



AN ASCENDING MULTI-ITEM AUCTION WITH FINANCIALLY CONSTRAINED BIDDERS

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ABSTRACT

Several heterogeneous items are to be sold to a group of potentially budget-constrained bidders. Every bidder has private knowledge of his own valuation of the items and his own budget. Due to budget constraints, bidders may not be able to pay up to their values and typically no Walrasian equilibrium exists. To deal with such markets, we propose the notion of ‘equilibrium under allotment’ and develop an ascending auction mechanism that always finds such an equilibrium assignment and a corresponding system of prices in finite time. The auction can be viewed as a novel generalization of the ascending auction of [Demange et al. \(1986\)](#) from settings without financial constraints to settings with financial constraints. We examine various strategic and efficiency properties of the auction and its outcome.

Keywords: Ascending auction, budget constraint, equilibrium under allotment.

JEL Classification Numbers: D44.

1. INTRODUCTION

Auctions are typically the most efficient mechanism for the allocation of private goods and have been used since antiquity for the sale of a variety

We are deeply grateful to Tommy Andersson for his detailed comments and also to many participants at various seminars and meetings for their helpful feedback. The current paper is a revision of the Tinbergen Institute Discussion Paper TI 2008-017/1 in the year of 2008.

of items. The academic study of auctions grew out of the pioneering work of [Vickrey \(1961\)](#) and has blossomed into an enormously important area of economic research over the last few decades. Standard auction theory assumes that all potential bidders are able to pay up to their values on the items for sale. However in reality buyers may face budget constraints for a variety of reasons and therefore may be unable to afford what the items are worth to them. As stressed by [Maskin \(2000\)](#) in his Marshall lecture, the consideration of financial constraints on buyers is particularly relevant and important in many developing countries, where auctions are used to privatize state assets for the promotion of efficiency, competition and development, but entrepreneurs may often be financially constrained. Financial constraints not only occur in developing countries but also in developed nations. In particular, [Che & Gale \(1998\)](#) have discussed a variety of situations where financial constraints may arise, ranging from an agent's moral hazard problem, business downturns and financial crises, to the acquisition decisions in many organizations which delegate to their purchasing units but impose budget constraints to control their spending, and to the case of salary caps in many professions where budget constraints are used to relax competition.

Financial constraints can pose a serious obstacle to the efficient allocation of resources. For instance, financial constraints seem to have played an important role in the outcome of auctions for selling spectrum licenses conducted in US (see [McMillan, 1994](#); [Salant, 1997](#)) and in European countries (see [Illing & Klüh, 2003](#)). In this paper, we study a general model in which a number of (indivisible) items are sold to a group of financially constrained bidders. Each bidder wants to consume at most one item. When no bidder faces a financial constraint, the model reduces to the well-known assignment model as studied by [Koopmans & Beckmann \(1957\)](#), [Shapley & Shubik \(1972\)](#), [Crawford & Knoer \(1981\)](#), [Leonard \(1983\)](#), and [Demange et al. \(1986\)](#) among others. Each bidder has private information about his values for the items and his budget and is unwilling to reveal such information for strategic reasons. In particular the auctioneer (seller) does not know the values and budgets of the bidders. It is well-known that even when a single item is auctioned, it is generally impossible to specify a mechanism which achieves full market efficiency when bidders face budget constraints. Of course, this observation also holds when there are multiple items for sale. Even worse, when bidders face financial constraints, a Walrasian equilibrium typically fails to exist,¹ and allocation

¹ Besides budget constraints, price rigidities or fixed prices can also cause the failure of Walrasian

mechanisms that perform well when no bidders face budget constraints, if applied, often result in highly inefficient outcomes.

A natural question is therefore whether an allocation mechanism can be designed that yields a reasonably efficient assignment of the items and a corresponding price system that supports the assignment.² In this paper we propose a general solution concept of equilibrium under allotment that gives a sufficiently efficient assignment of items and a supporting system of prices. More importantly we develop a dynamic auction mechanism that yields an equilibrium under allotment in finite time. The proposed auction can be seen as a novel generalization of the well-known ascending auction of [Demange et al. \(1986\)](#) (DGS auction in short) from settings without financial constraints to settings with financial constraints. It works as follows: the auctioneer starts with the seller's reservation price vector, that specifies the lowest admissible price for each item, and each bidder responds with the set of items demanded at those prices. The auctioneer adjusts prices upwards for a minimal set of overdemanded items until a system of prices is reached for which either a set of items is underdemanded or there is neither underdemand nor overdemand. In the first case precisely one item is assigned to a bidder that demanded that item at the previous price system. This bidder with item leaves the auction, while the remaining bidders are requested to report their demands for the remaining items at the previous prices of these items. We will show that at these prices either there is overdemand for some of the remaining items or there are a set of prices with neither overdemand nor underdemand. In the first situation the auctioneer continues by adjusting prices upwards for a minimal set of overdemanded items, until again a system of prices is reached for which either a set of items is underdemanded or there is neither underdemand nor overdemand. In case of underdemand again one item is assigned and the auction continues with the remaining bidders and items. As soon as there is neither underdemand nor overdemand, an equilibrium has been reached for all the remaining items.

An attractive feature of the auction is that it only requires the bidders to report their demands at price vectors along a finite path rather than their values or budgets. This property is very useful and practical, because businessmen

equilibrium; see for instance [Talman & Yang \(2008\)](#), [Hatfield et al. \(2012, 2016\)](#), [Andersson & Svensson \(2014, 2016\)](#), [Andersson et al. \(2015\)](#), and [Herings \(2015\)](#).

² It is impossible to achieve a Pareto-optimal outcome, because Walrasian equilibria simply may not exist due to budget constraints.

are in general reluctant to reveal their values, costs, or budgets. This also gives an explanation of why dynamic auctions like English and Dutch auctions are more popular than sealed-bid auctions like the Vickrey auction; see e.g., Rothkopf et al. (1990), Perry & Reny (2005), and Bergemann & Morris (2007). We show that when bidders face no budget constraints, the proposed auction reduces to the well-known DGS auction and thus maintains the DGS auction's strategic properties. In this case, the auction finds a Walrasian equilibrium and it is in the best interest of every bidder to bid truthfully. Moreover in the case where there are budget constraints, the auction might end up with an outcome in which a bidder does not receive his most preferred item given the prices at which the items are sold, because this item has been sold to some other bidder. A bidder that does not receive his most preferred item finds himself constrained on his ability to bid for that item. As shown in Borgs et al. (2005) it may be impossible to design truthful-bidding multi-unit auctions in the case of budget-constrained bidders. Indeed, a bidder that finds himself constrained in the outcome of the auction, might be able to attain a better outcome by misreporting his demands when he has an information advantage over other bidders. However, in the case of at most two items we demonstrate that bidders who receive their most preferred item will have no incentive to manipulate the auction. This property seems similar in essence to those found in the matching literature;³ see Dubins & Freedman (1981) and Roth & Sotomayor (1990). Another salient feature of the auction is that when a bidder feels himself constrained in his ability to influence the assignment of a particular item, then the price of this item equals the budget of another bidder who is actually assigned with this item and thus pays his full budget. We further demonstrate that the assignment of items and a system of prices generated by the auction yield a Pareto efficient allocation of the items and the money, when no bidder finds himself constrained.

