by changing "UP" to "DOWN" and/or "MINIMAL" to "MAXIMAL" according to Table 1 (a). For example, the algorithm GREEDYDOWNMINIMAL is obtained from GREEDYUPMINIMAL by changing Steps 0 and 1 to:

Step 0: Find a vector $p^\circ \in \mathrm{dom}\, g$ such that $\{q \mid q \leq p^\circ\} \cap \arg\min g \neq \emptyset$ and set $p := p^\circ$.
Step 1: Find the maximal minimizer $X \subseteq N$ of $g(p - \chi_X)$.

Starting with an initial vector p° large enough to ensure $\{q \mid q \leq p^\circ\} \cap \arg\min g \neq \emptyset$, this algorithm keeps decreasing the vector p. If g has the minimal minimizer $p^*_{\min}$, the algorithm stops when it reaches $p^*_{\min}$. The number of updates of p in GREEDYDOWNMINIMAL is exactly equal to $\|p^\circ - p^*_{\min}\|_\infty$ (Murota et al., 2016, Proposition 3.7). Table 1 (b) shows the output and the number of updates of p for the four algorithms.

In Section 8 we shall discuss connection of L$^\natural$-convex function minimization to iterative auctions. The algorithm GREEDYUPMINIMAL corresponds to ascending (English) auctions, and GREEDYDOWNMAXIMAL to descending (Dutch) auctions. In connection to two-phase (English–Dutch) auctions it is natural to consider two-phase algorithms for L$^\natural$-convex function minimization.

The combination of GREEDYUPMINIMAL and GREEDYDOWNMAXIMAL results in the following algorithm:

[42] We have $\hat{\mu}(p^\circ) = +\infty$ if there is no $p^* \in \arg\min g$ with $p^* \geq p^\circ$. It can be shown that $\hat{\mu}(p^\circ) \in \{\mu(p^\circ), +\infty\}$ holds for all $p^\circ \in \mathbb{Z}^n$; see Shioura (2017) for the proof.

Table 1: Algorithms GREEDY-{UP, DOWN}-{MINIMAL, MAXIMAL}

(a) Description of the algorithms

GREEDY		MINIMAL	MAXIMAL
UP	Step 0	p° such that $\{q \mid q \geq p^\circ\} \cap \arg\min g \neq \emptyset$	(i.e., $p^*_{\max} \geq p^\circ$)
	Step 1	minimal minimizer X of $g(p+\chi_X)$	maximal minimizer X of $g(p+\chi_X)$
DOWN	Step 0	p° such that $\{q \mid q \leq p^\circ\} \cap \arg\min g \neq \emptyset$	(i.e., $p^*_{\min} \leq p^\circ$)
	Step 1	maximal minimizer X of $g(p-\chi_X)$	minimal minimizer X of $g(p-\chi_X)$

(b) Output and the exact number of updates of p

GREEDY		MINIMAL	MAXIMAL
UP	Output	$p^*_{\min}$ if $p^*_{\min} \geq p^\circ$; otherwise $\min(\{q \mid q \geq p^\circ\} \cap \arg\min g)$	$p^*_{\max}$
	# Updates	$\|p^\circ - p^*_{\min}\|_\infty$ if $p^*_{\min} \geq p^\circ$; otherwise $\hat{\mu}(p^\circ)$	$\|p^\circ - p^*_{\max}\|_\infty$
DOWN	Output	$p^*_{\min}$	$p^*_{\max}$ if $p^*_{\max} \leq p^\circ$; otherwise $\max(\{q \mid q \leq p^\circ\} \cap \arg\min g)$
	# Updates	$\|p^\circ - p^*_{\min}\|_\infty$	$\|p^\circ - p^*_{\max}\|_\infty$ if $p^*_{\max} \leq p^\circ$; otherwise $\check{\mu}(p^\circ)$

p°: initial vector, $p^*_{\min}$: minimal minimizer of g, $p^*_{\max}$: maximal minimizer of g

$\hat{\mu}(p^\circ) = \min\{\|p^\circ - p^*\|_\infty \mid p^* \in \arg\min g,\ p^\circ \leq p^*\}$

$\check{\mu}(p^\circ) = \min\{\|p^\circ - p^*\|_\infty \mid p^* \in \arg\min g,\ p^\circ \geq p^*\}$

Algorithm TWOPHASEMINMAX

Step 0: Find a vector $p^\circ \in \mathrm{dom}\, g$ and set $p := p^\circ$. Go to Up Phase.

Up Phase:

Step U1: Find the minimal minimizer $X \subseteq N$ of $g(p+\chi_X)$.

Step U2: If $X = \emptyset$, then go to Down Phase.

Step U3: Set $p := p + \chi_X$ and go to Step U1.

Down Phase:

Step D1: Find the minimal minimizer $X \subseteq N$ of $g(p-\chi_X)$.

Step D2: If $X = \emptyset$, then output p and stop.

Step D3: Set $p := p - \chi_X$ and go to Step D1.

It can be shown from Theorem 7.1 (d) that, at the end of the up phase, the vector p satisfies the condition $\{q \mid q \leq p\} \cap \arg\min g \neq \emptyset$ required for an initial vector of GREEDYDOWNMAXIMAL. Therefore, the output of TWOPHASEMINMAX is guaranteed to be a minimizer of g. An upper bound on the number of updates of p is given in Murota et al. (2016, Theorem 4.13) which is improved to the following statement by Murota & Shioura (2016); see also Remark 7.8. Recall the definition of $\mu(p^\circ)$ from (7.24).

Theorem 7.17. *For any initial vector p°, the algorithm* TwoPhaseMinMax *terminates by outputting some minimizer of g. The number of updates of the vector p is bounded by $\mu(p^\circ)$ in the up phase and by $\mu(p^\circ)$ in the down phase; in total, bounded by $2\mu(p^\circ)$.*

For the analysis of the Vickrey–English–Dutch auction algorithm (Section 8.3), it is convenient to consider the combination of GreedyUpMinimal and GreedyDownMinimal. The resulting two-phase algorithm is called TwoPhaseMinMin, which is the same as TwoPhaseMinMax except that Step D1 is replaced by

Step D1: Find the maximal minimizer $X \subseteq N$ of $g(p - \chi_X)$.

An upper bound on the number of updates of p is given in Murota et al. (2016, Theorem 4.12), which is improved by Murota & Shioura (2016) to the following statement; see also Remark 7.9. Recall the notation $\|q\|_\infty^+ = \max(0, q_1, q_2, \ldots, q_n)$ for $q \in \mathbb{Z}^n$.

Theorem 7.18. *For any initial vector p°, the algorithm* TwoPhaseMinMin *terminates by outputting the minimal minimizer $p^*_{\min}$ of g, if $p^*_{\min}$ exists. The number of updates of the vector p is bounded by $\mu(p^\circ)$ in the up phase and is exactly equal to $\|p^\circ - p^*_{\min}\|_\infty^+$ in the down phase; in total, bounded by $\mu(p^\circ) + \|p^\circ - p^*_{\min}\|_\infty^+$.*

Remark 7.8. For the algorithm TwoPhaseMinMax, Murota et al. (2016, Theorem 4.13) show that the number of updates of p is bounded by $\eta(p^\circ, p^*) = \|p^\circ - p^*\|_\infty^+ + \|p^\circ - p^*\|_\infty^-$ in the up phase, by $2\eta(p^\circ, p^*)$ in the down phase, and in total by $3\eta(p^\circ, p^*)$, where p^* denotes the output of the algorithm. Theorem 7.17 gives an improved bound since $\eta(p^\circ, p^*) \geq \mu(p^\circ)$. Murota et al. (2013a, Theorem 3.2) state a bound for a two-phase auction algorithm, saying that the number of updates of p in TwoPhaseMinMax is bounded by $\mu(p^\circ)$ in the up phase, by $2\mu(p^\circ)$ in the down phase, and in total by $3\mu(p^\circ)$; see Murota et al. (2013b) for the proof. ∎

Remark 7.9. For the algorithm TwoPhaseMinMin, Murota et al. (2016, Theorem 4.12) show that the number of updates of p is bounded by $\eta(p^\circ, p^*_{\min}) = \|p^\circ - p^*_{\min}\|_\infty^+ + \|p^\circ - p^*_{\min}\|_\infty^-$ in the up phase, by $2\eta(p^\circ, p^*_{\min})$ in the down phase, and in total by $3\eta(p^\circ, p^*_{\min})$. Theorem 7.18 gives an improved bound since $\eta(p^\circ, p^*_{\min}) \geq \mu(p^\circ)$ and $\eta(p^\circ, p^*_{\min}) \geq \|p^\circ - p^*_{\min}\|_\infty^+$. ∎

Remark 7.10. Besides TwoPhaseMinMax and TwoPhaseMinMin, we can obtain other variants of two-phase algorithms by choosing appropriate combinations from among the algorithms Greedy-{Up, Down}-{Minimal, Maximal} listed in Table 1. ∎

7.4. Concluding remarks of section 7

In this paper we put more emphasis on $M^\natural$-concave functions and give $L^\natural$-convex functions only a secondary role as the conjugate of $M^\natural$-concave functions, though, in fact, they are equally important and play symmetric roles in discrete convex analysis.

The concept of L-convex functions is formulated by Murota (1998), compatibly with the accepted understanding of the relationship between submodularity and convexity expounded by Lovász (1983). Then $L^\natural$-convex functions are introduced by Fujishige & Murota (2000) as a variant of L-convex functions, together with the observation that they coincide with submodular integrally convex functions considered earlier by Favati & Tardella (1990). The concept of quasi L-convex functions is also introduced by Murota & Shioura (2003), in accordance with quasisupermodularity of Milgrom & Shannon (1994). L-convex functions in continuous variables are defined by Murota & Shioura (2000, 2004a), partly motivated by a phenomenon inherent in the network flow/tension problem described in Murota (2003, Section 2.2.1).

Recently, the concept of L-convex functions is extended to functions on graph structures, which are more general than $\mathbb{Z}^n$. See Kolmogorov (2011), Huber & Kolmogorov (2012), Fujishige (2014), and Hirai (2015, 2016a,b) for the recent development.

8. ITERATIVE AUCTIONS

This section presents a unified method of analysis for iterative auctions (dynamic auctions) by combining the Lyapunov function approach of Ausubel (2006) with discrete convex analysis. We are mainly concerned with the multi-item multi-unit model, where there are multiple indivisible goods for sale and each good may have several units. The bidders' valuation functions are assumed to have gross substitutes property. This section is mostly based on Murota et al. (2013a, 2016) with some new results from Murota & Shioura (2016).

8.1. Auction models and Walrasian equilibrium

Fundamental concepts about auctions are introduced here only briefly; see, e.g., Gul & Stacchetti (2000), Milgrom (2004), Cramton et al. (2006), and Blumrosen & Nisan (2007) for comprehensive accounts.

In the auction market, there are n types of items or goods, denoted by $N = \{1,2,\ldots,n\}$, and m bidders, denoted by $M = \{1,2,\ldots,m\}$, where $m \geq 2$. We have u_i units available for each item $i \in N$, where u_i is a positive integer. We denote the integer interval as $[\mathbf{0},u]_{\mathbb{Z}} = \{x \in \mathbb{Z}^n \mid \ \mathbf{0} \leq x \leq u\}$, where $u = (u_1,u_2,\ldots,u_n)$. Each vector $x \in [\mathbf{0},u]_{\mathbb{Z}}$ is called a *bundle*; a bundle $x = (x_1,x_2,\ldots,x_n)$ corresponds to a (multi-)set of items, where x_i represents the multiplicity of item $i \in N$. Each bidder $j \in M$ has his valuation function $f_j : [\mathbf{0},u]_{\mathbb{Z}} \to \mathbb{R}$; the number $f_j(x)$ represents the value of the bundle x worth to bidder j. The case with $u_i = 1$ for all $i \in N$ is referred to as *single-unit auction*, while the general case with $u \in \mathbb{Z}^n_{++}$ as *multi-unit auction*. Note that $[\mathbf{0},\mathbf{1}]_{\mathbb{Z}} = \{0,1\}^n$, where $\mathbf{1} = (1,1,\ldots,1)$. A further special case where each bidder is interested in getting at most one item is called *unit-demand auction*.

In an auction, we want to find an efficient allocation and market clearing prices. An *allocation* of items is defined as a set of bundles $x_1,x_2,\ldots,x_m \in [\mathbf{0},u]_{\mathbb{Z}}$ satisfying $\sum_{j=1}^{m} x_j = u$. Given a price vector $p \in \mathbb{R}^n_+$, each bidder $j \in M$ wants to have a bundle x which maximizes the value $f_j(x) - p^\top x$. For $j \in M$ and $p \in \mathbb{R}^n_+$, define

$$D_j(p) \quad = \quad D(p;f_j) = \arg\max\{f_j(x) - p^\top x \mid x \in [\mathbf{0},u]_{\mathbb{Z}}\}. \qquad (8.1)$$

We call the set $D_j(p) \subseteq [\mathbf{0},u]_{\mathbb{Z}}$ the *demand set*. The auctioneer wants to find a pair of a price vector p^* and an allocation $x_1^*,x_2^*,\ldots,x_m^*$ such that $x_j^* \in D_j(p^*)$ for all $j \in M$. Such a pair is called a *(Walrasian) equilibrium* and p^* is called a *(Walrasian) equilibrium price vector*.

Although the Walrasian equilibrium possesses several desirable properties, it does not always exist. Some condition has to be imposed on bidders' valuation functions before the existence of a Walrasian equilibrium can be guaranteed. Throughout this section we assume the following conditions for bidders' valuation functions f_j $(j = 1,2,\ldots,m)$:

(A0) f_j is monotone nondecreasing,
(A1) f_j is an M$^\natural$-concave function,
(A2) f_j takes integer values.

Recall from Sections 3.3 and 4.3 that a valuation function is M$^\natural$-concave if and only if it has the gross substitutes (GS) property (in its stronger form); see Theorems 3.7 and 4.6, in particular.

Remark 8.1. Whereas we are mainly concerned with the multi-unit model here, the single-unit model is treated more extensively in the literature, e.g., Kelso & Crawford (1982), Gul & Stacchetti (1999, 2000), Milgrom (2004), Blumrosen & Nisan (2007), Cramton et al. (2006), and Milgrom & Strulovici (2009). The method of analysis presented in this section remains meaningful and interesting also for the single-unit model. ∎

Remark 8.2. Iterative auctions for unit-demand auction are discussed extensively in the literature, e.g., Vickrey (1961), Demange et al. (1986), Mo et al. (1988), Sankaran (1994), Mishra & Parkes (2009), Andersson et al. (2013), and Andersson & Erlanson (2013). Specifically, the Vickrey–English auction by Demange et al. (1986), the Vickrey–Dutch auction by Mishra & Parkes (2009), and the Vickrey–English–Dutch auction by Andersson & Erlanson (2013) are such iterative auctions. Although these three algorithms are proposed independently of the iterative auction algorithms for the multi-unit model, it is possible to give a unified treatment of these iterative auction algorithms by revealing their relationship to the Lyapunov function approach (Section 8.3). ∎

8.2. Lyapunov function approach to iterative auctions

In this section we describe the Lyapunov function-based iterative auctions developed by Ausubel (2006) and Sun & Yang (2009). Our objective is to clarify the underlying mathematical structure with the aid of discrete convex analysis, and to derive sharp upper or exact bounds on the number of iterations in the iterative auctions.

For $j \in M$ and $p \in \mathbb{R}^n_+$, we define the *indirect utility function* $V_j : \mathbb{R}^n_+ \to \mathbb{R}$ by

$$V_j(p) \quad = \quad V(p; f_j) = \max\{f_j(x) - p^\top x \mid x \in [\mathbf{0}, u]_\mathbb{Z}\}, \tag{8.2}$$

and the *Lyapunov function* by

$$L(p) = \sum_{j=1}^{m} V_j(p) + u^\top p \qquad (p \in \mathbb{R}^n), \tag{8.3}$$

where the vector $u \in \mathbb{Z}^n_+$ represents the numbers of available units for items in N.