This paper connects directly with a number of papers concerned with auction design under budget constraints. The existing literature concentrates on *sealed-bid auctions* for selling a single item to many bidders, or two items to two bidders. In contrast we propose a dynamic auction for selling multiple items to many financially constrained bidders. Rothkopf (1977) is among the first to study some issues concerning sealed-bid auctions with budget

³ In the marriage matching, men have no incentive to manipulate in the men-proposing deferred acceptance procedure provided that women are honest, because every man is matched with his best possible partner.

constrained bidders. He investigates how such constraints may affect the best bids of a bidder. [Palfrey \(1980\)](#) analyzes a price discriminatory sealed-bid auction in a multiple item setting under budget constraints and gives a complete characterization of a Nash equilibrium in the case of two items or less and two bidders or less. [Pitchik & Schotter \(1988\)](#) study the equilibrium bidding behavior in sequential auctions for the sale of two items with budget constrained bidders. [Che & Gale \(1996, 1998\)](#) focus on single item auctions with budget constraints under incomplete information. They show that when bidders are subject to financial constraints, the well-known revenue equivalence theorem does not hold. In particular, [Che & Gale \(1998\)](#) provide several conditions under which first-price auctions yield higher expected revenue and social surplus than second-price auctions; see also [Krishna \(2002\)](#) and [Klemperer \(2004\)](#). [Laffont & Roberts \(1996\)](#) characterize an optimal sealed-bid auction in a single item setting under financial constraints.

[Maskin \(2000\)](#) studies the performance of second-price auctions and all-pay auctions and proposes a constrained-efficient sealed-bid auction for the sale of a single item when bidders are financially constrained. [Zheng \(2001\)](#) examines a single-object, first-price sealed-bid auction where budget constrained bidders have the possibility of defaulting on their bids. He shows that budget constraints and default risk together can have a highly significant impact on seller's profit, bidding behavior, and the likelihood of bankruptcy. [Benoît & Krishna \(2001\)](#) investigate simultaneous ascending auctions and sequential auctions for the sale of two items with budget constrained bidders. They compare the performance of both types of auctions when the two items are complements or substitutes; see also [Krishna \(2002\)](#). While concentrating on package auctions without budget constraints, [Ausubel & Milgrom \(2002\)](#) also briefly discuss the case of budget constraints and use the notion of core as a solution under the assumption that every bidder has strict preferences over finite choices. [Quintero Jaramillo \(2004\)](#) shows that a seller can benefit from offering small credit subsidies in an auction with financially constrained bidders. [Brusco & Lopomo \(2008, 2009\)](#) consider simultaneous ascending auctions of two identical objects and two bidders and show that even the slightest possibility of financial constraints may cause significant inefficiencies. [Pitchik \(2009\)](#) studies a sealed-bid sequential auction for selling two items to two bidders with budget constraints and incomplete information. [Hafalir et al. \(2012\)](#) examine a sealed-bid Vickrey auction for selling a divisible good that achieves near Pareto efficiency, weaker than Pareto efficiency. [Talman](#)

& Yang (2015) develop a dynamic auction for the assignment market with budget-constrained bidders that always finds a core allocation of the underlying economy, thus resulting in a Pareto efficient outcome.

This paper is organized as follows. Section 2 presents the model. Section 3 introduces the notion of equilibrium under allotment and other basic concepts and results. Section 4 describes and illustrates the ascending auction. Section 5 discusses the convergence of the auction process. Section 6 examines the outcome of the auction. Section 7 derives strategic and efficiency properties. Section 8 concludes. The appendix of the paper contains most of the proofs.

2. THE MODEL

A seller or auctioneer has n indivisible items for sale to a set of m financially constrained bidders. Let $N = \{1, \dots, n\}$ denote the set of the items for sale and $M = \{1, 2, \dots, m\}$ the set of bidders. In addition to the n real items there is a *dummy item*, denoted by 0. The dummy item 0 can be assigned to any number of bidders simultaneously, any real item $j \in N$ can be assigned to at most one bidder. The seller has for each real item $j \in N$ a nonnegative reservation price $c(j)$ below which the item will not be sold. By convention, the reservation price of the dummy good is known to be $c(0) = 0$. A price vector $p \in \mathbf{R}_+^{n+1}$ gives a price $p_j \geq 0$ for each item $j \in N \cup \{0\}$. A price vector $p \in \mathbf{R}_+^{n+1}$ is *feasible* if $p_j \geq c(j)$ for every $j \in N$ and $p_0 = 0$. Every bidder $i \in M$ attaches a (possibly negative) monetary value to each item in $N \cup \{0\}$ given by the valuation function $V^i: N \cup \{0\} \rightarrow \mathbf{R}$. Also by convention, the value of the dummy item for every buyer i is known to be $V^i(0) = 0$. It should be noted that a set $S \subseteq N$ of real items gives value $V^i(S) = \max_{j \in S} V^i(j)$ to bidder i , i.e., bidder i can utilize only one item and thus will never buy more than one real item. So, this is the well known assignment model as studied by Koopmans & Beckmann (1957), Shapley & Shubik (1972), Crawford & Knoer (1981), Leonard (1983), and Demange et al. (1986).

In this paper we generalize this standard model by considering the situation where each bidder i is initially endowed with a nonnegative amount of m^i units of money. Bidders are not allowed to have deficits on their money balances, so no bidder can afford an item j with a price p_j higher than his initial amount of money m^i . This means that unlike in the standard assignment model, the bidders are financially constrained by their initial money holdings m^i , $i \in M$. Since a bidder i is never willing to pay more than his valuation $V^i(j)$ for any

item j , his budget m^i is never binding when $m^i \geq \max_{j \in N} V^i(j)$. We say that bidder i is *financially constrained* if $m^i < \max_{j \in N} V^i(j)$, i.e., the valuation of bidder i for at least one item exceeds what he can afford, and that bidder i faces *no financial constraint* otherwise.

All values $V^i(j)$, $j \neq 0$, and m^i are private information and thus only bidder i knows his own values $V^i(j)$, $j \neq 0$ and his own budget m^i . Further it is assumed that all seller's reservation prices, and all valuations and money amounts of the bidders are integer values.

The utility of a bidder i possessing item j and money amount $x_i \geq 0$ is given by

$$U^i(j, x_i) = V^i(j) + x_i - m^i,$$

i.e., the utility is equal to the value of the item j plus the difference between his amount of money x_i and his initial amount m^i . So, $U^i(0, m^i) = 0$, i.e., the utility of bidder i is normalized to zero when he gets the dummy item 0 and his initial amount of money m^i . The utility of bidder i who buys item $j \in N \cup \{0\}$ against price $p_j \leq m^i$ is thus given by

$$U^i(j, m^i - p_j) = V^i(j) - p_j.$$

A *feasible assignment* π assigns to every bidder $i \in M$ precisely one item $\pi(i) \in N \cup \{0\}$ such that no real item $j \in N$ is assigned to more than one bidder. Note that a feasible assignment may assign the dummy good to several bidders and that a real item $j \in N$ is *unassigned* at π if there is no bidder i such that $\pi(i) = j$. Let $N_\pi = \{j \in N \mid j \neq \pi(i) \text{ for all } i \in M\}$, i.e, N_π is the set of real items that are not assigned to any bidder in π . A feasible assignment π^* is *socially efficient* if

$$\sum_{i \in M} V^i(\pi^*(i)) + \sum_{j \in N_{\pi^*}} c(j) \geq \sum_{i \in M} V^i(\pi(i)) + \sum_{j \in N_\pi} c(j)$$

for every feasible assignment π , so a socially efficient assignment maximizes the total value that can be obtained from allocating the items over all agents.

For feasible price vector $p \in \mathbf{R}_+^{n+1}$, the budget set of bidder i is given by

$$B^i(p) = \{j \in N \cup \{0\} \mid p_j \leq m^i\},$$

i.e., the budget set of bidder i at price system p is the set of all affordable items at p . Given a feasible price vector $p \in \mathbf{R}_+^{n+1}$, the demand set of bidder i is

defined by

$$D^i(p) = \{j \in B^i(p) \mid V^i(j) - p_j = \max_{k \in B^i(p)} (V^i(k) - p_k)\},$$

thus $D^i(p)$ is the collection of most preferred items at p by i within his budget set, i.e., an item $j \in N \cup \{0\}$ is in the demand set $D^i(p)$ if and only if it can be afforded at p and maximizes the surplus $V^i(k) - p_k$ over all affordable items k . When the demand set contains multiple items, then at the given prices of the items the bidder is indifferent between any two items in his demand set. Note that for any feasible p , the demand set $D^i(p) \neq \emptyset$, because $p_0 = 0 \leq m^i$ and thus the dummy item is always in the budget set $B^i(p)$. In fact this means that the bidder has always the possibility not to buy any real item.

A pair (p, π) of a feasible price vector p and a feasible assignment π is said to be *admissible* if $p_{\pi(i)} \leq m^i$ for all $i \in M$, i.e., every bidder i can afford to buy the item $\pi(i)$ assigned to him. Note that every admissible pair (p, π) yields the corresponding allocation (π, x) with $x_i = m^i - p_{\pi(i)} \geq 0$.

Definition 2.1. A Walrasian equilibrium (**WE**) is an admissible pair (p^*, π^*) such that

- (a) $\pi^*(i) \in D^i(p^*)$ for all $i \in M$,
- (b) $p_j^* = c(j)$ for every unassigned item $j \in N_{\pi^*}$.

If (p^*, π^*) is a WE, p^* is called a (Walrasian) equilibrium price vector and π^* a (Walrasian) equilibrium assignment. Because all values and money amounts are integer and the seller's reservation prices are nonnegative integers, it follows that if there exists an equilibrium price vector $p^* \in \mathbf{R}_+^{n+1}$, there must be an integral equilibrium price vector $p \in \mathbf{Z}_+^{n+1}$. Therefore we can restrict ourselves to the set \mathbf{Z}_+^{n+1} of nonnegative integer price vectors.

From [Shapley & Shubik \(1972\)](#) it is well known that in a situation without financial constraints a Walrasian equilibrium exists and every equilibrium assignment is socially efficient. To find an equilibrium some revealing mechanism is needed, because all valuations $V^i(j)$, $j \neq 0$, are private information. The well-known auctions proposed by [Crawford & Knoer \(1981\)](#) and [Demange et al. \(1986\)](#) are such mechanisms. In the remaining of this paper we call the auction introduced in the latter paper the DGS auction. In this literature the notion of overdemanded set of real items is used. A set $S \subseteq N$ of real items is *overdemanded* at a price vector $p \in \mathbf{R}^{n+1}$ if the number of bidders who demand goods only from this set is greater than the number of items in that set.

See the Appendix for a further discussion of this notion. The DGS auction is an ascending auction in which the auctioneer starts with the reservation price vector $p \in \mathbb{Z}_+^{n+1}$ given by $p_0 = 0$ and $p_j = c(j)$, $j \in N$. Then each bidder is required to report his demand set $D^i(p)$. When there is an overdemanded set of goods, the price of every item j in a *minimal* overdemanded set (i.e., no strict subset of this overdemanded set is overdemanded) is increased by one and the bidders have to resubmit their demands at this new price vector. The auction stops as soon as there are no overdemanded sets anymore. It is well-known that the DGS auction for the assignment model without financial constraints stops in a finite number of price adjustments with a unique minimal equilibrium price vector $p^{\min} \in \mathbb{Z}_+^{n+1}$, i.e., (i) there exists a feasible assignment π^* such that (p^{\min}, π^*) constitutes a Walrasian equilibrium and (ii) it holds that $p \geq p^{\min}$ for any other equilibrium price system $p \in \mathbb{R}_+^{n+1}$. Since the minimum Walrasian price vector corresponds to the Vickrey-Clarke-Groves payments (see [Leonard, 1983](#)), the DGS auction has truthful bidding in equilibrium. Also note that in the single item case, the DGS auction reduces to the English auction.

In case of financial constraints a Walrasian equilibrium is not guaranteed to exist, and when it does exist, the equilibrium (assignment) need not be socially efficient. The latter can be easily seen from an example with two bidders and one item. When $V^1(1) > V^2(1) > c(1) = 0$, then social efficiency requires to assign the item to bidder 1. Now, suppose that $m^1 < \min(m^2, V^2(1))$. Then there exists a Walrasian equilibrium, but at any equilibrium the item is assigned to bidder 2 at some (integer) price p_1 , $m^1 < p_1 \leq \min(m^2, V^2(1))$. So, all equilibria are socially inefficient.

The following example further shows that financial constraints may cause not only the nonexistence of a Walrasian equilibrium but also the failure of the DGS auction.⁴

Example 1. Consider a market with three bidders ($i = 1, 2, 3$) and two real items ($j = 1, 2$). The values of the bidders are shown in Table 1 and the seller's reservation price vector is $C = (c(0), c(1), c(2)) = (0, 0, 0)$.

Case 1 (No Budget Constraints). Then this market has a (unique) equilibrium assignment $\pi = (\pi(1), \pi(2), \pi(3)) = (0, 2, 1)$. The set of equilibrium

⁴ It should be noted that the DGS auction was not designed for the current setting with budget constraints but for the settings without budget constraints.

prices is given by

$$\{p \in \mathbf{R}^3 \mid p_0 = 0, 5 \leq p_1 \leq 6, 4 \leq p_2 \leq 5 \text{ with } p_1 = p_2 + 1\}.$$

The two equilibrium integer price vectors (for the real items) are $p^{\min} = (5, 4)$ and $p^{\max} = (6, 5)$. The DGS auction will find the equilibrium (π, p^{\min}) , realizing a social value of 11 and a revenue of 9 to the seller.

Case 2 (Budget Constraints). Let $(m^1, m^2, m^3) = (4, 3, 8)$ be the budgets of the bidders. Observe that all budgets are totally different across bidders. We show that the budget constraints fail the existence of a Walrasian equilibrium. Suppose to the contrary that there would be a Walrasian equilibrium price vector $p = (p_0, p_1, p_2)$. Clearly, we should have that $p_1 \leq 6$ and $p_2 \leq 5$, since otherwise no bidder demands a real item. When $p_1 = p_2 + 1$ we have that $D^3(p) = \{1, 2\}$. Further, when $p_1 = p_2 + 1 > 4$, then $D^1(p) = D^2(p) = \{0\}$ and there is underdemand, whereas when $p_1 = p_2 + 1 < 4$, then $D^1(p) = \{1\}$ and $D^2(p) = \{2\}$ and there is overdemand. So, there is no equilibrium with $p_1 = p_2 + 1$. When $p_1 < p_2 + 1$, then we have that $D^3(p) = \{1\}$. So, for equilibrium we must have that $p_1 > 4 = m^1$, otherwise also bidder 1 wants to have item 1. However, then $p_2 \geq p_1 - 1 > 3 = m^2$ and thus there is no demand for item 2. So, there cannot be an equilibrium with $p_1 < p_2 + 1$. Similarly, when $p_1 > p_2 + 1$, it holds that $D^3(p) = \{2\}$, implying that $p_2 > 3 = m^2$, otherwise also bidder 2 wants to have item 2. Then $p_1 > p_2 + 1 > 4 = m^1$ and thus there is no demand for item 1. Again there is no Walrasian equilibrium with $p_1 > p_2 + 1$. Hence, a Walrasian equilibrium does not exist. When one applies the DGS auction, first p_1 is increased from 0 to 1 and then both prices of the real items are increased simultaneously from $(1, 0)$ to $(4, 3)$. At each of these price systems p (with $p_0 = 0$) there is overdemand for both real items because, $D^1(p) = \{1\}$, $D^2(p) = \{2\}$ and $D^3(p) = \{1, 2\}$. However, at the next update we have $p = (0, 5, 4)$ (with $p_0 = 0$ the price of the dummy item) and the demand sets are $D^1(p) = \{0\}$, $D^2(p) = \{0\}$ and $D^3(p) = \{1, 2\}$. So, at prices $p_1 = 4$, $p_2 = 3$, each of the three bidders demands at least one of the items. As a result, the set $\{1, 2\}$ is a minimal overdemanded set and, according to the DGS auction, both prices are increased by one. However, at $p_1 = 5$ and $p_2 = 4$, only bidder 3 demands just one of the items (he is indifferent between both items). So, at these prices the seller wants to sell both items, but only one of the items is demanded. It shows that the DGS auction fails to allocate the items. At $p = (0, 3, 4)$ there is overdemand, while at the next update there is underdemand. \square

Table 1: Bidders' values on each item.