Under the assumptions (A0)–(A2) it can be shown[43] that there exists an equilibrium price vector p^* whose components are nonnegative integers. Henceforth we assume that the price vector p in iterative auctions is always chosen to be a nonnegative integer vector, i.e., $p \in \mathbb{Z}^n_+$. Accordingly, we regard V_j and L as integer-valued functions defined on nonnegative integers, i.e., $V_j : \mathbb{Z}^n_+ \to \mathbb{Z}$ and $L : \mathbb{Z}^n_+ \to \mathbb{Z}$.

The ascending auction algorithm based on the Lyapunov function (Ausubel, 2006; Sun & Yang, 2009) is as follows:

Algorithm ASCENDMINIMAL

Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n_+$ is an arbitrary vector satisfying $p^\circ \leq p^*_{\min}$ (e.g., $p^\circ = \mathbf{0}$).

Step 1: Find the minimal minimizer $X \subseteq N$ of $L(p + \chi_X)$.

Step 2: If $X = \emptyset$, then output p and stop.

Step 3: Set $p := p + \chi_X$ and go to Step 1.

The above algorithm can be interpreted in auction terms as follows:[44]

Algorithm ASCENDMINIMAL (in auction terms)

Step 0: The auctioneer sets $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n$ should satisfy $p^\circ \leq p^*_{\min}$.

Step 1: The auctioneer asks the bidders to report their demand sets $D_j(p)$ $(j \in M)$, and finds the minimal minimizer $X \subseteq N$ of $L(p + \chi_X)$.

Step 2: The auctioneer checks if $X = \emptyset$; if $X = \emptyset$ holds, then the auctioneer reports p as the final price vector and stop.

Step 3: The auctioneer sets $p := p + \chi_X$ and returns to Step 1.

The analysis of the algorithm ASCENDMINIMAL can be made transparent by using concepts and results from discrete convex analysis. Before presenting formal theorems, we enumerate the major mathematical ingredients.

- As pointed out by Ausubel (2006), the Walrasian equilibrium price vector can be characterized as a minimizer of the Lyapunov function L and an iterative auction algorithm can be understood as a minimization process of the Lyapunov function $L(p)$. See Theorem 8.1.

[43] The integrality follows from the fact that an integer-valued M$^\natural$-concave function f on $\mathbb{Z}^n$ has an integral subgradient (or supergradient) at every point x in dom f.

[44] See Ausubel (2006, Appendix B) for details about the implementation of Steps 2 and 3.

- The conjugate function of an $\mathrm{M}^\natural$-concave function is an $\mathrm{L}^\natural$-convex function, and vice versa (the conjugacy theorem in Section 7.2). Hence the indirect utility function V_j is an $\mathrm{L}^\natural$-convex function and therefore, the Lyapunov function L is an $\mathrm{L}^\natural$-convex function. See Theorem 8.2.

- The $\mathrm{L}^\natural$-convexity of the Lyapunov function L implies a nice combinatorial structure of the equilibrium prices. The set of the equilibrium prices is an $\mathrm{L}^\natural$-convex set (Remark 7.4), which is more special than just being a sublattice. See Theorem 8.3.

- The $\mathrm{L}^\natural$-convexity of the Lyapunov function L enables us to utilize general results on $\mathrm{L}^\natural$-convex function minimization (Section 7.3) to analyze the behavior of iterative auction algorithms, such as convergence to an equilibrium price and the number of iterations needed to reach the equilibrium price. See Theorem 8.4 as well as Theorem 8.9.

We now present the theorems substantiating the above-mentioned points. The conditions (A0)–(A2) are assumed implicitly in the following four theorems. Recall that a Walrasian equilibrium exists under these conditions. The first theorem is due to Ausubel (2006, Proposition 1); see Sun & Yang (2009, Lemma 1) for a more general result.

Theorem 8.1. *A vector $p \in \mathbb{Z}^n_+$ is an equilibrium price vector if and only if it is a minimizer of the Lyapunov function L.*

Proof. The key of the proof is the fact that the set of excess supply vectors at a price vector p, i.e., $\{u - \sum_{j=1}^{m} x_j \mid x_j \in D_j(p)\ (j = 1,2,\ldots,m)\}$, coincides with the set of subgradients of the Lyapunov function L at p; see Ausubel (2006). $\square$

Theorem 8.2.
(1) *For each $j \in M$, the indirect utility function V_j is an $L^\natural$-convex function.*
(2) *The Lyapunov function L is an $L^\natural$-convex function.*

Proof. (1) When regarded as $V_j : \mathbb{Z}^n_+ \to \mathbb{Z}$, the definition (8.2) of V_j shows that V_j is the conjugate function of f_j in the sense of (7.14). That is, $V_j = f_j^\triangledown$ in the notation of Section 7.2. Then Theorem 7.9 (2) shows the $\mathrm{L}^\natural$-convexity of V_j.

(2) In the definition (8.3) of L, each V_j is $\mathrm{L}^\natural$-convex by (1), and the linear term $u^\top p$ is obviously $\mathrm{L}^\natural$-convex. The sum of $\mathrm{L}^\natural$-convex functions is again $\mathrm{L}^\natural$-convex by Theorem 7.1. Hence the Lyapunov function L is $\mathrm{L}^\natural$-convex. $\square$

Theorem 8.3. *The equilibrium price vectors form a bounded L$^\natural$-convex set.*[45] *That is, for two equilibrium price vectors p^*, q^* and any nonnegative integer α, both $(p^* - \alpha\mathbf{1}) \vee q^*$ and $p^* \wedge (q^* + \alpha\mathbf{1})$ are equilibrium price vectors. In particular, the minimal equilibrium price vector $p^*_{\min}$ and the maximal equilibrium price vector $p^*_{\max}$ are uniquely determined.*

Proof. This follows from the L$^\natural$-convexity of the Lyapunov function (Theorem 8.2) and the L$^\natural$-convexity of the set of the minimizers (Remark 7.4); the boundedness is easily shown. □

Theorem 8.4. *For an initial vector p° with $p^\circ \leq p^*_{\min}$, the algorithm* ASCENDMINIMAL *outputs the minimal equilibrium price vector $p^*_{\min}$ and the number of updates of the price vector is exactly equal to $\|p^*_{\min} - p^\circ\|_\infty$.*

Proof. The Lyapunov function L is an L$^\natural$-convex function by Theorem 8.2, and the algorithm ASCENDMINIMAL is nothing but the algorithm GREEDYUPMINIMAL in Section 7.3 applied to L. Since the minimal minimizer of the Lyapunov function L is the minimal equilibrium price vector $p^*_{\min}$ by Theorem 8.1, the auction algorithm ASCENDMINIMAL yields the minimal equilibrium price vector $p^*_{\min}$. The number of updates of the price vector is equal to $\|p^*_{\min} - p^\circ\|_\infty$ by Theorem 7.16 (2). □

Theorem 8.4 is due to Murota et al. (2016), while the finite termination is noted in Ausubel (2006). The bound for the number of iterations in ASCENDMINIMAL is given as the ℓ_∞-distance from the initial price vector p° to the minimal equilibrium price vector $p^*_{\min}$. This implies, in particular, that the trajectory of the price vector generated by the ascending auction is the "shortest" path between the initial vector and the minimal equilibrium price vector.

8.2.1. *Variants of auction algorithms*

A variant of the ascending auction algorithm, called ASCENDMAXIMAL, is obtained through the application of the algorithm GREEDYUPMAXIMAL in Section 7.3 to the Lyapunov function L. Two other variants of the descending auction algorithm, called DESCENDMAXIMAL and DESCENDMINIMAL, are

[45] See Remark 7.4 for L$^\natural$-convex sets. If we consider real price vectors, the equilibrium price vectors form an L$^\natural$-convex polyhedron.

obtained through the application of the algorithms GREEDYDOWNMAXIMAL and GREEDYDOWNMINIMAL in Section 7.3 to the Lyapunov function L, where DESCENDMAXIMAL coincides with the descending auction algorithm in Ausubel (2006). The general results for L$^\natural$-convex function minimization summarized in Table 1 (b) in Section 7.3 imply the following exact bounds (Murota et al., 2016).

Theorem 8.5.
(1) *For an initial vector p° with $p^\circ \leq p^*_{\max}$, the algorithm* ASCENDMAXIMAL *outputs $p^*_{\max}$ and the number of updates of the price vector is exactly equal to $\|p^*_{\max} - p^\circ\|_\infty$.*
(2) *For an initial vector p° with $p^\circ \geq p^*_{\max}$, the algorithm* DESCENDMAXIMAL *outputs $p^*_{\max}$ and the number of updates of the price vector is exactly equal to $\|p^*_{\max} - p^\circ\|_\infty$.*
(3) *For any initial vector p° with $p^\circ \geq p^*_{\min}$, the algorithm* DESCENDMINIMAL *outputs $p^*_{\min}$ and the number of updates of the price vector is exactly equal to $\|p^*_{\min} - p^\circ\|_\infty$.*

A two-phase auction algorithm, consisting of an ascending auction phase followed by a descending phase, can be obtained by applying the algorithm TWOPHASEMINMAX in Section 7.3 to the Lyapunov function L. Another two-phase auction algorithm can be obtained from TWOPHASEMINMIN. Then Theorems 7.17 and 7.18 imply the following (Murota & Shioura, 2016).

Theorem 8.6.
(1) *For any initial vector p°, the two-phase algorithm* TWOPHASEMINMAX *outputs some equilibrium price p^*. The number of updates of the vector p is bounded by $\mu(p^\circ)$ in the ascending phase and by $\mu(p^\circ)$ in the descending phase; in total, bounded by $2\mu(p^\circ)$.*
(2) *For any initial vector p°, the two-phase algorithm* TWOPHASEMINMIN *outputs the minimal equilibrium price $p^*_{\min}$. The number of updates of the vector p is bounded by $\mu(p^\circ)$ in the ascending phase and is exactly equal to $\|p^\circ - p^*_{\min}\|^+_\infty$ in the descending phase; in total, bounded by $\mu(p^\circ) + \|p^\circ - p^*_{\min}\|^+_\infty$.*

Two-phase algorithms with more flexibility are given in Murota et al. (2013a), and Murota & Shioura (2016).

Remark 8.3. The algorithm TwoPhaseMinMax, when applied to valuation functions on $\{0,1\}^N$ (single-unit valuations), coincides with a special case of "Global Dynamic Double-Track (GDDT) procedure" proposed in Sun & Yang (2009). The "global Walrasian tâtonnement algorithm" proposed by Ausubel (2006) repeats ascending and descending phases until some equilibrium is found. Theorem 7.17 shows that the global Walrasian tâtonnement algorithm terminates after only one ascending phase and only one descending phase. Put differently, the behavior of the global Walrasian tâtonnement algorithm coincides with that of TwoPhaseMinMax. ∎

Remark 8.4. Besides TwoPhaseMinMax, we can obtain many variants of two-phase algorithms by choosing appropriate combinations from among the algorithms Greedy-{Up, Down}-{Minimal, Maximal} listed in Table 1. In Section 8.3, for example, we consider the combination of GreedyUpMinimal and GreedyDownMinimal. ∎

8.3. Unit-demand auctions

Fundamental multi-item unit-demand auction algorithms such as the Vickrey–English, Vickrey–Dutch, Vickrey–English–Dutch auctions can be reformulated in the framework of the Lyapunov function approach. In so doing we can derive bounds for the number of iterations in these auction algorithms from the corresponding results about $\mathrm{L}^\natural$-convex function minimization presented in Section 7.3.

The unit-demand auction model is a special case of the single-unit auction model, where each bidder is a *unit-demand* bidder, being interested in getting at most one item. We continue to use notations $N = \{1,2,\ldots,n\}$ for the set of items and $M = \{1,2,\ldots,m\}$ for the set of bidders. For each item i and each bidder j, we denote by v_{ji} the valuation of item i by bidder j, which is assumed to be a nonnegative integer, i.e., $v_{ji} \in \mathbb{Z}_+$. The valuation function $f_j : 2^N \to \mathbb{Z}_+$ of bidder j is given by

$$f_j(X) = \begin{cases} \max\{v_{ji} \mid i \in X\} & (\text{if } X \neq \emptyset), \\ 0 & (\text{if } X = \emptyset). \end{cases} \tag{8.4}$$

A valuation function of this form, often called a *unit-demand valuation*,[46] is a gross substitutes valuation, as pointed out by Gul & Stacchetti (1999). In other

[46] See, e.g., Cramton et al. (2006, Section 9.2.2) and Blumrosen & Nisan (2007, Definition 11.17).

words, a unit-demand valuation is $\mathrm{M}^\natural$-concave; see (3.21). We are interested in finding the minimal Walrasian equilibrium price vector $p^*_{\min} \in \mathbb{Z}^N_+$ by iterative auctions.

Fundamental iterative auction algorithms such as the Vickrey–English auction of Demange et al. (1986) (the variant by Mo et al. (1988) and Sankaran (1994), to be more specific), the Vickrey–Dutch auction of Mishra & Parkes (2009), and the Vickrey–English–Dutch auction of Andersson & Erlanson (2013) can be recast into the Lyapunov function-based framework. The following theorem is due to Murota et al. (2016, Theorem 5.5); the specific forms of the auction algorithms are described in Remark 8.5.

Theorem 8.7. *Let $L : \mathbb{Z}^N_+ \to \mathbb{Z}$ be the Lyapunov function associated with the unit-demand valuations* (8.4)*.*
(1) *For any initial price vector p° with $p^\circ \leq p^*_{\min}$, the sequence of price vectors p generated by the algorithm* VICKREY_ENGLISH *is the same as that of* GREEDYUPMINIMAL *applied to L.*
(2) *For any initial price vector p° with $p^\circ \geq p^*_{\min}$, the sequence of price vectors p generated by the algorithm* VICKREY_DUTCH *is the same as that of* GREEDYDOWNMINIMAL *applied to L.*
(3) *For any initial price vector p°, the sequence of price vectors p generated by the algorithm* VICKREY_ENGLISH_DUTCH *is the same as that of* TWOPHASEMINMIN *applied to L.*

Theorem 8.7 above is established on the basis of the following technical observations (Murota et al., 2016, Lemma 5.7), which relate the descending directions of the Lyapunov function with "sets in excess demand" (see Remark 8.5) used in the Vickrey–English, Vickrey–Dutch, Vickrey–English–Dutch auction algorithms.

Proposition 8.8. *Let $p \in \mathbb{Z}^N_+$ be a price vector.*
(1) *A set $X \subseteq N$ is the maximal set in excess demand at price p if and only if X is the minimal minimizer of $L(p+\chi_X) - L(p)$.*
(2) *A set $Z \subseteq \mathrm{supp}^+(p)$ is the maximal set in positive excess demand at price p if and only if $X = \mathrm{supp}^+(p) \setminus Z$ is the maximal minimizer of $L(p-\chi_X) - L(p)$.*

Theorem 8.7 enables us to resort to the general results for $\mathrm{L}^\natural$-convex function minimization in Section 7.3 to establish the following (exact or upper) bounds on the number of iterations in the unit-demand auction algorithms, where (1) and (2) are given in Andersson & Erlanson (2013, Corollary 2), and (3) is in Murota & Shioura (2016).

Theorem 8.9.
(1) *For any initial price vector* p° *with* $p^\circ \leq p^*_{\min}$, *the number of updates of the price vector in the algorithm* VICKREY_ENGLISH *is exactly equal to* $\|p^\circ - p^*_{\min}\|_\infty$.
(2) *For any initial price vector* p° *with* $p^\circ \geq p^*_{\min}$, *the number of updates of the price vector in the algorithm* VICKREY_DUTCH *is exactly equal to* $\|p^\circ - p^*_{\min}\|_\infty$.
(3) *For any initial price vector* p°, *the number of updates of the price vector in the algorithm* VICKREY_ENGLISH_DUTCH *is bounded by* $\mu(p^\circ)$ *in the ascending phase and is exactly equal to* $\|p^\circ - p^*_{\min}\|^+_\infty$ *in the descending phase; in total, bounded by* $\mu(p^\circ) + \|p^\circ - p^*_{\min}\|^+_\infty$.