Items	0	1	2
Bidder 1	0	5	0
Bidder 2	0	0	5
Bidder 3	0	6	5

Example 1 demonstrates clearly why under financial constraints an equilibrium does not need to exist. Without budget constraint, a bidder withdraws his demand for a real item when the price of the item becomes higher than the bidder's valuation. However, at the price equal to the valuation, the bidder is indifferent between the real and the dummy item (i.e., not buying an item). So, when at this price the real item belongs to the demand set, then also the dummy item belongs to it and the seller can choose between allocating the real item or the dummy item to the bidder. With budget constraint, by contrast, there are two possibilities that a bidder withdraws his demand. The first one is, as before, because the price rises above his valuation of the item. In this case, the dummy item is also in the demand set when the price is equal to the valuation. However, the second possibility is that the price is going to exceed the budget. Then, it might happen that at price equal to the budget, the bidder prefers the real item above every other item (and so the real item is the single item in his demand set), while the demand set does not contain the item anymore when the price is increased by only one. In the example this happens when the price system goes from $(4, 3)$ to $(5, 4)$. At $p_1 = 4$ the first bidder strictly prefers the first item to any other item (including the dummy item), while at $p_1 = 5$ the first item is not affordable anymore and bidder 1 only demands the dummy item (the same holds for bidder 2 with respect to item 2). So, with budget constraints it is possible that an overdemanded item (or set of items) becomes underdemanded when the price (prices) rises with only one unit. Because of this discontinuity of the demand sets, the Walrasian equilibrium fails to exist. Note that without budget constraints the change from overdemand to underdemand cannot happen, because then the bidder is indifferent between a real item and the dummy item when the price is equal to the reservation value.

The change from overdemand to underdemand in Case 2 of Example 1 is also the reason why the DGS auction fails to work properly. Rather than follow the DGS auction precisely (the auction requires to increase the prices of all items in a minimal overdanded set), one might consider the possibility to rise only one of the prices at (4,3). However, this is not of any help. For instance, when only p_1 increases from 4 to 5, then at (5,3) there is no demand for item 1, whereas both bidders 2 and 3 demand item 2. So, item 1 is underdemanded and item 2 is still overdanded. Then increasing p_2 from 3 to 4, gives again the situation as described in the example. Similarly, when first p_2 is increased from 3 to 4, then at (4,4) there is no demand for item 2, whereas both bidders 1 and 3 demand item 1. So, anyway the procedure ends up with prices (5,4) at which bidders 1 and 2 demand the dummy item and bidder 3 is indifferent between the two real items. Of course, it is then possible to assign either item 1 or item 2 to bidder 3. In the first case, bidder 3 pays 5 to the seller who keeps item 2, realizing a social value of 6. In the second case, bidder 3 pays 4 to the seller who keeps item 1, realizing a social value of 5. Both assignments result in a loss of efficiency, because bidders 1 and 2 are willing to pay for the unassigned item, but they don't receive it. This brings us to the central issue of this paper: the design of an auction for markets with financially constrained bidders.

3. EQUILIBRIUM UNDER ALLOTMENT

A possible way out of market situations in which the Walrasian equilibrium does not exist and thus the DGS auction cannot work properly is as follows: as soon as underdemand appears, one may allot an item from the chosen minimal overdanded set at the previous price system to one of the bidders who demanded that item at that price system, for instance, by having a lottery between these bidders. The bidder to whom the item is allotted, has to pay the price of the item at the previous price system. Of course, allotting the item to one of these bidders implies that the item cannot be assigned to the others who demanded also the item at the same price. So, the auctioneer can only accept one of the bids but has to decline all other equal bids. In Case 2 of Example 1 the auctioneer might accept one of the bids at price system with $p_1 = 4$ and $p_2 = 3$, for instance, by allotting item 2 to bidder 2 against $p_2 = 3$. Then bidder 2 leaves the auction with item 2 and the auction continues with the bidders 1 and 3 and item 1, resulting in a price $p_1 = 5$ at which only bidder

3 demands item 1. This outcome yields a social value of 11 and a revenue of 8 to the seller, resulting in a much better outcome than the one given at the end of the previous section. However, note that this outcome can only sustain because the bid of bidder 3 for item 2 has been declined. In summary, this procedure generates the outcome (p^*, π^*) where $p^* = (p_0^*, p_1^*, p_2^*) = (0, 5, 3)$ and $\pi^* = (\pi^*(1), \pi^*(2), \pi^*(3)) = (0, 2, 1)$. Observe that at prices p^* , bidder 1 gets his best-liked item 0 and pays nothing, bidder 2 gets his best-liked item 2 and pays $p_2^* = 3$ equal to his budget $m^2 = 3$, whereas bidder 3 gets item 1 (second-best) rather than his most-preferred item 2, and pays $p_1^* = 5$. So, bidder 3 finds himself rationed at this outcome on item 2, and bidder 2 who receives item 2 pays his full budget $m^2 = 3$.

The reasoning above gives us a clue to the introduction of an equilibrium under allotment and the design of a dynamic auction. The necessity to decline bids of some bidders while accepting an equal bid of one bidder induces a situation of rationing. After all, any bidder who leaves the auction with a net surplus lower than the net surplus that could have been obtained from an item j when paying the same price as what the bidder paid to which the item was allotted, feels himself a posterior rationed on the demand of such an item j . To explore this observation, we adapt the Walrasian equilibrium by incorporating the concept of an *allotment scheme* $R = (R^1, \dots, R^m)$ where, for $i \in M$, the vector $R^i \in \{0, 1\}^{n+1}$ is a *rationing vector* yielding which goods bidder i can demand and for which goods offers of bidder i will be declined. That is, $R_j^i = 1$ means that bidder i is allowed to demand good j , while $R_j^i = 0$ means that bidder i is not allowed to demand good $j \in N$. When $R_j^i = 0$, we say that bidder i is *rationed* on his demand for item j . If a bidder is not rationed on any item, we say that he is *unrationed*. Since the dummy item is always available for every bidder i , we have that $R_0^i = 1$ for all i . Given a rationing vector R^i with $R_j^i = 0$ for item j , the vector R_{-j}^i denotes the same R^i but allows bidder i to demand item j by ignoring $R_j^i = 0$. At a feasible price vector p and an allotment scheme $R = (R^1, \dots, R^m)$, the demand set of bidder $i \in M$ is given by

$$D^i(p, R^i) = \{j \in N \mid R_j^i = 1, p_j \leq m^i \text{ and } V^i(j) - p_j = \max_{\{k \in N \cup \{0\} \mid p_k \leq m^i \text{ and } R_k^i = 1\}} (V^i(k) - p_k)\}.$$

We now introduce the notion of *equilibrium under allotment* for the assignment model with financially constrained bidders.

Definition 3.1. An equilibrium under allotment (p, π, R) on a market with financially constrained bidders consists of an admissible pair (p, π) and an allotment scheme R such that

- (i) $\pi(i) \in D^i(p, R^i)$ for all $i \in M$;
- (ii) $p_j = c(j)$ for any unassigned item $j \in N_\pi$;
- (iii) If $R_j^i = 0$ for some i , then (a) $j \in D^i(p, R_{-j}^i)$ and (b) there exists $h \in M \setminus \{i\}$ with $\pi(h) = j$ and $m^h = p_j$.

Conditions (i) and (ii) correspond to Conditions (a) and (b) of the definition of the Walrasian equilibrium and are straightforward. In (iii) conditions on the allotment scheme are specified.⁵ First, (iiia) says that any rationing is binding, i.e., a bidder that is rationed on some item, demands the item if the rationing on that item is dropped. Second, (iiib) states that rationing on an item can only occur if the item is sold to some other bidder and that this bidder pays his full budget for the item and thus cannot afford a higher price. Together the conditions imply that it is impossible to drop any of the rations and that in an equilibrium under allotment the seller extracts all the money from the buyer that is assigned a rationed item. In an equilibrium under allotment the prices of the unrationed items are fully competitive. However, the prices of items for which some of the bidders are rationed are not competitive prices in the sense that at these prices there is still overdemand for these items. However, as Example 1 shows, rising these prices results in underdemand and henceforth items with prices above the reservation prices of the seller but nevertheless unsold. When there is no rationing in the equilibrium, i.e., $R_j^i = 1$ for all $i \in M$ and $j \in N$, the equilibrium under allotment is simply a Walrasian equilibrium.

Parallel to the well-known equilibrium existence theorem of [Shapley & Shubik \(1972\)](#) on the assignment market *without financial constraints*, we can establish the following existence theorem on the assignment market *with financial constraints*.

Theorem 3.2. *The assignment model with financially constrained bidders has at least one equilibrium under allotment.*

⁵ These conditions may be seen as the counterparts of standard rationing conditions in fix-price literature, see e.g., [Drèze \(1975\)](#) and [van der Laan \(1980\)](#).

In the next Section we design an ascending auction that always finds an equilibrium under allotment, thus providing a constructive proof for Theorem 3.2. To describe the auction and prove its convergence, we introduce the notions of overdemanded and underdemanded sets and give some of its properties.

For a set of real items $S \subseteq N$, and a price vector $p \in \mathbf{R}_+^{n+1}$, define the *lower inverse demand* set of S at p by

$$D_S^-(p) = \{i \in M \mid D^i(p) \subseteq S\},$$

i.e., this is the set of bidders who demand only items in S . Note that S is a subset of real items, so any bidder i in the lower inverse demand set does not demand the dummy item and thus has a strict positive surplus $V^i(j) - p_j$ for any item j in his demand set $D^i(p)$. We also define the *upper inverse demand* of S at p by

$$D_S^+(p) = \{i \in M \mid D^i(p) \cap S \neq \emptyset\},$$

i.e., this is the set of bidders that demand at least one of the items in S . Clearly, the lower inverse demand set is a subset of the upper inverse demand set. Let $|A|$ stand for the cardinality of a finite set A .

Definition 3.3.

1. A set of real items $S \subseteq N$ is *overdemanded* at price vector $p \in \mathbf{R}_+^{n+1}$ if $|D_S^-(p)| > |S|$. An overdemanded set S is said to be *minimal* if no strict subset of S is overdemanded.
2. A set of real items $S \subseteq N$ is *underdemanded* at price vector $p \in \mathbf{R}_+^{n+1}$ if (i) $S \subseteq \{j \in N \mid p_j > c(j)\}$ and (ii) $|D_S^+(p)| < |S|$. An underdemanded set S is said to be *minimal* if no strict subset of S is an underdemanded set.

The notion of minimal overdemanded set is due to Demange et al. (1986) and the notion of minimal underdemanded set can be found in Mishra & Talman (2006) and is used in a slightly different way by Sotomayor (2002). We further say that an item j is *overpriced* if $\{j\}$ is a minimal underdemanded set, i.e., no bidder has item j in his demand set. So, a minimal underdemanded set S either contains at least two (not overpriced) items, or has an overpriced item as its single element.

In the next three lemmas we give some properties, the proofs of the lemmas are relegated to the Appendix. The first lemma states that for every nonempty subset S of a minimal overdemanded set O at p , the number of bidders in the lower inverse demand set $D_O^-(p)$ that demand at least one item of S is at least

equal to the number of items in S plus the difference between $|D_O^-(p)|$ and $|O|$ and thus is at least one more than the number of items in S .

Lemma 3.4. *Let O be a minimal overdemanded set of items at a price vector p . Then, for every nonempty subset S of O , we have*

$$|\{i \in D_O^-(p) \mid D^i(p) \cap S \neq \emptyset\}| \geq |S| + |D_O^-(p)| - |O|.$$

The next corollary follows immediately.

Corollary 3.5. *For every item in a minimal overdemanded set O at p , there are at least two bidders in $D_O^-(p)$ (actually the number is $|D_O^-(p)| - |O| + 1 \geq 2$) demanding that item.*

The next lemma shows that the number of bidders in the upper inverse demand set of a minimal underdemanded set is precisely one less than the number of items in the set and that any bidder in the upper inverse demand set demands at least two items from the minimal underdemanded set.

Lemma 3.6. *Let U be a minimal underdemanded set of items at a price vector p . Then $|D_U^+(p)| = |U| - 1$ and the demand set $D^i(p)$ of every bidder $i \in D_U^+(p)$ contains at least two elements of U .⁶*

[Mishra & Talman \(2006, Th. 1\)](#) establishes the next result for the case without financial constraints. In fact, this result holds true no matter whether there are financial constraints or not.

Lemma 3.7. *There is a Walrasian equilibrium at $p \in \mathbf{R}_+^{n+1}$ if and only if at p no set of items is overdemanded and no set of items is underdemanded.*

4. AN ASCENDING AUCTION

In this section we introduce an ascending auction which extends the DGS auction to the current setting with financial constraints. The auction starts with $p_j = c(j)$ for each real item $j \in N$ and $p_0 = 0$. In the first round the prices of the items in some minimal overdemanded set are increased. In the DGS

⁶ This lemma, the lemma and corollary above were already introduced in the first draft of this paper ([van der Laan & Yang, 2008](#)) and have found applications elsewhere; see e.g., [Andersson et al. \(2013\)](#) and [Andersson et al. \(2015\)](#).

auction for the model without financial constraints this continues as long as there is overdemand. As soon as there is no overdemand, the auction ends up with an equilibrium price system and an assignment. However, as Example 1 has shown, in case of financial constraints it might happen that an increase of the prices of the items in a (minimal) overdemanded set results in a situation with underdemand. To deal with such situations, in the modified auction precisely one item is allocated each time when a price increase results in an underdemanded set. Roughly speaking, the auctioneer starts by announcing the seller's reservation prices of the real items and requires the bidders to respond with their demand sets. If there is overdemand without any underdemanded set of items, then the prices of the items in a minimal overdemanded set are increased with one and the bidders are required again to report their demand sets. This continues until a situation is reached in which there is either an underdemanded set of items, or there is neither overdemand nor underdemand. When the first case happens, then precisely one of the items in the chosen minimal overdemanded set at the previous price system is sold against its price in this system to one of the bidders who had the item in his demand set; see Step 4 in the auction below for a detailed description. This bidder with the item leaves the market, after which the auctioneer recalls the previous prices for the remaining items and requires the remaining bidders to resubmit their demands for the remaining items at these prices. This continues until either all items are sold subsequently or a situation is reached at which there is neither overdemand nor underdemand. Then there is an equilibrium for the remaining items and bidders.