Proof. We prove the claims to illustrate the use of the general results in Section 7.3. (1) follows from Theorem 8.7 (1) and Theorem 7.16 (2). (2) follows from Theorem 8.7 (2) and Table 1 (b). (3) follows from Theorem 8.7 (3) and Theorem 7.18. □

Remark 8.5. The Vickrey–English, Vickrey–Dutch, Vickrey–English–Dutch auction algorithms are described here, following Andersson & Erlanson (2013) and Andersson et al. (2013). Denote by 0 an artificial item (null-item) which has no value (i.e., $v_{j0} = 0$ for all $j \in M$) and is available in an infinite number of units. For each bidder $j \in M$ and a price vector $p \in \mathbb{Z}^N_+$, define $D_j(p) \subseteq N \cup \{0\}$ by

$$\begin{aligned} D_j(p) &= \arg\max\{v_{ji} - p_i \mid i \in N \cup \{0\}\} \\ &= \{i \in N \cup \{0\} \mid v_{ji} - p_i \geq v_{ji'} - p_{i'} \ (\forall i' \in N \cup \{0\})\}, \end{aligned}$$

where $p_0 = 0$. For an item set $Y \subseteq N$ and a price vector $p \in \mathbb{Z}^N_+$, define

$$O(Y,p) = \{j \in M \mid D_j(p) \subseteq Y\},$$
$$U(Y,p) = \{j \in M \mid D_j(p) \cap Y \neq \emptyset\}.$$

The set $O(Y,p)$ consists of bidders who only demand items in Y at price p, while $U(Y,p)$ is the set of bidders who demand some item in Y at price p. Obviously, $O(Y,p) \subseteq U(Y,p)$. A set $X \subseteq N$ is said to be *in excess demand* at price p if it satisfies

$$|U(Y,p) \cap O(X,p)| > |Y| \qquad (\emptyset \neq \forall Y \subseteq X).$$

For each price vector p there uniquely exists a maximal set in excess demand.[47] The Vickrey-English auction algorithm due to Mo et al. (1988) and Sankaran (1994), a variant of the one in Demange et al. (1986), is as follows:

Algorithm VICKREY_ENGLISH
Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}_+^N$ is an arbitrary vector satisfying $p^\circ \leq p^*_{\min}$ (e.g., $p^\circ = \mathbf{0}$).
Step 1: Find the maximal set $X \subseteq N$ in excess demand at price p.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p + \chi_X$ and go to Step 1.

The Vickrey-Dutch auction algorithm refers to the variants of the sets $D_j(p)$ and $O(Y, p)$ defined as

$$D_j^+(p) = D_j(p) \cap \mathrm{supp}^+(p),$$
$$O^+(Y, p) = \{j \in M \mid D_j^+(p) \subseteq Y\}.$$

A set $X \subseteq N$ is said to be *in positive excess demand* at price p if $X \subseteq \mathrm{supp}^+(p)$ and

$$|U(Y, p) \cap O^+(X, p)| > |Y| \qquad (\emptyset \neq \forall Y \subseteq X).$$

For each price vector p there uniquely exists a maximal set in positive excess demand.[48] The Vickrey–Dutch auction by Mishra & Parkes (2009) is as follows:

Algorithm VICKREY_DUTCH
Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}_+^N$ is an arbitrary vector satisfying $p^\circ \geq p^*_{\min}$.
Step 1: Find the maximal set $Z \subseteq N$ in positive excess demand at price p, and set $X := \mathrm{supp}^+(p) \setminus Z$.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p - \chi_X$ and go to Step 1.

The Vickrey–English–Dutch auction by Andersson & Erlanson (2013) is a combination of the Vickrey–English and Vickrey–Dutch auctions, as follows:

[47] See Mo et al. (1988, Proposition 1), Andersson & Erlanson (2013, Proposition 1), and Andersson et al. (2013, Theorem 1).
[48] See Andersson & Erlanson (2013, Theorem 2).

Algorithm VICKREY_ENGLISH_DUTCH

Step 0: Set $p := p^{\circ}$, where $p^{\circ} \in \mathbb{Z}_{+}^{N}$ is an arbitrary vector. Go to Ascending Phase.

Ascending Phase:

Step A1: Find the maximal set $X \subseteq N$ in excess demand at price p.

Step A2: If $X = \emptyset$, then go to Descending Phase.

Step A3: Set $p := p + \chi_X$ and go to Step A1.

Descending Phase:

Step D1: Find the maximal set $Z \subseteq N$ in positive excess demand at price p, and set $X := \mathrm{supp}^{+}(p) \setminus Z$.

Step D2: If $X = \emptyset$, then output p and stop.

Step D3: Set $p := p - \chi_X$ and go to Step D1.

∎

8.4. Concluding remarks of section 8

Use of discrete convex analysis in the Lyapunov function approach is also conceived by Drexl & Kleiner (2015). Besides the basic form of ascending auction, the paper proposes and analyzes the "singleton-based tâtonnement" which reflects a certain practice in auction design. It also discusses the double-track adjustment process of Sun & Yang (2009) as an application of the framework of Section 8.2; the underlying key fact here is that gross substitutes and complements are represented by twisted M$^{\natural}$-concave functions (Section 3.5). Lehmann et al. (2006) shows a connection between discrete convex analysis and combinatorial auctions. Sun & Yang (2014) proposes a dynamic auction for multiple complementary goods that goes beyond the framework discussed in this paper.

9. INTERSECTION AND SEPARATION THEOREMS

9.1. Separation theorem

The *duality* principle in convex analysis can be expressed in a number of different forms. One of the most appealing statements is in the form of the separation theorem, which asserts the existence of a separating affine function $y = \alpha^{*} + \langle p^{*}, x \rangle$ for a pair of convex and concave functions. In application to economic problems, the separating vector p^{*} gives the equilibrium price.

In the continuous case we have the following.

Theorem 9.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $h : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be convex and concave functions, respectively* (*satisfying certain regularity conditions*). *If*

$$f(x) \geq h(x) \qquad (\forall x \in \mathbb{R}^n),$$

there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^n$ such that

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (\forall x \in \mathbb{R}^n).$$

In the discrete case we are concerned with functions defined on integer points: $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ and $h : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$. A *discrete separation theorem* means a statement like:

> For any $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ and $h : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ belonging to certain classes of functions, if $f(x) \geq h(x)$ for all $x \in \mathbb{Z}^n$, then there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^n$ such that
>
> $$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (\forall x \in \mathbb{Z}^n).$$
>
> Moreover, if f and h are integer-valued, there exist integer-valued $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbb{Z}^n$.

In application to economic problems, the separating vector p^* in a discrete separation theorem often gives the equilibrium price in markets with indivisible goods.

Discrete separation theorems capture deep combinatorial properties in spite of the apparent similarity to the separation theorem in the continuous case. In this connection we note the following facts that indicate the difficulty inherent in discrete separation theorems.[49] Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex-extensible function, with the convex closure $\overline{f}$. Also let $h : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a concave-extensible function, with the concave closure $\overline{h}$. In the following statements, $\not\Longrightarrow$ stands for "does not imply."

1. $f(x) \geq h(x)$ $(\forall x \in \mathbb{Z}^n)$ $\not\Longrightarrow$ $\overline{f}(x) \geq \overline{h}(x)$ $(\forall x \in \mathbb{R}^n)$.
2. $f(x) \geq h(x)$ $(\forall x \in \mathbb{Z}^n)$ $\not\Longrightarrow$ existence of $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^n$.
3. existence of $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^n$ $\not\Longrightarrow$ existence of $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbb{Z}^n$.

[49] See Murota (2003, Examples 1.5 and 1.6) for concrete examples.

It is known that discrete separation theorems hold for M$^\natural$-convex/M$^\natural$-concave functions and for L$^\natural$-convex/L$^\natural$-concave functions. The M$^\natural$-separation theorem (Theorem 9.2) is shown by Murota (1996a, 1998, 1999) in terms of M-convex/concave functions, and the L$^\natural$-separation theorem (Theorem 9.3) by Murota (1998) in terms of L-convex/concave functions. The assumptions of the theorems refer to the convex and concave conjugate functions of f and h defined, respectively, by[50]

$$f^\bullet(p) = \sup\{\langle p,x\rangle - f(x) \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{R}^n), \tag{9.1}$$
$$h^\circ(p) = \inf\{\langle p,x\rangle - h(x) \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{R}^n). \tag{9.2}$$

Theorem 9.2 (M$^\natural$-separation theorem)**.** *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an M$^\natural$-convex function and $h : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function such that* $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} h \neq \emptyset$ *or* $\mathrm{dom}_{\mathbb{R}} f^\bullet \cap \mathrm{dom}_{\mathbb{R}} h^\circ \neq \emptyset$. *If $f(x) \geq h(x)$ $(\forall x \in \mathbb{Z}^n)$, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^n$ such that*

$$f(x) \geq \alpha^* + \langle p^*, x\rangle \geq h(x) \qquad (\forall x \in \mathbb{Z}^n).$$

Moreover, if f and h are integer-valued, there exist integer-valued $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbb{Z}^n$.

Theorem 9.3 (L$^\natural$-separation theorem)**.** *Let $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an L$^\natural$-convex function and $k : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be an L$^\natural$-concave function such that* $\mathrm{dom}_{\mathbb{Z}} g \cap \mathrm{dom}_{\mathbb{Z}} k \neq \emptyset$ *or* $\mathrm{dom}_{\mathbb{R}} g^\bullet \cap \mathrm{dom}_{\mathbb{R}} k^\circ \neq \emptyset$. *If $g(p) \geq k(p)$ $(\forall p \in \mathbb{Z}^n)$, there exist $\beta^* \in \mathbb{R}$ and $x^* \in \mathbb{R}^n$ such that*

$$g(p) \geq \beta^* + \langle p, x^*\rangle \geq k(p) \qquad (\forall p \in \mathbb{Z}^n).$$

Moreover, if g and k are integer-valued, there exist integer-valued $\beta^* \in \mathbb{Z}$ and $x^* \in \mathbb{Z}^n$.

As an immediate corollary of the M$^\natural$-separation theorem we can obtain an optimality criterion for the problem of maximizing the sum of two M$^\natural$-concave functions, which we call the *M$^\natural$-concave intersection problem*. Note that the sum of M$^\natural$-concave functions is no longer M$^\natural$-concave and Theorem 4.4 does not apply. Recall the notation $f[-p](x) = f(x) - \langle p, x\rangle$.

[50] We have $f^\bullet(p) = -f^\triangle(-p)$ and $h^\circ(p) = -h^\triangledown(p)$ in the notation of (7.12) and (7.13).

Theorem 9.4 (M$^\natural$-concave intersection theorem)**.** *For M$^\natural$-concave functions $f_1, f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ and a point $x^* \in \mathrm{dom}_{\mathbb{Z}} f_1 \cap \mathrm{dom}_{\mathbb{Z}} f_2$ we have*

$$f_1(x^*) + f_2(x^*) \geq f_1(x) + f_2(x) \qquad (\forall x \in \mathbb{Z}^n)$$

if and only if there exists $p^* \in \mathbb{R}^n$ such that

$$\begin{aligned} f_1[-p^*](x^*) &\geq f_1[-p^*](x) \qquad (\forall x \in \mathbb{Z}^n), \\ f_2[+p^*](x^*) &\geq f_2[+p^*](x) \qquad (\forall x \in \mathbb{Z}^n). \end{aligned}$$

These conditions are equivalent, respectively, to

$$\begin{aligned} f_1[-p^*](x^*) &\geq f_1[-p^*](x^* + \chi_i - \chi_j) \qquad (\forall i, j \in \{0, 1, \ldots, n\}), \\ f_2[+p^*](x^*) &\geq f_2[+p^*](x^* + \chi_i - \chi_j) \qquad (\forall i, j \in \{0, 1, \ldots, n\}), \end{aligned}$$

and for such p^* we have

$$\arg\max_{\mathbb{Z}} (f_1 + f_2) = \arg\max_{\mathbb{Z}} f_1[-p^*] \cap \arg\max_{\mathbb{Z}} f_2[+p^*].$$

Moreover, if f_1 and f_2 are integer-valued, we can choose integer-valued $p^* \in \mathbb{Z}^n$.

An extension of the M$^\natural$-concave intersection theorem is given in Theorem 10.4, which constitutes the technical pivot in the Fujishige–Tamura model that unifies the stable marriage and the assignment game (see Remark 10.1).

Remark 9.1. Three different proofs are available for the M$^\natural$-concave intersection theorem. The original proof (Murota, 1996a) is based on the reduction of the M$^\natural$-concave intersection problem to the M-convex submodular flow problem; see Remark 12.2 in Section 12.1. Then Theorem 9.4 is derived from the negative-cycle optimality criterion (Theorem 12.2) for the M-convex submodular flow problem. The second proof is based on the reduction to the discrete separation theorem, which is proved by the polyhedral-combinatorial method using the (standard) separation theorem in convex analysis; see the proof of Murota (2003, Theorem 8.15). The third proof Murota (2004b) is a direct constructive proof based on the successive shortest path algorithm. ∎

9.2. Fenchel duality

Another expression of the duality principle is in the form of the Fenchel duality. This is a min-max relation between a pair of convex and concave functions and their conjugate functions. Such a min-max theorem is computationally useful in that it affords a certificate of optimality.

We start with the continuous case. For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, the convex conjugate $f^{\bullet} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by[51]

$$f^{\bullet}(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{R}^n\} \qquad (p \in \mathbb{R}^n). \tag{9.3}$$

For $h : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, the concave conjugate $h^{\circ} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$h^{\circ}(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbb{R}^n\} \qquad (p \in \mathbb{R}^n). \tag{9.4}$$

Theorem 9.5. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $h : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be convex and concave functions, respectively (satisfying certain regularity conditions). Then*

$$\inf\{f(x) - h(x) \mid x \in \mathbb{R}^n\} = \sup\{h^{\circ}(p) - f^{\bullet}(p) \mid p \in \mathbb{R}^n\}.$$

We now turn to the discrete case. For any functions $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ and $h : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$, we define the discrete versions of (9.3) and (9.4) as

$$\begin{aligned} f^{\bullet}(p) &= \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{Z}^n), &(9.5)\\ h^{\circ}(p) &= \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{Z}^n). &(9.6)\end{aligned}$$

Then we have a chain of inequalities:

$$\begin{array}{ccc} \inf\{f(x) - h(x) \mid x \in \mathbb{Z}^n\} & & \sup\{h^{\circ}(p) - f^{\bullet}(p) \mid p \in \mathbb{Z}^n\} \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\geq$} \\ \inf\{\overline{f}(x) - \overline{h}(x) \mid x \in \mathbb{R}^n\} & \geq & \sup\{\overline{h}^{\circ}(p) - \overline{f}^{\bullet}(p) \mid p \in \mathbb{R}^n\}, \end{array} \tag{9.7}$$

where $\overline{f}$ and $\overline{h}$ are the convex and concave closures of f and h, respectively, and $\overline{f}^{\bullet}$ and $\overline{h}^{\circ}$ are defined by (9.3) for $\overline{f}$ and (9.4) for $\overline{h}$. We observe that

1. The second inequality ($\geq$) in the middle of (9.7) is in fact an equality ($=$) (under mild regularity conditions) by the Fenchel duality theorem in convex analysis (Theorem 9.5);

[51] We have $f^{\bullet}(p) = -f^{\triangle}(-p)$ and $h^{\circ}(p) = -h^{\triangledown}(p)$ in the notation of (7.10) and (7.11).