At each round t of the auction a new price system p^t is announced with the vector of the seller's reservation prices $p^1 = C = (c(0), c(1), \dots, c(n)) \in \mathbb{Z}^{n+1}$ at the first round $t = 1$. During the auction process the set of bidders and the set of items are shrinking, so accordingly these sets and also the notions of price vector, demand set and (minimal) overdemanded and underdemanded sets all have to be adapted. We denote by N^t and M^t the set of real items and the set of bidders respectively that are still involved at round t , meaning that the set of items $N \setminus N^t$ has been assigned to the set of bidders $M \setminus M^t$ before round t . Accordingly, p^t is a vector of $|N^t| + 1$ nonnegative integer prices with $p_0^t = 0$ the price of the dummy item and p_j^t the price of real item j , $j \in N^t$. Correspondingly, the budget set and the demand set of some bidder $h \in M^t$ at round t are given by

$$B^h(p^t) = \{j \in N^t \cup \{0\} \mid p_j^t \leq m^h\},$$

and

$$D^h(p^t) = \{j \in B^h(p^t) \mid V^h(j) - p_j^t = \max_{k \in B^h(p^t)} (V^h(k) - p_k^t)\}.$$

Note again that $0 \in B^h(p^t)$ for every p^t and thus $B^h(p^t)$ is never empty.

The Ascending Auction

Step 1 (Initialization): Set $t := 1$, $p^t := C$, $N^t := N$ and $M^t := M$. Go to Step 2.

Step 2: Every bidder $i \in M^t$ reports his demand set $D^i(p^t) \subseteq N^t \cup \{0\}$. If there exists an underdemanded set at p^t , go to Step 4. Otherwise, go to Step 3.

Step 3: If there is no overdemanded set at p^t , then go to step 5. Otherwise, the auctioneer chooses a minimal overdemanded set $O^t \subseteq N^t$ of items. Then set $p_j^{t+1} := p_j^t + 1$ for every $j \in O^t$, $p_j^{t+1} := p_j^t$ for every $j \in (N^t \setminus O^t) \cup \{0\}$, $M^{t+1} := M^t$ and $N^{t+1} := N^t$. Set $t := t + 1$ and return to Step 2.

Step 4: Let $U^t \subseteq N^t$ be a minimal underdemanded set. Then take some item $k \in U^t \cap O^{t-1}$ and bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O^{t-1}\}$ such that $k \in D^h(p^{t-1})$ and $k \notin D^h(p^t)$. Assign item k to bidder h against price p_k^{t-1} . Set $M^{t+1} := M^t \setminus \{h\}$ and $N^{t+1} := N^t \setminus \{k\}$. If $N^{t+1} = \emptyset$, the auction stops, otherwise let $p_j^{t+1} := p_j^{t-1}$ for all $j \in N^{t+1} \cup \{0\}$. Set $t := t + 1$ and return to Step 2.

Step 5: There is a feasible assignment π^t for N^t, M^t , such that (p^t, π^t) is a Walrasian equilibrium for N^t, M^t . Item $\pi^t(i) \in N^t \cup \{0\}$ is assigned to bidder $i \in M^t$ against price p_k^t , $k = \pi^t(i)$, and the auction stops.

We now explain each step in more detail and then provide an example to illustrate how the auction actually operates. In Step 1, the auctioneer announces a set of items for sale and sets the starting prices equal to the reservation prices.

In Step 2, each bidder is asked to report his demand set for the available items at the current prices. Based on the reported demands from the bidders, the auctioneer checks if there is any underdemanded set of items. If so, then

Step 4 will be performed. Otherwise, the auctioneer goes to Step 3 and checks whether there is any overdemanded set of items. If not, the auction goes to Step 5. In case there is overdemand, the auctioneer chooses a minimal overdemanded set of items and goes to the next round. In this round the price of every item in the chosen minimal overdemanded set is increased by one unit, the price of any other item remains constant and Step 2 will be performed again.

In Step 4, the auctioneer first chooses a minimal underdemanded set. Then she selects precisely one item, say item k , that belonged to the minimal overdemanded set that was chosen in Step 2 at the previous round $t - 1$ and that also belongs to the minimal underdemanded set at the current round t . This item k is assigned to a bidder h satisfying (i) his demand set at $t - 1$ was a subset of the minimal overdemanded set, (ii) who demanded the item k at the previous round $t - 1$, and (iii) who does not demand item k anymore at the current round t . This bidder h pays the price p_k^{t-1} of item k at the previous round and leaves the auction with the item k . When no real items are left, the auction stops. Otherwise, the auction moves to the next round $t + 1$ with the remaining items and bidders and all the remaining items are set equal to the prices in round $t - 1$. Step 2 will be performed again. When the auction reaches Step 5, then according to Lemma 3.7 a Walrasian equilibrium has been reached for the remaining set of items and bidders and the auction terminates.

It should be noted that there will be at least one remaining bidder when the auction returns to Step 2 from Step 4. Clearly, this is true when the number of bidders m is larger than the number of items n , because in Step 4 always precisely one bidder leaves with one item. When $m \leq n$, it might happen that at certain round the auction returns from Step 4 to Step 2 with only one bidder. Obviously then overdemand cannot occur in Step 2. In the next section we prove that underdemand can never occur in Step 2 when the auction returned from Step 4 in the previous round. So, when after Step 4 the auction returns to Step 2 with precisely one bidder, then neither underdemand nor overdemand can occur and the auction goes to Step 5.

Example 2. Consider a market with five bidders (1, 2, 3, 4, 5) and four real items (1, 2, 3, 4). The initial endowment vector of money is given by $m = (m^1, m^2, m^3, m^4, m^5) = (3, 4, 3, 5, 4)$ and bidders' values are given in Table 2. The seller's reservation price vector is given by

$$C = (c(0), c(1), c(2), c(3), c(4)) = (0, 2, 2, 2, 2).$$

Table 2: Bidders' values on each item.

Items	0	1	2	3	4
Bidder 1	0	4	8	5	7
Bidder 2	0	7	6	8	3
Bidder 3	0	5	5	9	7
Bidder 4	0	9	4	6	2
Bidder 5	0	6	5	4	10

Without financial constraints this market has a unique socially efficient assignment $\pi^* = (\pi^*(1), \pi^*(2), \pi^*(3), \pi^*(4), \pi^*(5)) = (2, 0, 3, 1, 4)$, yielding total social value $\sum_{i \in M} V^i(\pi^*(i)) = 36$. The ascending DGS auction finds a minimal equilibrium price vector $p^* = (0, 7, 6, 8, 6)$ and the socially efficient allocation π^* within a finite number of rounds. The seller's revenue generated by the auction is 27.

In the current situation with financial constraints, the bidders cannot afford to buy items at these minimal equilibrium prices. To find an equilibrium under allotment we apply the new ascending auction described above. The price vectors, demand sets and other relevant data generated by the auction are given in Table 3. Since $p_0^t = 0$ for all t , these prices are deleted from the vectors p^t in the second column of Table 3. In the first seven rounds the auction operates in the same way as the DGS auction. Both auctions start at round $t = 1$ with price vector $p^1 = (0, 2, 2, 2, 2)$ (Step 1). Then, in Step 2, bidders report their demand sets: $D^1(p^1) = \{2\}$, $D^2(p^1) = \{3\}$, $D^3(p^1) = \{3\}$, $D^4(p^1) = \{1\}$ and $D^5(p^1) = \{4\}$. There is no underdemand and the auction goes to Step 3. The set $S = \{3\}$ is a minimal overdemanded set and the auctioneer adjusts p^1 to $p^2 = (0, 2, 2, 3, 2)$, after which the process returns to Step 2. Proceeding with alternating Steps 2 and 3, both auctions generate at round 6 price vector $p^6 = (0, 3, 3, 4, 4)$. At this price vector there is overdemand for the items 1 and 2 (there are three bidders for the two items) and, according to Step 3, the prices of the items 1 and 2 are increased. However, at the new price vector $p^7 = (0, 4, 4, 4, 4)$, there is no demand anymore for item 2, i.e., item 2 is overpriced. Now the DGS auction breaks down without reaching an equilibrium. In fact, due to the financial constraints a Walrasian equilibrium does not exist. Of course, in this final round 7 of the DGS auction the auctioneer can still decide to allocate item 1 to the unique bidder 4 having

1 in his demand set, item 3 to the unique bidder 2 and item 4 to the unique bidder 5. However, item 2 is not allocated and the remaining bidders 1 and 3 don't get any real item. The resulting allocation gives a total value of $V^2(3) + V^4(1) + V^5(4) + c(2) = 29$ and is not socially efficient. The seller's revenue from this ad-hoc termination of the DGS auction is only 12 and her total revenue is $12 + c(2) = 14$.

When faced with the overpriced item 2 at round 7, in the new auction the auctioneer continues with Step 4 and assigns item 2 randomly to one of the bidders 1 and 3. Note that both bidders demand item 2 at p^6 and that their demand sets $D^h(p^6)$, $h = 1, 3$, are subsets of the minimal overdemanded set $O^6 = \{1, 2\}$. Further both bidders do not demand item 2 at p^7 . Suppose item 2 is assigned to bidder 1. Then this bidder pays $p_2^6 = 3$ and leaves the auction with item 2. Then round 8 starts with $M^8 = \{2, 3, 4, 5\}$ and $N^8 = \{1, 3, 4\}$, the auctioneer adjusts p^7 to $p^8 = (p_0, p_1, p_3, p_4) = (0, 3, 4, 4)$ (with the same prices as in round 6 for the three remaining real items), and the process returns to Step 2. At p^8 , item 1 is (a minimal) overdemanded (set) and its price is increased to $p_1^9 = 4$. At round 9 there is neither overdemand nor underdemand and the auction goes to Step 5, in which the dummy item 0 is assigned to bidder 3 (who pays nothing) and the items 1, 3 and 4 to the bidders 4 at $p_1^9 = 4$, 2 at $p_2^9 = 4$, and 5 at $p_5^9 = 4$ respectively. This assignment and these prices form a Walrasian equilibrium for the sets $N^9 = \{1, 3, 4\}$ of real items and $M^9 = \{2, 3, 4, 5\}$ of bidders that are still available in round 9.

The final price system $p^* = (p_0, p_1, p_2, p_3, p_4) = (0, 4, 3, 4, 4)$ and assignment $\pi^* = (\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)) = (2, 3, 0, 1, 4)$ form an equilibrium under allotment with allotment scheme $R^* = (R^1, R^2, R^3, R^4, R^5)$, where $R_2^{*3} = 0$ and $R_j^{*i} = 1$ for all $(i, j) \neq (3, 2)$. This equilibrium yields a total value of $\sum_{i \in M} V^i(\pi(i)) = 35$, which is slightly less than the value 36 of the Walrasian equilibrium allocation. Recall that there is no Walrasian equilibrium at all in this example due to budget constraints. At (p^*, π^*, R^*) , the bidders 1, 2, 4 and 5 get their most preferred item. However, bidder 3 gets the dummy item, but prefers and can afford item 2, but this item has been allotted to bidder 1. When in round 7 item 2 should have been assigned to bidder 3 instead of bidder 1, the auction would realize a total value of 32. In both cases the seller's revenue from the auction is 15, which is also her total revenue, because all items are sold. \square

Table 3: The data generated by the auction in Example 4.

Round	p^t	N^t	M^t	$D^1(p^t)$	$D^2(p^t)$	$D^3(p^t)$	$D^4(p^t)$	$D^5(p^t)$	O^t
1	(2,2,2,2)	{1,2,3,4}	{1,2,3,4,5}	{2}	{3}	{3}	{1}	{4}	{3}
2	(2,2,3,2)	{1,2,3,4}	{1,2,3,4,5}	{2}	{1,3}	{3}	{1}	{4}	{1,3}
3	(3,2,4,2)	{1,2,3,4}	{1,2,3,4,5}	{2}	{1,2,3}	{4}	{1}	{4}	{4}
4	(3,2,4,3)	{1,2,3,4}	{1,2,3,4,5}	{2}	{1,2,3}	{4}	{1}	{4}	{4}
5	(3,2,4,4)	{1,2,3,4}	{1,2,3,4,5}	{2}	{1,2,3}	{2}	{1}	{4}	{2}
6	(3,3,4,4)	{1,2,3,4}	{1,2,3,4,5}	{2}	{1,3}	{1,2}	{1}	{4}	{1,2}
7	(4,4,4,4)	{1,2,3,4}	{1,2,3,4,5}	{0}	{3}	{0}	{1}	{4}	
8	(3,4,4)	{1,3,4}	{2,3,4,5}		{1,3}	{1}	{1}	{4}	{1}
9	(4,4,4)	{1,3,4}	{2,3,4,5}		{3}	{0}	{1}	{4}	

5. CONVERGENCE

In this section we show that the auction is well-designed, i.e., all steps are feasible and the auction stops in finitely many rounds. The proofs of all lemmas of this section are given in the Appendix.

First, observe that each time when Step 4 is performed an item is assigned to some of the bidders and both the set of bidders and the set of items decrease by one. So, when $m \leq n$, at each round t we have that $|M^t| \leq |N^t|$. We show that in this case the auction always stops in Step 5. When $m > n$, then at each round t we have that $|M^t| > |N^t|$. In this case the auction stops either in Step 4 when $N^{t+1} = \emptyset$ or in Step 5. In the first case all items are assigned sequentially in a number of n Steps 4, in the latter case the auction reaches a round in which there is neither overdemand nor underdemand. Then, according to Lemma 3.7, there is a Walrasian equilibrium for the sets of remaining items and bidders, showing the feasibility of Step 5. Clearly, also the Steps 1-3 are feasible. So to show feasibility, we only need to consider Step 4.

The auction starts in Step 1 with all prices equal to the seller's reservation prices. At this starting price system there is no underdemand, because by Definition 3.3.2 an item can only be underdemanded when its price is above its seller's reservation price. So, at the starting price vector p^1 in round $t = 1$, either the auction goes to Step 5 and stops, or there is overdemand. In the latter case, a sequence of alternating Steps 2 and Steps 3 is performed with in each Step 3 an increase of the prices of all items in a minimal overdemand set, until there is neither underdemand nor overdemand and the auction goes to Step 5, or items become underdemanded and the auction goes to Step 4. So, when in some round t , the auction goes to Step 4 for the first time, then in round $t - 1$ the prices in some minimal overdemand set, say O^{t-1} , were increased. We prove that this holds in any round t in which the auction goes to Step 4, i.e., when there is underdemand in some round t , then there was overdemand at round $t - 1$ and thus, when the auction reaches Step 4 in round t , then in round $t - 1$ the prices of the items in some minimal overdemand set O^{t-1} were increased. In Step 4 an item k in the intersection of some minimal underdemanded set U^t and the set O^{t-1} is selected and assigned to a bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O^{t-1}\}$ satisfying $k \in D^h(p^{t-1}) \setminus D^h(p^t)$. The next two lemmas state that there indeed exist such an item k and bidder h .

Lemma 5.1. *Let U be a minimal underdemanded set at prices p^t in some round t and let O be the chosen minimal overdemand set at the previous*

round $t - 1$. Then $U \cap O \neq \emptyset$.

Lemma 5.2. *Let U be a minimal underdemanded set at prices p^t in some round t and let O be the chosen minimal overdemanded set at the previous round $t - 1$. Then there exist item k and bidder h satisfying the requirements of Step 4.*

In the special case of $U^t = \{k\}$ with $k \in O^{t-1}$, i.e., the single item k in U^t is overpriced at p^t , we have that no bidder is demanding k at p^t . Hence, any bidder h with $D^h(p^{t-1}) \subseteq O^{t-1}$ and having item k in his demand set $D^h(p^{t-1})$ can be selected. Note that according to Corollary 3.5, there are at least two of such bidders.