2. The first inequality ($\overset{\vee}{|}$) in the left of (9.7) can be strict (i.e., $\neq$) even when f is convex-extensible and h is concave-extensible, and similarly for the third inequality ($\overset{|}{\wedge}$) in the right. See Examples 9.1 and 9.2 below.[52]

Example 9.1. For $f,h : \mathbb{Z}^2 \to \mathbb{Z}$ defined as

$$f(x_1,x_2) = |x_1 + x_2 - 1|, \qquad h(x_1,x_2) = 1 - |x_1 - x_2|$$

we have $\inf\{f - h\} = 0$, $\inf\{\overline{f} - \overline{h}\} = -1$. The conjugate functions (9.5) and (9.6) are given by

$$f^{\bullet}(p_1,p_2) = \begin{cases} p_1 & ((p_1,p_2) \in S) \\ +\infty & (\text{otherwise}), \end{cases} \qquad h^{\circ}(p_1,p_2) = \begin{cases} -1 & ((p_1,p_2) \in T) \\ -\infty & (\text{otherwise}) \end{cases}$$

with $S = \{(-1,-1),(0,0),(1,1)\}$ and $T = \{(-1,1),(0,0),(1,-1)\}$. Hence $\sup\{h^{\circ} - f^{\bullet}\} = h^{\circ}(0,0) - f^{\bullet}(0,0) = -1 - 0 = -1$. Then (9.7) reads as

$$\underset{(0)}{\inf\{f - h\}} \;>\; \underset{(-1)}{\inf\{\overline{f} - \overline{h}\}} \;=\; \underset{(-1)}{\sup\{\overline{h}^{\circ} - \overline{f}^{\bullet}\}} \;=\; \underset{(-1)}{\sup\{h^{\circ} - f^{\bullet}\}}.$$

∎

Example 9.2. For $f,h : \mathbb{Z}^2 \to \mathbb{Z}$ defined as

$$f(x_1,x_2) = \max(0, x_1 + x_2), \qquad h(x_1,x_2) = \min(x_1,x_2)$$

we have $\inf\{f - h\} = \inf\{\overline{f} - \overline{h}\} = 0$. The conjugate functions (9.5) and (9.6) are given as $f^{\bullet} = \delta_S$ and $h^{\circ} = -\delta_T$ in terms of the (convex) indicator functions[53] of $S = \{(0,0),(1,1)\}$ and $T = \{(1,0),(0,1)\}$. Since $S \cap T = \emptyset$, the function $h^{\circ} - f^{\bullet}$ is identically equal to $-\infty$, whereas $\sup\{\overline{h}^{\circ} - \overline{f}^{\bullet}\} = 0$ since $\overline{f}^{\bullet} = \delta_{\overline{S}}$, $\overline{h}^{\circ} = -\delta_{\overline{T}}$ and $\overline{S} \cap \overline{T} = \{(1/2,1/2)\}$. Then (9.7) reads as

$$\underset{(0)}{\inf\{f - h\}} \;=\; \underset{(0)}{\inf\{\overline{f} - \overline{h}\}} \;=\; \underset{(0)}{\sup\{\overline{h}^{\circ} - \overline{f}^{\bullet}\}} \;>\; \underset{(-\infty)}{\sup\{h^{\circ} - f^{\bullet}\}}.$$

∎

The Fenchel-type duality holds for M$^{\natural}$-convex/M$^{\natural}$-concave functions and L$^{\natural}$-convex/L$^{\natural}$-concave functions. The Fenchel-type duality theorem originates in Murota (1996a) (see also Murota, 1998) and formulated into the following form in Murota (2003). The essence of the theorem is the assertion that the first and third inequalities in (9.7) are in fact equalities for M$^{\natural}$-convex/M$^{\natural}$-concave functions and L$^{\natural}$-convex/L$^{\natural}$-concave functions.

[52] These examples are taken from Murota (2009).

[53] $\delta_S(p) = 0$ for $p \in S$ and $= +\infty$ for $p \notin S$.

Theorem 9.6 (Fenchel-type duality theorem)**.**
(1) *Let* $f : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ *be an integer-valued* $M^\natural$*-convex function and* $h : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$ *be an integer-valued* $M^\natural$*-concave function such that* $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} h \neq \emptyset$ *or* $\mathrm{dom}_{\mathbb{Z}} f^\bullet \cap \mathrm{dom}_{\mathbb{Z}} h^\circ \neq \emptyset$*, where* $f^\bullet$ *and* h° *are defined by* (9.5) *and* (9.6)*. Then we have*

$$\inf\{f(x) - h(x) \mid x \in \mathbb{Z}^n\} = \sup\{h^\circ(p) - f^\bullet(p) \mid p \in \mathbb{Z}^n\}. \tag{9.8}$$

If this common value is finite, the infimum and the supremum are attained.
(2) *Let* $g : \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$ *be an integer-valued* $L^\natural$*-convex function and* $k : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$ *be an integer-valued* $L^\natural$*-concave function such that* $\mathrm{dom}_{\mathbb{Z}} g \cap \mathrm{dom}_{\mathbb{Z}} k \neq \emptyset$ *or* $\mathrm{dom}_{\mathbb{Z}} g^\bullet \cap \mathrm{dom}_{\mathbb{Z}} k^\circ \neq \emptyset$*, where* $g^\bullet$ *and* k° *are defined by* (9.5) *and* (9.6)*. Then we have*

$$\inf\{g(p) - k(p) \mid p \in \mathbb{Z}^n\} = \sup\{k^\circ(x) - g^\bullet(x) \mid x \in \mathbb{Z}^n\}. \tag{9.9}$$

If this common value is finite, the infimum and the supremum are attained.

The Fenchel-type duality theorem can be formulated for real-valued functions $f, g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ and $h, k : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ as well; see Murota (2003, Theorem 8.21).

Remark 9.2. For the Fenchel-type duality, the two functions must be consistent with respect to the types ($M^\natural$ or $L^\natural$). In Example 9.1, f is $M^\natural$-convex and h is $L^\natural$-concave. This is also the case in Example 9.2. ∎

Remark 9.3. Whereas the $L^\natural$-separation and $M^\natural$-separation theorems are parallel or conjugate to each other in their statements, the Fenchel-type duality theorem is self-conjugate, in that the substitution of $f = g^\bullet$ and $h = k^\circ$ into (9.8) results in (9.9) by virtue of the biconjugacy $g = (g^\bullet)^\bullet$ and $k = (k^\circ)^\circ$ (Theorem 7.9). With the knowledge of M-/L-conjugacy (Section 7.2), these three duality theorems are almost equivalent to one another; once one of them is established, the other two theorems can be derived by relatively easy formal calculations. ∎

9.3. Concluding remarks of section 9

The significance of the duality theorems of this section in combinatorial optimization is mentioned here. Frank's discrete separation theorem (Frank,

1982) for submodular/supermodular set functions is a special case of the L$^\natural$-separation theorem. Frank's weight splitting theorem (Frank, 1981) for the weighted matroid intersection problem is a special case of the M$^\natural$-concave intersection problem. Edmonds's intersection theorem (Edmonds, 1970) for (poly) matroids in the integral case is a special case of the Fenchel-type duality (Theorem 9.6 (1)). Fujishige's Fenchel-type duality theorem (Fujishige, 1984) for submodular set functions is a special case of Theorem 9.6 (2). Murota (2003, Section 8.2.3) gives more details.

10. STABLE MARRIAGE AND ASSIGNMENT GAME

Two-sided matching (Roth & Sotomayor, 1990; Abdulkadiroğlu & Sönmez, 2013) affords a fairly general framework in game theory, including the stable matching of Gale & Shapley (1962) and the assignment model of Shapley & Shubik (1972) as special cases. An even more general framework has been proposed by Fujishige & Tamura (2007) in which the existence of an equilibrium is established on the basis of a novel duality-related property of M$^\natural$-concave functions. The results of Fujishige & Tamura (2007) are described in this section.[54]

10.1. Fujishige–Tamura model

Let P and Q be finite sets and put

$$E = P \times Q = \{(i,j) \mid i \in P, j \in Q\},$$

where we think of P as a set of workers and Q as a set of firms, respectively. We suppose that worker i works at firm j for x_{ij} units of time, gaining a salary s_{ij} per unit time. Then the *labor allocation* is represented by an integer vector

$$x = (x_{ij} \mid (i,j) \in E) \in \mathbb{Z}^E$$

and the salary by a real vector $s = (s_{ij} \mid (i,j) \in E) \in \mathbb{R}^E$. We are interested in the stability of a pair (x,s) in the sense to be made precise later.

For $i \in P$ and $j \in Q$ we put

$$E_{(i)} = \{i\} \times Q = \{(i,j) \mid j \in Q\}, \qquad E_{(j)} = P \times \{j\} = \{(i,j) \mid i \in P\},$$

[54] This section is based on Murota (2009, Section 11.10).

and for a vector y on E we denote by $y_{(i)}$ and $y_{(j)}$ the restrictions of y to $E_{(i)}$ and $E_{(j)}$, respectively. For example, for the labor allocation x we obtain

$$x_{(i)} = (x_{ij} \mid j \in Q) \in \mathbb{Z}^{E_{(i)}}, \qquad x_{(j)} = (x_{ij} \mid i \in P) \in \mathbb{Z}^{E_{(j)}}$$

and this convention also applies to the salary vector s to yield $s_{(i)}$ and $s_{(j)}$.

It is supposed that for each $(i,j) \in E$ lower and upper bounds on the salary s_{ij} are given, denoted by $\underline{\pi}_{ij} \in \mathbb{R} \cup \{-\infty\}$ and $\overline{\pi}_{ij} \in \mathbb{R} \cup \{+\infty\}$, where $\underline{\pi}_{ij} \leq \overline{\pi}_{ij}$. A salary s is called *feasible* if $\underline{\pi}_{ij} \leq s_{ij} \leq \overline{\pi}_{ij}$ for all $(i,j) \in E$. We put

$$\underline{\pi} = (\underline{\pi}_{ij} \mid (i,j) \in E) \in (\mathbb{R} \cup \{-\infty\})^E, \qquad \overline{\pi} = (\overline{\pi}_{ij} \mid (i,j) \in E) \in (\mathbb{R} \cup \{+\infty\})^E.$$

Each agent (worker or firm) $k \in P \cup Q$ evaluates his/her state $x_{(k)}$ of labor allocation in monetary terms through a function $f_k : \mathbb{Z}^{E_{(k)}} \to \mathbb{R} \cup \{-\infty\}$. Here the effective domain $\mathrm{dom}\, f_k = \{z \in \mathbb{Z}^{E_{(k)}} \mid f_k(z) > -\infty\}$ is assumed to satisfy the following natural condition:

$$\mathrm{dom}\, f_k \text{ is bounded and hereditary, with unique minimal element } \mathbf{0}, \tag{10.1}$$

where $\mathrm{dom}\, f_k$ being hereditary means that $\mathbf{0} \leq z \leq y \in \mathrm{dom}\, f_k$ implies $z \in \mathrm{dom}\, f_k$. In what follows we always assume that x is feasible in the sense that

$$x_{(i)} \in \mathrm{dom}\, f_i \quad (i \in P), \qquad x_{(j)} \in \mathrm{dom}\, f_j \quad (j \in Q).$$

A pair (x,s) of feasible allocation x and feasible salary s is called an *outcome*.

Example 10.1. The *stable marriage problem* can be formulated as a special case of the present setting. Put $\underline{\pi} = \overline{\pi} = \mathbf{0}$ and define $f_i : \mathbb{Z}^{E_{(i)}} \to \mathbb{R} \cup \{-\infty\}$ for $i \in P$ and $f_j : \mathbb{Z}^{E_{(j)}} \to \mathbb{R} \cup \{-\infty\}$ for $j \in Q$ as

$$\begin{aligned} f_i(y) &= \begin{cases} a_{ij} & (y = \chi_j, j \in Q), \\ 0 & (y = \mathbf{0}), \\ -\infty & (\text{otherwise}), \end{cases} \\ f_j(z) &= \begin{cases} b_{ij} & (z = \chi_i, i \in P), \\ 0 & (z = \mathbf{0}), \\ -\infty & (\text{otherwise}), \end{cases} \end{aligned} \tag{10.2}$$

where the vector $(a_{ij} \mid j \in Q) \in \mathbb{R}^Q$ represents (or, is an encoding of) the preference of "man" $i \in P$ over "women" Q, and $(b_{ij} \mid i \in P) \in \mathbb{R}^P$ the preference of "woman" $j \in Q$ over "men" P. Then a matching X is stable if and only if $(x,s) = (\chi_X, \mathbf{0})$ is stable in the present model. ∎

Example 10.2. The *assignment model* is a special case where $\underline{\pi} = (-\infty, \ldots, -\infty)$, $\overline{\pi} = (+\infty, \ldots, +\infty)$ and the functions f_i and f_j are of the form of (10.2) with some $a_{ij}, b_{ij} \in \mathbb{R}$ for all $i \in P, j \in Q$. ∎

10.2. Market equilibrium

Given an outcome (x, s) the payoff of worker $i \in P$ is defined to be the sum of his/her evaluation of $x_{(i)}$ and the total income from firms:

$$f_i(x_{(i)}) + \sum_{j \in Q} s_{ij} x_{ij} \qquad (=: (f_i + s_{(i)})(x_{(i)})). \tag{10.3}$$

Similarly, the payoff of firm $j \in Q$ is defined as

$$f_j(x_{(j)}) - \sum_{i \in P} s_{ij} x_{ij} \qquad (=: (f_j - s_{(j)})(x_{(j)})). \tag{10.4}$$

Each agent ($i \in P$ or $j \in Q$) naturally wishes to maximize his/her payoff function.[55]

A *market equilibrium* is defined as an outcome (x, s) that is stable under reasonable actions (i) by each worker i, (ii) by each firm j, and (iii) by each worker-firm pair (i, j). To be specific, we say that (x, s) is stable with respect to $i \in P$ if

$$(f_i + s_{(i)})(x_{(i)}) = \max\{(f_i + s_{(i)})(y) \mid y \leq x_{(i)}\}. \tag{10.5}$$

Similarly, (x, s) is said to be stable with respect to $j \in Q$ if

$$(f_j - s_{(j)})(x_{(j)}) = \max\{(f_j - s_{(j)})(z) \mid z \leq x_{(j)}\}. \tag{10.6}$$

In technical terms (x, s) is said to satisfy the *incentive constraint* if it satisfies (10.5) and (10.6).

The stability of (x, s) with respect to (i, j) is defined as follows. Suppose that worker i and firm j think of a change of their contract to a new salary $\alpha \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]_{\mathbb{R}}$ and a new working time of $\beta \in \mathbb{Z}_+$ units. Worker i will be happy with this contract if there exists $y \in \mathbb{Z}^{E_{(i)}}$ such that

$$y_j = \beta, \qquad y_k \leq x_{ik} \quad (k \in Q \setminus \{j\}), \tag{10.7}$$

$$(f_i + s_{(i)})(x_{(i)}) < (f_i + (s_{(i)}^{-j}, \alpha))(y), \tag{10.8}$$

[55] We have $(f_i + s_{(i)})(x_{(i)}) = f_i[+s_{(i)}](x_{(i)})$ and $(f_j - s_{(j)})(x_{(j)}) = f_j[-s_{(j)}](x_{(j)})$ in the notation of (4.20).

where $(s_{(i)}^{-j}, \alpha)$ denotes the vector $s_{(i)}$ with its j-th component replaced by α. Note that y means the new labor allocation of worker i with an increased payoff given on the right-hand side of (10.8). Similarly, firm j is motivated to make the new contract if there exists $z \in \mathbb{Z}^{E_{(j)}}$ such that

$$z_i = \beta, \qquad z_k \leq x_{kj} \quad (k \in P \setminus \{i\}), \tag{10.9}$$

$$(f_j - s_{(j)})(x_{(j)}) < (f_j - (s_{(j)}^{-i}, \alpha))(z), \tag{10.10}$$

where $(s_{(j)}^{-i}, \alpha)$ is the vector $s_{(j)}$ with its i-th component replaced by α. Then we say that (x,s) is stable with respect to (i,j) if there exists no (α, β, y, z) that simultaneously satisfies (10.7), (10.8), (10.9) and (10.10).

We now define an outcome (x,s) to be *stable* if, for every $i \in P$, $j \in Q$, (x,s) is (i) stable with respect to i, (ii) stable with respect to j, and (iii) stable with respect to (i,j). This is our concept of market equilibrium.

A remarkable fact, found by Fujishige & Tamura (2007), is that a market equilibrium exists if the functions f_k are M$^\natural$-concave.

Theorem 10.1. *Assume that $\underline{\pi} \leq \overline{\pi}$ and, for each $k \in P \cup Q$, f_k is an M$^\natural$-concave function satisfying* (10.1)*. Then a stable outcome $(x,s) \in \mathbb{Z}^E \times \mathbb{R}^E$ exists. Furthermore, we can take an integral $s \in \mathbb{Z}^E$ if $\underline{\pi} \in (\mathbb{Z} \cup \{-\infty\})^E$, $\overline{\pi} \in (\mathbb{Z} \cup \{+\infty\})^E$, and f_k is integer-valued for every $k \in P \cup Q$.*

10.3. Technical ingredients

The technical ingredients of the above theorem can be divided into the following two theorems, due to Fujishige & Tamura (2007). Note also that sufficiency part of Theorem 10.2 (which we need here) is independent of M$^\natural$-concavity.

Theorem 10.2. *Under the same assumption as in Theorem 10.1 let x be a feasible allocation. Then (x,s) is a stable outcome for some s if and only if there exist $p \in \mathbb{R}^E$, $u = (u_{(i)} \mid i \in P) \in (\mathbb{Z} \cup \{+\infty\})^E$ and $v = (v_{(j)} \mid j \in Q) \in (\mathbb{Z} \cup \{+\infty\})^E$ such that*

$$x_{(i)} \in \arg\max\{(f_i + p_{(i)})(y) \mid y \leq u_{(i)}\}, \tag{10.11}$$

$$x_{(j)} \in \arg\max\{(f_j - p_{(j)})(z) \mid z \leq v_{(j)}\}, \tag{10.12}$$

$$\underline{\pi} \leq p \leq \overline{\pi}, \tag{10.13}$$

$$(i,j) \in E, u_{ij} < +\infty \implies p_{ij} = \underline{\pi}_{ij}, v_{ij} = +\infty, \tag{10.14}$$

$$(i,j) \in E, v_{ij} < +\infty \implies p_{ij} = \overline{\pi}_{ij}, u_{ij} = +\infty. \tag{10.15}$$

Moreover, (x, p) is a stable outcome for any (x, p, u, v) satisfying the above conditions.

Theorem 10.3. *Under the same assumption as in Theorem 10.1 there exists (x, p, u, v) that satisfies (10.11)–(10.15). Furthermore, we can take an integral $p \in \mathbb{Z}^E$ if $\underline{\pi} \in (\mathbb{Z} \cup \{-\infty\})^E$, $\overline{\pi} \in (\mathbb{Z} \cup \{+\infty\})^E$, and f_k is integer-valued for every $k \in P \cup Q$.*

It is worth while noting that the essence of Theorem 10.3 is an intersection-type theorem for a pair of M$^\natural$-concave functions, Theorem 10.4 below, due to Fujishige & Tamura (2007). Indeed Theorem 10.3 can be derived easily from Theorem 10.4 applied to

$$f_P(x) = \sum_{i \in P} f_i(x_{(i)}), \qquad f_Q(x) = \sum_{j \in Q} f_j(x_{(j)}). \tag{10.16}$$

Theorem 10.4. *Assume $\underline{\pi} \leq \overline{\pi}$ for $\underline{\pi} \in (\mathbb{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbb{R} \cup \{+\infty\})^E$, and let $f, g : \mathbb{Z}^E \to \mathbb{R} \cup \{-\infty\}$ be M$^\natural$-concave functions such that the effective domains are bounded and hereditary, with unique minimal element $\mathbf{0}$. Then there exist $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$, $p \in \mathbb{R}^E$, $u \in (\mathbb{Z} \cup \{+\infty\})^E$ and $v \in (\mathbb{Z} \cup \{+\infty\})^E$ such that*

$$x \in \arg\max\{(f + p)(y) \mid y \leq u\}, \tag{10.17}$$
$$x \in \arg\max\{(g - p)(z) \mid z \leq v\}, \tag{10.18}$$
$$\underline{\pi} \leq p \leq \overline{\pi}, \tag{10.19}$$
$$e \in E, u_e < +\infty \implies p_e = \underline{\pi}_e, v_e = +\infty, \tag{10.20}$$
$$e \in E, v_e < +\infty \implies p_e = \overline{\pi}_e, u_e = +\infty. \tag{10.21}$$

Furthermore, we can take an integral $p \in \mathbb{Z}^E$ if $\underline{\pi} \in (\mathbb{Z} \cup \{-\infty\})^E$, $\overline{\pi} \in (\mathbb{Z} \cup \{+\infty\})^E$, and f and g are integer-valued.

Remark 10.1. Two special cases of Theorem 10.4 are worth mentioning.

- The first case is where $\underline{\pi} = (-\infty, \ldots, -\infty)$ and $\overline{\pi} = (+\infty, \ldots, +\infty)$. In this case, (10.19) is void, and we must have $u_e = v_e = +\infty$ for all $e \in E$ by (10.20) and (10.21). Therefore, the assertion of Theorem 10.4 reduces to: There exist $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and $p \in \mathbb{R}^E$ such that $x \in \arg\max(f + p)$ and $x \in \arg\max(g - p)$, which coincides with the M$^\natural$-concave intersection theorem (Theorem 9.4).

- The second case is where $\underline{\pi} = \overline{\pi} = \mathbf{0}$, which corresponds to the discrete concave stable marriage model of Eguchi et al. (2003). Let w be a vector such that $y \leq w$ for all $y \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$. By (10.19) we must have $p_e = 0$ for all $e \in E$. For each $e \in E$, we must have $u_e = +\infty$ or $v_e = +\infty$ (or both) by (10.20) and (10.21). Therefore, the assertion of Theorem 10.4 reduces to: There exist $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$, $u \in \mathbb{Z}^E$, and $v \in \mathbb{Z}^E$ such that $w = u \vee v$, $x \in \arg\max\{f(y) \mid y \leq u\}$, and $x \in \arg\max\{g(z) \mid z \leq v\}$. This is the main technical result of Eguchi et al. (2003) that implies the existence of a stable allocation in their model. ∎

10.4. Concluding remarks of section 10

The Fujishige–Tamura model contains several recently proposed matching models such as Eriksson & Karlander (2000), Fleiner (2001), Sotomayor (2002) as well as Eguchi & Fujishige (2002), Eguchi et al. (2003), Fujishige & Tamura (2006) as special cases. In particular, the hybrid model of Eriksson & Karlander (2000), with flexible and rigid agents, is a special case where P and Q are partitioned as $P = P_\infty \cup P_0$ and $Q = Q_\infty \cup Q_0$, and $\underline{\pi}_{ij} = -\infty$, $\overline{\pi}_{ij} = +\infty$ for $(i, j) \in P_\infty \times Q_\infty$ and $\underline{\pi}_{ij} = \overline{\pi}_{ij} = 0$ for other (i, j). Realistic constraints on matchings such as lower quotas can be expressed in terms of matroids (Fleiner, 2001; Fleiner & Kamiyama, 2016; Kojima et al., 2014; Goto et al., 2016; Yokoi, 2016).

11. VALUATED ASSIGNMENT PROBLEM

As we have seen in Sections 3.6 and 6.2, M$^\natural$-concave set functions are amenable to (bipartite) graph structures. As a further step in this direction we describe the valuated (independent) assignment problem, introduced by Murota (1996b,c). In contrast to the original formulation of the problem in terms of valuated matroids (or M-convex set functions), we present here a reformulation in terms of M$^\natural$-concave set functions for the convenience of applications to economics and game theory.

11.1. Problem description

The problem we consider is the following:[56]

[56] This problem is a variant of the valuated independent assignment problem.

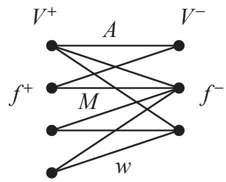

Figure 4: Valuated assignment problem

[M$^\natural$-concave matching problem] Given a bipartite graph $G = (V^+, V^-; A)$, a pair of M$^\natural$-concave set functions $f^+ : 2^{V^+} \to \mathbb{R} \cup \{-\infty\}$ and $f^- : 2^{V^+} \to \mathbb{R} \cup \{-\infty\}$, and a weight function $w : A \to \mathbb{R}$ (see Fig. 4), find a matching M $(\subseteq A)$ that maximizes

$$w(M) + f^+(\partial^+ M) + f^-(\partial^- M), \tag{11.1}$$

where $w(M) = \sum\{w(a) \mid a \in M\}$, and $\partial^+ M$ (resp., $\partial^- M$) denotes the set of the vertices in V^+ (resp., V^-) incident to M. For (11.1) to be finite, we have implicit constraints that

$$\partial^+ M \in \mathrm{dom}\, f^+, \qquad \partial^- M \in \mathrm{dom}\, f^-. \tag{11.2}$$

In applications the empty set often belongs to $\mathrm{dom}\, f^+$ (resp., $\mathrm{dom}\, f^-$), in which case $\mathrm{dom}\, f^+$ (resp., $\mathrm{dom}\, f^-$) forms the family of independent sets of a matroid. If $f^+ \equiv 0$ and $f^- \equiv 0$ (with $\mathrm{dom}\, f^+ = 2^{V^+}$ and $\mathrm{dom}\, f^- = 2^{V^-}$), this problem coincides with the conventional weighted matching problem.

An important special case of the M$^\natural$-concave matching problem arises from a very special underlying graph $G_\equiv = (V^+, V^-; A_\equiv)$ that represents a one-to-one correspondence between V^+ and V^-. In other words, given a pair of M$^\natural$-concave set functions $f_1, f_2 : 2^V \to \mathbb{R} \cup \{-\infty\}$ and a weight function $w : V \to \mathbb{R}$, let V^+ and V^- be disjoint copies of V and $A_\equiv = \{(v^+, v^-) \mid v \in V\}$, where $v^+ \in V^+$ and $v^- \in V^-$ denote the copies of $v \in V$. The given functions f_1 and f_2 are regarded as set functions on V^+ and V^-, respectively. Then we obtain the following problem:

[M$^\natural$-concave intersection problem] Given a pair of M$^\natural$-concave set functions $f_1, f_2 : 2^V \to \mathbb{R} \cup \{-\infty\}$ and a weight function $w : V \to \mathbb{R}$, find a subset X that maximizes

$$w(X) + f_1(X) + f_2(X), \tag{11.3}$$

where $w(X) = \sum_{v \in X} w(v)$.

11.2. Optimality criterion by potentials

We show the optimality criterion for the M$^\natural$-concave matching problem in terms of potentials, where a *potential* means a function $p : V^+ \cup V^- \to \mathbb{R}$ (or a vector $p \in \mathbb{R}^{V^+ \cup V^-}$) on the vertex set $V^+ \cup V^-$. In the following theorem due to Murota (1996c) (see also Murota, 2000b, Theorem 5.2.39), the statement (1) refers to the existence of an appropriate potential, whereas its reformulation in (2) reveals the duality nature.[57] For each arc $a = (u,v) \in A$, $\partial^+ a$ denotes the initial (tail) vertex of a, and $\partial^- a$ the terminal (head) vertex of a, i.e., $\partial^+ a = u \in V^+$ and $\partial^- a = v \in V^-$, where all the arcs are assumed to be directed from V^+ to V^-.

Theorem 11.1 (Potential criterion)**.** *Let M be a matching in $G = (V^+, V^-; A)$ satisfying* (11.2) *for the M$^\natural$-concave matching problem to maximize* (11.1).
(1) *M is an optimal matching if and only if there exists a potential $p : V^+ \cup V^- \to \mathbb{R}$ such that*

(i) $w(a) - p(\partial^+ a) + p(\partial^- a) \begin{cases} \leq 0 & (a \in A), \\ = 0 & (a \in M), \end{cases}$

(ii) *$\partial^+ M$ is a maximizer of $f^+[+p^+]$,*

(iii) *$\partial^- M$ is a maximizer of $f^-[-p^-]$,*

where p^+ and p^- are the restrictions of p to V^+ and V^-, respectively, and $f^+[+p^+]$ and $f^-[-p^-]$ are defined by

$$f^+[+p^+](X) = f^+(X) + \sum\{p(u) \mid u \in X\} \qquad (X \subseteq V^+),$$
$$f^-[-p^-](Y) = f^-(Y) - \sum\{p(v) \mid v \in Y\} \qquad (Y \subseteq V^-).$$

(2)

$$\begin{aligned} &\max_M \{w(M) + f^+(\partial^+ M) + f^-(\partial^- M)\} = \\ &\min_p \{\max(f^+[+p^+]) + \max(f^-[-p^-]) \mid \\ &w(a) - p(\partial^+ a) + p(\partial^- a) \leq 0 \ (a \in A)\}. \end{aligned}$$

(3) *If f^+, f^- and w are all integer-valued, the potential p in* (1) *and* (2) *can be chosen to be integer-valued.*
(4) *Let p be a potential that satisfies* (i)–(iii) *in* (1) *for some (optimal) matching $M = M_0$. A matching M' is optimal if and only if it satisfies* (i)–(iii) *(with M replaced by M').*

[57] Compare the identity in (2) with the Fenchel-type duality in Theorem 9.6.

In connection to (ii) and (iii) in (1) in Theorem 11.1, Theorem 3.4 shows:

$$X \in \arg\max(f^+[+p^+]) \iff$$
$$\begin{cases} f^+(X) - f^+(X-u+v) + p(u) - p(v) \geq 0 & (\forall u \in X,\ \forall v \in V^+ \setminus X), \\ f^+(X) - f^+(X-u) + p(u) \geq 0 & (\forall u \in X), \\ f^+(X) - f^+(X+v) - p(v) \geq 0 & (\forall v \in V^+ \setminus X), \end{cases} \tag{11.4}$$

$$Y \in \arg\max(f^-[-p^-]) \iff$$
$$\begin{cases} f^-(Y) - f^-(Y-u+v) - p(u) + p(v) \geq 0 & (\forall u \in Y,\ \forall v \in V^- \setminus Y), \\ f^-(Y) - f^-(Y-u) - p(u) \geq 0 & (\forall u \in Y), \\ f^-(Y) - f^-(Y+v) + p(v) \geq 0 & (\forall v \in V^- \setminus Y). \end{cases} \tag{11.5}$$

These expressions are crucial in deriving the second optimality criterion (Theorem 11.3) in Section 11.3 and in designing efficient algorithms for the M$^\natural$-concave matching problem.

The optimality condition for the M$^\natural$-concave intersection problem (11.3) deserves a separate statement in the form of weight splitting, though it is an immediate corollary of the above theorem.

Theorem 11.2 (Weight splitting for M$^\natural$-concave intersection)**.**
(1) *A subset $X \subseteq V$ maximizes $w(X) + f_1(X) + f_2(X)$ if and only if there exist $w_1, w_2 : V \to \mathbb{R}$ such that*

(i) [*"weight splitting"*] $w(v) = w_1(v) + w_2(v)$ $(v \in V)$,

(ii) *X is a maximizer of $f_1[+w_1]$,*

(iii) *X is a maximizer of $f_2[+w_2]$.*

(2) $\max\limits_X \{w(X) + f_1(X) + f_2(X)\}$
$$= \min_{w_1,w_2} \{\max(f_1[+w_1]) + \max(f_2[+w_2]) \mid w(v) = w_1(v) + w_2(v)\ (v \in V)\}.$$

(3) *If f_1, f_2 and w are all integer-valued, we may assume that $w_1, w_2 : V \to \mathbb{Z}$.*

11.3. Optimality criterion by negative-cycles

As the second criterion for optimality we describe the negative-cycle criterion. First we need to introduce the auxiliary graph $G_M = (\tilde{V}, A_M)$ associated with a matching M satisfying $\partial^+ M \in \operatorname{dom} f^+$ and $\partial^- M \in \operatorname{dom} f^-$ in (11.2). Define $X = \partial^+ M$ and $Y = \partial^- M$.

The vertex set $\tilde{V}$ of the auxiliary graph G_M is given by $\tilde{V} = V^+ \cup V^- \cup \{s^+, s^-\}$, where s^+ and s^- are new vertices often referred to as "source vertex" and "sink vertex" respectively. The arc set A_M consists of nine disjoint parts:

$$A_M = (A^\circ \cup M^\circ) \cup (A^+ \cup F^+ \cup S^+) \cup (A^- \cup F^- \cup S^-) \cup R,$$

where[58]

$$\begin{array}{rcll} A^\circ &=& \{a \mid a \in A\} & \text{(copy of } A\text{)}, \\ M^\circ &=& \{\overline{a} \mid a \in M\} & (\overline{a}\text{: reorientation of } a), \\ A^+ &=& \{(u,v) \mid u \in X,\ v \in V^+ \setminus X,\ X - u + v \in \mathrm{dom}\, f^+\}, & \\ F^+ &=& \{(u,s^+) \mid u \in X\}, & \\ S^+ &=& \{(s^+,v) \mid v \in V^+ \setminus X\}, & \\ A^- &=& \{(v,u) \mid u \in Y,\ v \in V^- \setminus Y,\ Y - u + v \in \mathrm{dom}\, f^-\}, & \\ F^- &=& \{(s^-,u) \mid u \in Y\}, & \\ S^- &=& \{(v,s^-) \mid v \in V^- \setminus Y\}, & \\ R &=& \{(s^-,s^+)\}. & \end{array} \tag{11.6}$$

The arc length $\ell_M(a)$ for $a \in A_M$ is defined by

$$\ell_M(a) = \begin{cases} -w(a) & (a \in A^\circ), \\ w(\overline{a}) & (a = (u,v) \in M^\circ,\ \overline{a} = (v,u) \in M), \\ f^+(X) - f^+(X - u + v) & (a = (u,v) \in A^+), \\ f^+(X) - f^+(X - u) & (a = (u,s^+) \in F^+), \\ f^+(X) - f^+(X + v) & (a = (s^+,v) \in S^+), \\ f^-(Y) - f^-(Y - u + v) & (a = (v,u) \in A^-), \\ f^-(Y) - f^-(Y - u) & (a = (s^-,u) \in F^-), \\ f^-(Y) - f^-(Y + v) & (a = (v,s^-) \in S^-), \\ 0 & (a = (s^-,s^+) \in R). \end{cases} \tag{11.7}$$

A directed cycle in G_M of a negative length with respect to the arc length ℓ_M is called a *negative cycle*. As is well known in network flow theory, there exists no negative cycle in (G_M, ℓ_M) if and only if there exists a potential $p : \tilde{V} \to \mathbb{R}$ such that

$$\ell_M(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \qquad (a \in A_M), \tag{11.8}$$

where $\partial^+ a$ denotes the initial (tail) vertex of a, and $\partial^- a$ the terminal (head) vertex of a. With the use of (11.4), (11.5) and (11.8), Theorem 11.1 is translated into the following theorem; see Remark 11.1. This theorem gives an optimality criterion in terms of negative cycles; see Murota (1996c) and Murota (2000b, Theorem 5.2.42).

[58] The *reorientation* of an arc $a = (u,v)$ means the arc (v,u), to be denoted as $\overline{a}$.

Theorem 11.3 (Negative-cycle criterion). *In the $M^\natural$-concave matching problem to maximize (11.1), a matching M satisfying (11.2) is optimal if and only if there exists in the auxiliary graph G_M no negative cycle with respect to the arc length ℓ_M.*

Remark 11.1. The condition (11.8) for $a \in (F^+ \cup S^+) \cup (F^- \cup S^-)$ refers to $p(s^+)$ and $p(s^-)$, while the potential p in Theorem 11.1 is defined only on $V^+ \cup V^-$. To derive (11.8) from Theorem 11.1 we may define $p(s^+) = p(s^-) = 0$. Indeed, the conditions imposed on $p(s^+)$ by (11.8) are

$$f^+(X) - f^+(X-u) + p(u) - p(s^+) \geq 0 \qquad (u \in X),$$
$$f^+(X) - f^+(X+v) + p(s^+) - p(v) \geq 0 \qquad (v \in V^+ \setminus X),$$

which are satisfied by (11.4) if $p(s^+) = 0$. Similarly for $p(s^-)$. ∎

11.4. Concluding remarks of section 11

Theorems 11.1 and 11.3 contain several standard results in matroid optimization, such as Frank's weight splitting theorem (Frank, 1981) for the weighted matroid intersection problem. The proofs of Theorems 11.1 and 11.3 can be found in Murota (1996c) and Murota (2000b, Section 5.2). There are two key lemmas, called "upper-bound lemma" and "unique-max lemma," which capture the essential properties inherent in M-concavity. On the basis of these optimality criteria efficient algorithms can be designed for the $M^\natural$-concave matching problem. For algorithmic issues, see Murota (1996b) and Murota (2000b, Section 6.2).

The valuated matching problem treated in this section is generalized to the submodular flow problem in Section 12.

12. SUBMODULAR FLOW PROBLEM

12.1. Submodular flow problem

Let $G = (V, A)$ be a directed graph with vertex set V and arc set A. Suppose that each arc $a \in A$ is associated with upper-capacity $\overline{c}(a)$, lower-capacity $\underline{c}(a)$, and cost $\gamma(a)$ per unit flow. Furthermore, for each vertex $v \in V$, the amount of flow supply at v is specified by $x(v)$.

The minimum cost flow problem is to find a flow $\xi = (\xi(a) \mid a \in A)$ that minimizes the total cost $\langle \gamma, \xi \rangle_A = \sum_{a \in A} \gamma(a)\xi(a)$ subject to the capacity constraint $\underline{c}(a) \leq \xi(a) \leq \overline{c}(a)$ $(a \in A)$ and the supply specification. Here the supply specification means a constraint that the boundary $\partial\xi$ of ξ defined by

$$\partial\xi(v) = \sum\{\xi(a) \mid a \in \delta^+ v\} - \sum\{\xi(a) \mid a \in \delta^- v\} \qquad (v \in V) \tag{12.1}$$

should be equal to a given value $x(v)$, where $\delta^+ v$ and $\delta^- v$ denote the sets of arcs leaving (going out of) v and entering (coming into) v, respectively. We can interpret $x(v) = \partial\xi(v)$ as the net amount of flow entering the network at v from outside.

We consider the integer flow problem, which is described by an integer-valued upper-capacity $\overline{c} : A \to \mathbb{Z} \cup \{+\infty\}$, an integer-valued lower-capacity $\underline{c} : A \to \mathbb{Z} \cup \{-\infty\}$, a real-valued cost function $\gamma : A \to \mathbb{R}$, and an integer supply vector $x : V \to \mathbb{Z}$, where it is assumed that $\overline{c}(a) \geq \underline{c}(a)$ for each $a \in A$. The variable to be optimized is an integral flow $\xi : A \to \mathbb{Z}$.

[Minimum cost flow problem MCFP (linear arc cost)][59]

$$\begin{aligned} \text{Minimize} \quad & \Gamma_0(\xi) = \sum_{a \in A} \gamma(a)\xi(a) && (12.2)\\ \text{subject to} \quad & \underline{c}(a) \leq \xi(a) \leq \overline{c}(a) \qquad (a \in A), && (12.3)\\ & \partial\xi = x, && (12.4)\\ & \xi(a) \in \mathbb{Z} \qquad (a \in A). && (12.5) \end{aligned}$$

A generalization of the minimum cost flow problem MCFP is obtained by relaxing the supply specification $\partial\xi = x$ to the constraint that the flow boundary $\partial\xi$ should belong to a given subset B of $\mathbb{Z}^V$ representing "feasible" or "admissible" supplies:[60]

$$\partial\xi \in B. \tag{12.6}$$

Such problem is called the *submodular flow problem*, if B is an M-convex set (integral base polyhedron; see Remark 4.2).[61] This problem is introduced by Edmonds & Giles (1977).

[59] MCFP stands for Minimum Cost Flow Problem.

[60] By the flow conservation law, the sum of the components of $\partial\xi$ is equal to zero, i.e., $\partial\xi(V) = 0$, for any flow ξ. Accordingly we assume that B is contained in the hyperplane $\{x \in \mathbb{R}^V \mid x(V) = 0\}$.

[61] In the conventional formulation (Fujishige, 2005, Chapter III), the M-convex set B is given by an integer-valued submodular set function that describes B; see also Murota (2003, Section 4.4).

[Submodular flow problem MSFP_1 (linear arc cost)][62]

$$\text{Minimize} \qquad \Gamma_1(\xi) = \sum_{a \in A} \gamma(a)\xi(a) \tag{12.7}$$

$$\text{subject to} \qquad \underline{c}(a) \leq \xi(a) \leq \overline{c}(a) \qquad (a \in A), \tag{12.8}$$

$$\partial\xi \in B, \tag{12.9}$$

$$\xi(a) \in \mathbb{Z} \qquad (a \in A). \tag{12.10}$$

A further generalization of the problem is obtained by introducing a cost function for the flow boundary $\partial\xi$ rather than merely imposing the constraint $\partial\xi \in B$. Namely, with a function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ we add a new term $f(\partial\xi)$ to the objective function, thereby imposing constraint $\partial\xi \in B = \mathrm{dom}\, f$ implicitly. If the function f is M-convex, the generalized problem is called the *M-convex submodular flow problem*, introduced by Murota (1999).

[M-convex submodular flow problem MSFP_2 (linear arc cost)]

$$\text{Minimize} \qquad \Gamma_2(\xi) = \sum_{a \in A} \gamma(a)\xi(a) + f(\partial\xi) \tag{12.11}$$

$$\text{subject to} \qquad \underline{c}(a) \leq \xi(a) \leq \overline{c}(a) \qquad (a \in A), \tag{12.12}$$

$$\partial\xi \in \mathrm{dom}\, f, \tag{12.13}$$

$$\xi(a) \in \mathbb{Z} \qquad (a \in A). \tag{12.14}$$

The special case of the M-convex submodular flow problem MSFP_2 with a $\{0, +\infty\}$-valued f reduces to the submodular flow problem MSFP_1.

A still further generalization is possible by replacing the linear arc cost in Γ_2 with a separable convex function. Namely, using univariate convex functions[63] $f_a : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ $(a \in A)$, we consider $\sum_{a \in A} f_a(\xi(a))$ instead of $\sum_{a \in A} \gamma(a)\xi(a)$ to obtain MSFP_3 below.

[M-convex submodular flow problem MSFP_3 (nonlinear arc cost)]

$$\text{Minimize} \qquad \Gamma_3(\xi) = \sum_{a \in A} f_a(\xi(a)) + f(\partial\xi) \tag{12.15}$$

$$\text{subject to} \qquad \xi(a) \in \mathrm{dom}\, f_a \qquad (a \in A), \tag{12.16}$$

$$\partial\xi \in \mathrm{dom}\, f, \tag{12.17}$$

$$\xi(a) \in \mathbb{Z} \qquad (a \in A). \tag{12.18}$$

[62] MSFP stands for M-convex Submodular Flow Problem. We use denotation MSFP_i with $i = 1, 2, 3$ to indicate the hierarchy of generality in the problems.

[63] $f_a(t-1) + f_a(t+1) \geq 2 f_a(t)$ for all integers t.

Obviously, MSFP_2 is a special case of MSFP_3 with

$$f_a(t) = \begin{cases} \gamma(a)t & (t \in [\underline{c}(a), \overline{c}(a)]_{\mathbb{Z}}), \\ +\infty & (\text{otherwise}). \end{cases} \tag{12.19}$$

Conversely, MSFP_3 can be put into the form MSFP_2; see Remark 12.1.

Remark 12.1. Problem MSFP_3 on $G = (V, A)$ can be written in the form of MSFP_2 on a larger graph $\tilde{G} = (\tilde{V}, \tilde{A})$. We replace each arc $a = (u, v) \in A$ with a pair of arcs, $a^+ = (u, v_a^-)$ and $a^- = (v_a^+, v)$, where v_a^+ and v_a^- are newly introduced vertices. Accordingly, we have $\tilde{A} = \{a^+, a^- \mid a \in A\}$ and $\tilde{V} = V \cup \{v_a^+, v_a^- \mid a \in A\}$. For each $a \in A$ we consider a function $\tilde{f}_a : \mathbb{Z}^2 \to \mathbb{R} \cup \{+\infty\}$ given by

$$\tilde{f}_a(t, s) = \begin{cases} f_a(t) & (t + s = 0), \\ +\infty & (\text{otherwise}), \end{cases}$$

and define $\tilde{f} : \mathbb{Z}^{\tilde{V}} \to \mathbb{R} \cup \{+\infty\}$ by

$$\tilde{f}(\tilde{x}) = \sum_{a \in A} \tilde{f}_a(\tilde{x}(v_a^+), \tilde{x}(v_a^-)) + f(\tilde{x}|_V) \qquad (\tilde{x} \in \mathbb{Z}^{\tilde{V}}),$$

where $\tilde{x}|_V$ denotes the restriction of $\tilde{x}$ to V. For a flow $\tilde{\xi} : \tilde{A} \to \mathbb{Z}$, we have $\tilde{\xi}(a^+) = \tilde{\xi}(a^-)$ if $(\partial \tilde{\xi}(v_a^+), \partial \tilde{\xi}(v_a^-)) \in \mathrm{dom}\, \tilde{f}_a$. Problem MSFP_3 is thus reduced to MSFP_2 with objective function $\tilde{\Gamma}_2(\tilde{\xi}) = \tilde{f}(\partial \tilde{\xi})$, where the function $\tilde{f}$ is M-convex. ∎

Remark 12.2. The M$^\natural$-concave intersection problem (Section 9.1) can be formulated as an M-convex submodular flow problem. Suppose we want to maximize the sum $f_1(x) + f_2(x)$ of two M$^\natural$-concave functions $f_1, f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$. Let $\tilde{f}_1, \tilde{f}_2 : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \{-\infty\}$ be the associated M-concave functions; see (4.18). We consider an M-convex submodular flow problem on the bipartite graph $G = (V_1 \cup V_2, A)$ in Fig. 5, where $V_i = \{v_{i0}, v_{i1}, \ldots, v_{in}\}$ for $i = 1, 2$ and $A = \{(v_{1j}, v_{2j}) \mid j = 0, 1, \ldots, n\}$. The boundary cost function $f : \mathbb{Z}^{V_1} \times \mathbb{Z}^{V_2} \to \mathbb{R} \cup \{+\infty\}$ is defined by $f(x_1, x_2) = -\tilde{f}_1(x_1) - \tilde{f}_2(-x_2)$ for $x_1 \in \mathbb{Z}^{V_1}$ and $x_2 \in \mathbb{Z}^{V_2}$, which is an M-convex function. The arc costs are identically zero and no capacity constraints are imposed ($\gamma(a) = 0$, $\overline{c}(a) = +\infty$, $\underline{c}(a) = -\infty$ for all $a \in A$). Since $x_1 = -x_2$ if $(x_1, x_2) = \partial \xi$ for a flow ξ in this network, this M-convex submodular flow problem is equivalent to the problem of maximizing $f_1(x) + f_2(x)$. Theorem 9.4 for the M-convex intersection problem can be regarded as a special case of Theorem 12.1 for the M-convex submodular flow problem. ∎

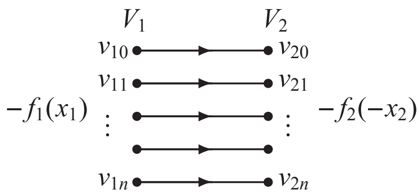

Figure 5: M-convex submodular flow problem for M$^\natural$-concave intersection problem

In subsequent sections we show optimality criteria for the M-convex submodular flow problem in terms of potentials and negative cycles.

12.2. Optimality criterion by potentials

We show the optimality criterion for the M-convex submodular flow problem MSFP$_3$ in terms of potentials. A *potential* means a function $p : V \to \mathbb{R}$ (or a vector $p \in \mathbb{R}^V$) on the vertex set V. The *coboundary* of a potential p is a function $\delta p : A \to \mathbb{R}$ on the arc set A defined by

$$\delta p(a) = p(\partial^+ a) - p(\partial^- a) \qquad (a \in A), \tag{12.20}$$

where, for each arc $a \in A$, $\partial^+ a$ denotes the initial (tail) vertex of a and, $\partial^- a$ the terminal (head) vertex of a. The following theorem is due to Murota (1999); see also Murota (2003, Section 9.4).

Theorem 12.1 (Potential criterion)**.** *Consider the M-convex submodular flow problem* MSFP$_3$.
(1) *For a feasible flow $\xi : A \to \mathbb{Z}$, two conditions* (OPT) *and* (POT) *below are equivalent.*

(OPT) ξ *is an optimal flow.*

(POT) *There exists a potential $p : V \to \mathbb{R}$ such that*[64]

(i) $\xi(a) \in \arg\min f_a[+\delta p(a)]$ *for every $a \in A$, and*

(ii) $\partial\xi \in \arg\min f[-p]$.

(2) *Suppose that a potential $p : V \to \mathbb{R}$ satisfies* (i) *and* (ii) *above for an optimal flow ξ. A feasible flow ξ' is optimal if and only if*

[64] By notation (4.20), $f_a[+\delta p(a)]$ means the function defined as $f_a[+\delta p(a)](t) = f_a(t) + (p(\partial^+ a) - p(\partial^- a))t$ for all $t \in \mathbb{Z}$.

(i) $\xi'(a) \in \arg\min f_a[+\delta p(a)]$ *for every* $a \in A$, *and*
(ii) $\partial\xi' \in \arg\min f[-p]$.

(3) *If the cost functions* f_a $(a \in A)$ *and* f *are integer-valued, there exists an integer-valued potential* $p : V \to \mathbb{Z}$ *in* (POT). *Moreover, the set of integer-valued optimal potentials,*

$$\Pi^* = \{p \mid p : \textit{integer-valued optimal potential}\},$$

is an L-convex set.[65]

In connection to (i) and (ii) in (POT) in Theorem 12.1, note the equivalences:

$$\begin{aligned}
&\xi(a) \in \arg\min f_a[+\delta p(a)] \iff \text{for } d = \pm 1\\
&f_a(\xi(a)+d) - f_a(\xi(a)) + d[p(\partial^+ a) - p(\partial^- a)] \geq 0, \qquad (12.21)\\
&\partial\xi \in \arg\min f[-p] \iff\\
&\Delta f(\partial\xi; v, u) + p(u) - p(v) \geq 0 \qquad (\forall u, v \in V), \qquad (12.22)
\end{aligned}$$

where

$$\Delta f(z; v, u) = f(z + \chi_v - \chi_u) - f(z) \qquad (z \in \operatorname{dom} f; u, v \in V). \qquad (12.23)$$

These expressions are crucial in deriving the second optimality criterion (Theorem 12.2) in Section 12.3 and in designing efficient algorithms for the M-convex submodular flow problem.

12.3. Optimality criterion by negative cycles

The optimality of an M-convex submodular flow can also be characterized by the nonexistence of negative cycles in an auxiliary network. This fact leads to the cycle-cancelling algorithm. We consider the M-convex submodular flow problem MSFP_2 that has a linear arc cost. This is not restrictive, since MSFP_3 can be put in the form of MSFP_2 (Remark 12.1).

For a feasible flow $\xi : A \to \mathbb{Z}$ we define an auxiliary network as follows. Let $G_\xi = (V, A_\xi)$ be a directed graph with vertex set V and arc set $A_\xi =$

[65] A nonempty set $P \subseteq \mathbb{Z}^n$ is called an *L-convex set* if it is an L$^\natural$-convex set (Remark 7.4) such that $p \in P$ implies $p + \mathbf{1}, p - \mathbf{1} \in P$. See Murota (2003, Chapter 5) for details.

$A_\xi^\circ \cup B_\xi^\circ \cup C_\xi$ consisting of three disjoint parts:

$$\begin{aligned} A_\xi^\circ &= \{a \mid a \in A,\ \xi(a) < \overline{c}(a)\}, \\ B_\xi^\circ &= \{\overline{a} \mid a \in A,\ \underline{c}(a) < \xi(a)\} \qquad (\overline{a}\text{: reorientation of } a), \\ C_\xi &= \{(u,v) \mid u,v \in V,\ u \neq v,\ \partial\xi - (\chi_u - \chi_v) \in \mathrm{dom}\, f\}. \end{aligned} \tag{12.24}$$

We define an arc length function $\ell_\xi : A_\xi \to \mathbb{R}$ by

$$\ell_\xi(a) = \begin{cases} \gamma(a) & (a \in A_\xi^\circ), \\ -\gamma(\overline{a}) & (a \in B_\xi^\circ,\ \overline{a} \in A), \\ \Delta f(\partial\xi; v, u) & (a = (u,v) \in C_\xi). \end{cases} \tag{12.25}$$

We refer to (G_ξ, ℓ_ξ) as the auxiliary network.

A directed cycle in G_ξ of a negative length with respect to the arc length ℓ_ξ is called a *negative cycle*. As is well known in network flow theory, there exists no negative cycle in (G_ξ, ℓ_ξ) if and only if there exists a potential $p : V \to \mathbb{R}$ such that

$$\ell_\xi(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \qquad (a \in A_\xi). \tag{12.26}$$

With the use of (12.21), (12.22) and (12.26), Theorem 12.1 is translated into the following theorem which gives an optimality criterion in terms of negative cycles; see Murota (1999) and also Murota (2003, Section 9.5).

Theorem 12.2 (Negative-cycle criterion)**.** *For a feasible flow $\xi : A \to \mathbb{Z}$ to the M-convex submodular flow problem* MSFP$_2$*, the conditions* (OPT) *and* (NNC) *below are equivalent.*

(OPT) *ξ is an optimal flow.*

(NNC) *There exists no negative cycle in the auxiliary network (G_ξ, ℓ_ξ) with ℓ_ξ of* (12.25).

12.3.1. Cycle cancellation

The negative-cycle optimality criterion states that the existence of a negative cycle implies the non-optimality of a feasible flow. This suggests the possibility of improving a non-optimal feasible flow by the cancellation of a suitably chosen negative cycle.

Suppose that negative cycles exist in the auxiliary network (G_ξ, ℓ_ξ) for a feasible flow ξ, where the arc length ℓ_ξ is defined by (12.25). Choose a negative cycle having the smallest number of arcs, and let Q $(\subseteq A_\xi)$ be the set of its arcs. Modifying the flow ξ along Q by a unit amount we obtain a new flow $\overline{\xi}$ defined by

$$\overline{\xi}(a) = \begin{cases} \xi(a)+1 & (a \in Q \cap A_\xi^\circ), \\ \xi(a)-1 & (\overline{a} \in Q \cap B_\xi^\circ), \\ \xi(a) & (\text{otherwise}). \end{cases} \tag{12.27}$$

The following theorem[66] shows that the updated flow $\overline{\xi}$ is a feasible flow with an improvement in the objective function in (12.11):

$$\Gamma_2(\xi) = \sum_{a\in A} \gamma(a)\xi(a) + f(\partial\xi).$$

Theorem 12.3. *For a feasible flow $\xi : A \to \mathbb{Z}$ to the M-convex submodular flow problem* MSFP$_2$*, let Q be a negative cycle having the smallest number of arcs in (G_ξ, ℓ_ξ). Then $\overline{\xi}$ in* (12.27) *is a feasible flow and*

$$\Gamma_2(\overline{\xi}) \leq \Gamma_2(\xi) + \ell_\xi(Q) < \Gamma_2(\xi). \tag{12.28}$$

12.4. Concluding remarks of section 12

On the basis of the optimality criteria in Theorems 12.1 and 12.2 we can design efficient algorithms for the M-convex submodular flow problem, where the expressions (12.21) and (12.22) are crucial. For algorithmic issues, see Murota (1999), Murota (2003, Section 10.4), Iwata & Shigeno (2003), Murota & Tamura (2003a), and Iwata et al. (2005).

13. DISCRETE FIXED POINT THEOREM

Discrete fixed point theorems in discrete convex analysis originate in the theorem of Iimura et al. (2005) based on Iimura (2003), which is described in this section. Subsequent development and other types of discrete fixed point theorems are mentioned in Section 13.5.

[66] The inequality (12.28) is by no means obvious. See Murota (1999) and Murota (2003, Section 10.4) for the proof.

13.1. Discrete fixed point theorem

To motivate the discrete fixed point theorem of Iimura et al. (2005), we first take a glimpse at Kakutani's fixed point theorem.

Let S be a subset of $\mathbb{R}^n$ and F be a set-valued mapping (correspondence) from S to itself, which is denoted as $F : S \to\to S$ (or $F : S \to 2^S$). A point $x \in S$ satisfying $x \in F(x)$ is said to be a *fixed point* of F. Kakutani's fixed point theorem reads as follows.

Theorem 13.1. *A set-valued mapping $F : S \to\to S$, where $S \subseteq \mathbb{R}^n$, has a fixed point if*

(a) *S is a bounded closed convex subset of $\mathbb{R}^n$,*
(b) *For each $x \in S$, $F(x)$ is a nonempty closed convex set, and*
(c) *F is upper-hemicontinuous.*

In the discrete fixed point theorem (Theorem 13.2 below) we are concerned with $F : S \to\to S$, where S is a subset of $\mathbb{Z}^n$. The three conditions (a) to (c) in Theorem 13.1 above are "discretized" as follows.

- Condition (a) assumes that the domain of definition S is nicely-shaped or well-behaved. In the discrete case we assume S to be "integrally convex."

- Condition (b) assumes that each value $F(x)$ is nicely-shaped or well-behaved. In the discrete case we assume that $F(x) = \overline{F(x)} \cap \mathbb{Z}^n$, where $\overline{F(x)}$ denotes the convex hull of $F(x)$.

- Condition (c) assumes that mapping F is continuous in some appropriate sense. In the discrete case we assume F to be "direction-preserving."

The key concepts, "integrally convex set" and "direction-preserving mapping," are explained in Section 13.2. The discrete fixed point theorem of Iimura et al. (2005) is the following.

Theorem 13.2. *A set-valued mapping $F : S \to\to S$, where $S \subseteq \mathbb{Z}^n$, has a fixed point if*

(a) *S is a nonempty finite integrally convex subset of $\mathbb{Z}^n$,*
(b) *For each $x \in S$, $F(x)$ is nonempty and $F(x) = \overline{F(x)} \cap \mathbb{Z}^n$, and*
(c) *F is direction-preserving.*

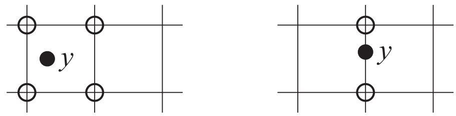

Figure 6: Integral neighbor $N(y)$ of y ($\circ$: point of $N(y)$)

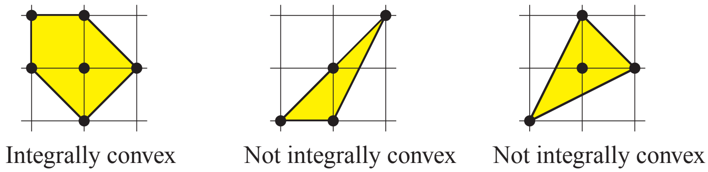

Figure 7: Concept of integrally convex sets

13.2. Integrally convex set and direction-preserving mapping

13.2.1. Integrally convex set

The *integral neighborhood* of a point $y \in \mathbb{R}^n$ is defined as

$$N(y) = \{z \in \mathbb{Z}^n \mid \|z - y\|_\infty < 1\}. \tag{13.1}$$

See Fig. 6. A set $S \subseteq \mathbb{Z}^n$ is said to be *integrally convex* if

$$y \in \overline{S} \implies y \in \overline{S \cap N(y)} \tag{13.2}$$

for any $y \in \mathbb{R}^n$ (Favati & Tardella, 1990). Figure 7 illustrates this concept. We have $S = \overline{S} \cap \mathbb{Z}^n$ for an integrally convex set S. It is known that $\mathrm{L}^\natural$-convex sets and $\mathrm{M}^\natural$-convex sets are integrally convex. See Murota (2003, Section 3.4) and Moriguchi et al. (2016) for more about integral convexity.

13.2.2. Direction-preserving mapping

Let S be a subset of $\mathbb{Z}^n$ and $F : S \to\to S$ be a set-valued mapping (correspondence) from S to S. For $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ we denote by $\pi(x) = (\pi_1(x), \ldots, \pi_n(x)) \in \mathbb{R}^n$ the projection of x to $\overline{F(x)}$; see Fig. 8. This means that

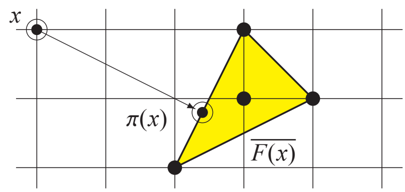

Figure 8: Projection $\pi(x)$ with $\sigma(x) = \mathrm{sign}\,(\pi(x) - x) = (+1, -1)$

$\pi(x)$ is the point of $\overline{F(x)}$ that is nearest to x with respect to the Euclidean norm. We define the direction sign vector $\sigma(x) \in \{+1, 0, -1\}^n$ as

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_n(x)) = (\mathrm{sign}\,(\pi_1(x) - x_1), \ldots, \mathrm{sign}\,(\pi_n(x) - x_n)),$$

where

$$\mathrm{sign}\,(y) = \begin{cases} +1 & (y > 0), \\ 0 & (y = 0), \\ -1 & (y < 0). \end{cases}$$

Then we say that F is *direction-preserving* if for all $x, z \in S$ with $\|x - z\|_\infty \leq 1$ it holds that

$$\sigma_i(x) > 0 \implies \sigma_i(z) \geq 0 \qquad (i = 1, \ldots, n). \tag{13.3}$$

Note that this is equivalent to saying that $\sigma_i(x)\sigma_i(z) \neq -1$ for each $i = 1, \ldots, n$ if $x, z \in S$ and $\|x - z\|_\infty \leq 1$. Being direction-preserving is interpreted as being "continuous" in the discrete setting.

13.3. Illustrative examples

Example 13.1. The significance of being direction-preserving is most transparent in the case of $n = 1$. Let $S = [a, b]_{\mathbb{Z}}$ be an integer interval with $a, b \in \mathbb{Z}$ and $a \leq b$. Consider $F : S \to\to S$ represented as $F(x) = [\alpha(x), \beta(x)]_{\mathbb{Z}}$, where $\alpha(x), \beta(x) \in \mathbb{Z}$ and $a \leq \alpha(x) \leq \beta(x) \leq b$. The projection $\pi(x)$ and the direction sign vector $\sigma(x)$ are given by

$$\pi(x) = \begin{cases} x & (\alpha(x) \leq x \leq \beta(x)), \\ \alpha(x) & (x \leq \alpha(x) - 1), \\ \beta(x) & (x \geq \beta(x) + 1), \end{cases} \qquad \sigma(x) = \begin{cases} 0 & (\alpha(x) \leq x \leq \beta(x)), \\ +1 & (x \leq \alpha(x) - 1), \\ -1 & (x \geq \beta(x) + 1). \end{cases}$$

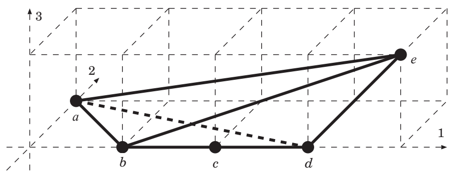

Figure 9: Necessity of the assumption of integral convexity

Suppose that F is direction-preserving, which means $\sigma(x)\sigma(x+1) \neq -1$ for all x with $a \leq x < b$. There are three possibilities:

(i) $\sigma(x) = +1$ for all $x \in S$,
(ii) $\sigma(x) = -1$ for all $x \in S$,
(iii) $\sigma(x) = 0$ for some $x \in S$.

In the first case (i) we must have $x+1 \leq \alpha(x) \leq b$ for all $x \in S$, but this is impossible for $x = b$. Similarly, the second case (ii) is not possible, either. Therefore, we must have the third case (iii), and then the x satisfying $\sigma(x) = 0$ is a fixed point of F. ∎

Example 13.2. The assumption (a) of integral convexity in Theorem 13.2 cannot be weakened to the "hole-free" property: $S = \overline{S} \cap \mathbb{Z}^n$. Let $n = 3$ and consider a subset S of $\mathbb{Z}^3$ (Fig. 9) given by

$$S = \{a = (0,1,0),\ b = (1,0,0),\ c = (2,0,0),\ d = (3,0,0),\ e = (4,0,1)\},$$

which is not integrally convex, but satisfies $S = \overline{S} \cap \mathbb{Z}^n$. Define $F : S \to\to S$ by

$$F(a) = F(b) = \{e\}, \quad F(c) = \{a,e\}, \quad F(d) = F(e) = \{a\}.$$

For each $x \in S$, $F(x)$ is a nonempty subset of S satisfying $F(x) = \overline{F(x)} \cap \mathbb{Z}^n$. Furthermore, F is direction-preserving. Indeed we have

$$\begin{array}{lcrrrlcrrr}
\pi(a) - a & = & (\ 4, & -1, & 1), & \sigma(a) & = & (+1, & -1, & +1), \\
\pi(b) - b & = & (\ 3, & 0, & 1), & \sigma(b) & = & (+1, & 0, & +1), \\
\pi(c) - c & = & (\ 0, & 1/2, & 1/2), & \sigma(c) & = & (\ 0, & +1, & +1), \\
\pi(d) - d & = & (-3, & 1, & 0), & \sigma(d) & = & (-1, & +1, & 0), \\
\pi(e) - e & = & (-4, & 1, & -1), & \sigma(e) & = & (-1, & +1, & -1)
\end{array}$$

and the condition (13.3) holds for every pair (x,z) with $\|x-z\|_\infty \leq 1$, i.e., for $(x,z) = (a,b),(b,c),(c,d),(d,e)$. Thus, F meets the conditions (b) and (c) in Theorem 13.2, but F has no fixed point. ∎

13.4. Proof outline

The proof of Theorem 13.2 consists of the following three major steps; the reader is referred to Iimura et al. (2005) for the detail.

1. An integrally convex set S has a simplicial decomposition $\mathscr{T}$ with a nice property. For each $y \in \mathbb{R}^n$ contained in the convex hull of S, let $T(y)$ denote the smallest simplex in $\mathscr{T}$ that contains y. Then the simplicial decomposition $\mathscr{T}$ has the property that all the vertices of $T(y)$ belong to the integral neighborhood $N(y)$ of y. That is, the set of the vertices of $T(y)$, to be denoted by $V(y)$, is given as $V(y) = T(y) \cap N(y)$.

2. With reference to the simplicial decomposition $\mathscr{T}$, we define a piecewise linear extension, say, f of the projection π by

$$f(y) = \sum_{x \in V(y)} \lambda_x \pi(x) \qquad (y = \sum_{x \in V(y)} \lambda_x x, \quad \sum_{x \in V(y)} \lambda_x = 1, \quad \lambda_x \geq 0).$$

   By Brouwer's fixed point theorem applied to $f : \overline{S} \to \overline{S}$, we obtain a fixed point $y^* \in \overline{S}$ of f, i.e., $y^* = f(y^*)$.

3. From the equations

$$\sum_{x \in V(y^*)} \lambda_x (\pi(x) - x) = \sum_{x \in V(y^*)} \lambda_x \pi(x) - \sum_{x \in V(y^*)} \lambda_x x = f(y^*) - y^* = \mathbf{0}$$

   and the assumption of F being direction-preserving, we see that $\pi(x) - x = \mathbf{0}$ for some $x \in V(y^*)$. Let x^* be such a point in $V(y^*)$. Then x^* is a fixed point of F, since $x^* = \pi(x^*) \in \overline{F(x^*)}$, from which follows $x^* \in \overline{F(x^*)} \cap \mathbb{Z}^n = F(x^*)$ by condition (b).

13.5. Concluding remarks of section 13

The discrete fixed point theorem initiated by Iimura (2003) and Iimura et al. (2005) aims at a discrete version of Brouwer's fixed point theorem. Related work in this direction includes Laan van der et al. (2006), Danilov & Koshevoi (2007), Chen & Deng (2006, 2008, 2009), Yang (2008, 2009), Talman & Yang (2009), Iimura & Yang (2009), Iimura (2010), Deng et al. (2011), Laan van der et al. (2010, 2011), and Iimura et al. (2012). Discrete fixed point theorems are used successfully in showing the existence of a competitive equilibrium under indivisibility, a pure Nash equilibrium with discrete strategy sets, etc.

Efforts are made to weaken the condition (c) of "direction preserving" in Theorem 13.2. Weaker conditions called "locally gross direction preserving" and "simplicially locally gross direction preserving" are considered by Yang (2008, 2009), Iimura & Yang (2009), and Iimura (2010). Further variants are found in Talman & Yang (2009), Laan van der et al. (2011), and Iimura et al. (2012). These studies, however, share the framework of mappings and correspondences defined on integrally convex sets or their simplicial divisions.

The proof of Theorem 13.2 by Iimura et al. (2005) is not constructive, relying on Brouwer's fixed point theorem. Constructive proofs are given by Laan van der et al. (2006) and Laan van der et al. (2011). Computational complexity of finding a fixed point for direction-preserving mappings is discussed by Chen & Deng (2006, 2008, 2009) and Deng et al. (2011).

Another type of (discrete) fixed point theorem, the lattice-theoretical fixed point theorem of Tarski (1955), is a powerful tool used extensively in economics and game theory; see Milgrom & Roberts (1990), Vives (1990), and Topkis (1998). For stable matchings, use and power of Tarski's fixed point theorem are demonstrated by Adachi (2000), Fleiner (2003), and Farooq et al. (2012). It may be said, however, that Tarski's fixed point theorem is rather independent of discrete convex analysis.

Yet another type of discrete fixed point theorems are considered in the literature, including Robert (1986), Shih & Dong (2005), Richard (2008), Yang (2008), Sato & Kawasaki (2009) and Kawasaki et al. (2013).

14. OTHER TOPICS

14.1. Matching market and economy with indivisible goods

Since the seminal paper by Kelso & Crawford (1982), the concept of gross substitutes with its variants has turned out to be pivotal in discussing matching market and economy with indivisible goods. The literature includes, e.g., Roth & Sotomayor (1990), Bikhchandani & Mamer (1997), Gul & Stacchetti (1999), Ausubel & Milgrom (2002), Fujishige & Yang (2003), Milgrom (2004), Hatfield & Milgrom (2005), Ausubel (2006), Sun & Yang (2006, 2009), Milgrom & Strulovici (2009), Hatfield et al. (2016).

Application of discrete convex analysis to economics was started by Danilov et al. (1998, 2001) for the Walrasian equilibrium of indivisible markets (see also Murota, 2003, chapter 11). The interaction between economics and discrete convex analysis was reinforced decisively by the observation of Fujishige & Yang (2003) that M$^\natural$-concavity (of set functions) is equivalent to the gross substitutes property (Theorem 3.7 in Section 3.3). This equivalence is extended to functions in integer variables (Section 4.3). While the reader is referred to Tamura (2004) and Murota (2003, chapter 11) for this earlier development, we mention more recent papers below.

As described in Section 10, the Fujishige-Tamura model of two-sided matching markets, proposed by Fujishige & Tamura (2006, 2007), is a common generalization of the stable marriage model (Gale & Shapley, 1962) and the assignment game (Shapley & Shubik, 1972).

Inoue (2008) uses the property of M$^\natural$-convex sets that they are closed under (Minkowski) summation, to show that the weak core in a finite exchange economy is nonempty if every agent's upper contour set is M$^\natural$-convex. Kojima et al. (2014) present a unified treatment of two-sided matching markets with a variety of distributional constraints that can be represented by M$^\natural$-concave functions. It is shown that the generalized deferred acceptance algorithm is strategy-proof and yields a stable matching. Yokote (2016) considers a market in which each buyer demands at most one unit of commodity and each seller produces multiple units of several types of commodities. The core and the competitive equilibria are shown to exist and coincide under the assumption that the cost function of each seller is M$^\natural$-convex.

Algorithmic aspects of Walrasian equilibria are investigated by Paes Leme & Wong (2016) in a general setting, in which the algorithms from discrete convex analysis are singled out as efficient methods for the gross substitutes case. See also Paes Leme (2014), Murota & Tamura (2003a) and Murota (2003, Section 11.5).

14.2. Trading networks

M$^\natural$-concavity plays a substantial role in the modeling and analysis of vertical trading networks (supply chain networks) introduced by Ostrovsky (2008) and further studied by Hatfield et al. (2013), Fleiner (2014), Fleiner et al. (2015), Ikebe et al. (2015), Ikebe & Tamura (2015), and Candogan et al. (2016) in more general settings.

In a trading network, an agent is identified with a vertex (node) of the network. In-coming arcs to a vertex represent the trades in which the agent acts as a buyer and out-going arcs represent the trades in which the agent acts as a seller. Each vertex v of the network is associated with a choice function C_v and/or a valuation function f_v of the agent, defined on the set $U_v \cup W_v$ of the arcs incident to v, where U_v is the set of in-coming arcs to the vertex v and W_v is the set of out-going arcs from v. In particular, the function f_v is a set function on $U_v \cup W_v$ in the single-unit case, whereas it is a function on $\mathbb{Z}^{U_v \cup W_v}$ in the multi-unit case.

In the single-unit case, Ostrovsky (2008) identifies the key property of a choice function, called the same-side substitutability (SSS) and the cross-side complementarity (CSC), which are discussed in Section 3.5. These properties are satisfied by the choice function induced from a unique-selecting twisted M$^\natural$-concave valuation function f_v, with twisting by W_v; see Theorem 3.13. The multi-unit case is treated by Ikebe & Tamura (2015). The conditions (SSS) and (CSC) are generalized to (SSS-CSC$^1[\mathbb{Z}]$) and (SSS-CSC$^2[\mathbb{Z}]$), and these conditions are shown to be satisfied by the choice function induced from a unique-selecting twisted M$^\natural$-concave valuation f_v; see Theorem 4.14 in Section 4.5.

Discrete convex analysis is especially relevant and useful when valuation functions and the price vector p are explicitly involved in the model as in Hatfield et al. (2013); Ikebe et al. (2015); Candogan et al. (2016). Specifically, we can use the results from discrete convex analysis as follows:

- The existence of a competitive equilibrium (Hatfield et al., 2013, Definition 3) can be proved with the aid of the M$^\natural$-concave intersection theorem (Theorem 9.4).
- The lattice structure of the equilibrium price vectors can be shown through the conjugacy relationship between M$^\natural$-concavity and L$^\natural$-convexity (Section 7.2).
- The equivalence of chain stability and stability can be established with the aid of the negative-cycle criterion for the M-convex submodular flow problem (Theorem 12.2). Recall from Remark 12.2 that the M$^\natural$-concave intersection problem can be formulated as an M-convex submodular flow problem.
- Fundamental computational problems for a trading network, such as checking stability, computing a competitive equilibrium, and maximizing

the welfare, can often be solved with the aid of algorithms known in discrete convex analysis, such as those for maximizing $M^\natural$-concave functions and for solving the M-convex submodular flow problem. See Candogan et al. (2016) as well as Murota & Tamura (2003a), Murota (2003, Chapter 11), and Ikebe et al. (2015).

14.3. Congestion games

Congestion games (Rosenthal, 1973), which are equivalent to (exact) "finite" potential games (Monderer & Shapley, 1996), are a class of games possessing a Nash equilibrium in pure strategies. There are various generalizations of potential games, such as: ordinal and generalized ordinal (Monderer & Shapley, 1996) and best-response (Voorneveld, 2000) potential games. For algorithmic aspects of congestion games, we refer to Roughgarden (2007) and Tardos & Wexler (2007).

Recently, a connection is made by Fujishige et al. (2015) between congestion games on networks and discrete convex analysis. It has been known (Fotakis, 2010) that for every congestion game on an extension-parallel network, considered by Holzman & Law-yone (2003), any best-response sequence reaches a pure Nash equilibrium of the game in n steps, where n is the number of players. It is pointed out by Fujishige et al. (2015) that the fast convergence of best-response sequences is a consequence of $M^\natural$-convexity of the associated potential function, which is a laminar convex function and hence is $M^\natural$-convex; see (4.35) in Section 4.6.

In economics, potential games on some subset of a Euclidean space are more widely studied. A maximizer of (some sort of) potential function is a Nash equilibrium. We also have the converse if the potential function is "concave," since local optimality implies the global optimality there. Ui (2006, 2008) studies the condition for a local maximizer of a function on the integer lattice to become a global maximizer of the function as well, with application to best-response potential games on the integer lattice. In Ui (2008), it is shown that a condition analogous to midpoint concavity, called "larger midpoint property," is sufficient for the equivalence of local optimality and global optimality, and shows the equivalence of a Nash equilibrium and a maximizer of the best-response potential function. A more general condition for the equivalence of local and global optimality is studied in Ui (2006), along with its relation to M-, L-, $L^\natural$-, and $M^\natural$-convex functions.

14.4. Integrally concave games

Another study on the games on the integer lattice $\mathbb{Z}^n$ is found in Iimura & Watanabe (2014), which deals with n-person symmetric games with integrally concave payoff functions defined on the n-product of a finite integer interval. Here, the integral concavity is in the sense of Favati & Tardella (1990); see also Murota (2003, Section 3.4). It is shown that every game in this class of games has a (not necessarily symmetric) Nash equilibrium, which is located within a unit distance from the diagonal of strategy space. Although assuming concavity on the entire strategy space is somewhat stringent, this result generalizes the result of Cheng et al. (2004) that every n-person symmetric "two-strategy" game has a (not necessarily symmetric) Nash equilibrium, because any real-valued function on the n-product of a doubleton is integrally concave. A further generalization has been made by Iimura & Watanabe (2016), which implies the existence of an equilibrium in discrete Cournot game with concave industry revenue, convex cost, and nonincreasing inverse demand.

14.5. Unimodularity and tropical geometry

Unimodular coordinate transformations are a natural operation for discrete convexity; see Sun & Yang (2008) and Baldwin & Klemperer (2016). In Section 4.7 we have mentioned that a function f is twisted M$^\natural$-concave if and only if it is represented as $f(x) = g(Ux)$ with $U = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ for some M$^\natural$-concave function g. Another such example is a class of multimodular functions in Hajek (1985) which are used in discrete-event control (Altman et al., 2000). A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *multimodular* if the function $\tilde{f} : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ defined by $\tilde{f}(x_0, x) = f(x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1})$ for $x_0 \in \mathbb{Z}$ and $x \in \mathbb{Z}^n$ is submodular in $n+1$ variables. This means that f is multimodular if and only if the function $g(x) = f(Dx)$ is L$^\natural$-convex, where $D = (d_{ij} \mid 1 \leq i, j \leq n)$ is a bidiagonal matrix defined by $d_{ii} = 1$ $(i = 1,\ldots,n)$ and $d_{i+1,i} = -1$ $(i = 1,\ldots,n-1)$. This matrix D is unimodular, and its inverse D^{-1} is an integral matrix with $(D^{-1})_{ij} = 1$ for $i \geq j$ and $(D^{-1})_{ij} = 0$ for $i < j$. Therefore, a function f is multimodular if and only if it is represented as $f(x) = g(Ux)$ with $U = D^{-1}$ for some L$^\natural$-convex function g.

The fundamental role of unimodularity for discrete convexity, beyond unimodular coordinate transformations, is investigated in Danilov & Koshevoy (2004) under the name of "unimodular systems." An application of unimodular systems to competitive equilibrium is found in Danilov et al. (2001).

Another recent topic, of a similar flavor, is tropical geometry. Baldwin & Klemperer (2016) investigate indivisibility issues in terms of tropical geometry. The Ricardian theory of international trade is treated by Shiozawa (2015), mechanism design by Crowell & Tran (2016), and dominant strategy implementation by Weymark (2016). The interaction of tropical geometry with economics may yield unexpected results.[67]

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