The next lemma shows that any time when in some round $t + 1$ the auction enters Step 2 after in round t an item k has been assigned to some bidder h by Step 4, there will be no underdemand of items. So, when the auction arrives in Step 2 after Step 4, then the auction goes always to Step 3. Then, either there is neither overdemand nor underdemand and the auction goes to Step 5 (and stops), or there is overdemand and the prices of items in some minimal overdemanded set are increased. This guarantees that any time when the auction goes to Step 4, prices in some minimal overdemanded set were increased in the previous round. Recall that when in round $t + 1$ Step 2 is reached from Step 4, the price vector p^{t+1} is equal to the price vector p^{t-1} , except that some item k has been deleted.

Lemma 5.3. *Let U be a minimal underdemanded set in round t that appears after in round $t - 1$ the prices of the items in a minimal overdemanded set O were increased, and let, in Step 4, $k \in U \cap O$ be the item assigned to some bidder $h \in \{i \in M^t \mid D^i(p^{t-1}) \subseteq O\}$ such that $k \in D^h(p^{t-1})$, but $k \notin D^h(p^t)$. When the auction proceeds to round $t + 1$ and returns to Step 2, then there will be no underdemanded set of items.*

The final lemma states that when in Step 4 an item has been assigned, the new set of bidders M^{t+1} cannot become empty. This is obvious when the number of bidders is bigger than the number of items. However, it also holds when the set of bidders is at most equal to the number of items. The reason is that when the auction returns from Step 4 to Step 2 with precisely one bidder, the auction goes to Step 5 and stops.

Lemma 5.4. *When in some round t an item k is assigned to some bidder $h \in M^t$ at Step 4 of the auction, then $|M^{t+1}| \geq 1$.*

Now we come to present the convergence theorem for the auction.

Theorem 5.5. *The auction terminates with a feasible assignment and a price system in a finite number of rounds.*

Proof. The auction starts in Step 1 with all prices equal to the seller's reservation prices and the auction goes to Step 2. Now, $p_j = c(j)$ for all j and thus, by definition, there cannot be underdemand and the auction goes to Step 3. When there is also no overdemand, the auction goes to Step 5 and stops. Otherwise, the prices of all items in a minimal overdemanded set are increased and the auction returns to Step 2. The auction continues with alternating Steps 2 and 3 until there is neither overdemand nor underdemand and the auction goes to Step 5 and stops, or underdemand arises for the first time. Since the value of any item to any bidder i is finite and any initial endowment m^i is also finite, one of these cases occurs within a finite number of rounds. When the auction goes to Step 4 and assigns an item k to some bidder h . By Lemmas 5.1 and 5.2 this step is feasible. After that the auction either stops in Step 4 because all items are assigned or, according to Lemma 5.4 returns to Step 2 with at least one remaining bidder. According to Lemma 5.3 there is no underdemand when the auction returns to Step 2 after Step 4. Hence, either there is neither overdemand nor underdemand and the auction goes to Step 5 and stops, or there is overdemand again. Then, similarly as above, within a finite number of rounds again one item is assigned in Step 4, or the auction goes to Step 5 and stops. Repeating this every time after the auction returns in Step 2 after Step 4, it follows that the auction terminates in finitely many rounds, because the number of items is finite.

When the auction stops in Step 4, all items are assigned to different bidders and the auction ends up with a feasible assignment and price system. When the auction stops in Step 5 in some round t , then according to Lemma 3.7 there is a Walrasian equilibrium assignment with respect to the set of items N^t and the set of bidders M^t . Together with the items that have been assigned already before in Step 4, this Walrasian assignment forms a feasible assignment for N and M . Hence the auction terminates with a feasible assignment and a price system in finitely many rounds. \square

6. THE OUTCOME OF THE AUCTION

According to Theorem 5.5 the auction finds a feasible assignment in finitely many rounds. In this section we prove that the feasible assignment and the resulting price system induces an equilibrium under allotment. Let π^* be the assignment resulting from the auction, i.e., $\pi^*(i) = k$ for some $k \in N$ when bidder i was assigned an item in either Step 4 or Step 5, and $\pi^*(i) = 0$ otherwise; and let p^* be the resulting price vector, i.e., when item k is assigned, then p_k^* is the price at which item k is assigned to some bidder h , otherwise p_k^* is the price of the item in the round t in which the auction stops in Step 5. When an item k is assigned at Step 4, then the item is underdemand and thus $p_k^t > c(k)$; and because the auction starts with the reservation price vector C , we have $p_k^t \geq c(k)$ for all items $k \in N^t$ when in round t the auction stops at Step 5. Note that $p_0^t = c(0) = 0$ for all t . Hence $p_k^* = p_k^{t-1} = p_k^t - 1 \geq c(k)$ when item k is assigned in round t by Step 4, $p_k^* = p_k^t \geq c(k)$ for any item k that is assigned in the final round t by Step 5 and $p_0^* = c(0)$ and thus p^* is feasible. Further, when a bidder gets assigned an item in either Step 4 or 5, then the item is in his demand set and thus every bidder i can afford to buy the item $\pi^*(i)$ assigned to him. Hence (p^*, π^*) is admissible. We further define the allotment scheme R^* as follows. For $i \in M$, define R^{i*} by

$$R_k^{i*} = \begin{cases} 0 & \text{if } k \in A^i, \\ 1 & \text{otherwise,} \end{cases} \quad (6.1)$$

where $A^i = \{j \in N \setminus \pi^*(i) \mid p_j^* \leq m^i \text{ and } V^i(j) - p_j^* > V^i(\pi^*(i)) - p_{\pi^*(i)}^*\}$.

Theorem 6.1. *The admissible pair (p^*, π^*) and the allotment scheme R^* yield an equilibrium under allotment (p^*, π^*, R^*) .*

Proof. We have shown above that (p^*, π^*) is an admissible pair. So, it remains to prove that the conditions (i)-(iii) of Definition 3.1 hold. To prove (i), first consider a bidder i that got assigned an item k in Step 4 at some round t against price p_k^{t-1} . Then according to Step 4,

$$k \in D^i(p^{t-1}) = \{j \in N^{t-1} \mid p_j^{t-1} \leq m^i, V^i(j) - p_j^{t-1} = \max_{\ell \in B^i(p^{t-1})} (V^i(\ell) - p_\ell^{t-1})\}$$

where $B^i(p^{t-1}) = \{\ell \in N^{t-1} \cup \{0\} \mid p_\ell^{t-1} \leq m^i\}$. After item k has been assigned to bidder i in round t , the auction continues with Step 2 in round

$t + 1$ with the remaining set of items $N^{t+1} = N^{t-1} \setminus \{k\}$. Since at any stage $\tau \geq t + 1$, $p_j^\tau \geq p_j^{t-1}$ for all $j \in N^{t+1}$, it follows that

$$V^i(k) - p_k^* \geq V^i(j) - p_j^*, \text{ for all } j \in N^{t+1} \text{ with } p_j^* \leq m^i.$$

Further, observe that any $j \in N \setminus N^{t-1}$ has been assigned in some round $\tau \leq t - 1$, before in round t the item k is assigned to bidder i . According to (6.1) we have that $R_j^{*i} = 0$ for all $j \in N \setminus N^{t-1}$ satisfying $p_j^* \leq m^i$ and $V^i(j) - p_j^* > V^i(k) - p_k^*$. Hence $k \in D^i(p^*, R^{*i})$. Second we consider a bidder i who was assigned item k in Step 5 in the final round t . Such a bidder i has item k in his demand set $D^i(p^t)$ with respect to the items in N^t . Again, for any $j \in N \setminus N^t$ that was assigned before in some round $\tau \leq t - 1$, we have that $R_j^{*i} = 0$ when $p_j^* \leq m^i$ and $V^i(j) - p_j^* > V^i(k) - p_k^*$. Hence, also in this case we have that $k \in D^i(p^*, R^{*i})$.

To prove (ii), observe that when an item k is not assigned to a bidder i , then k belongs to the set N^t when the auction stops in Step 5 in the final round t . Then there is neither underdemand nor overdemand and, according to Lemma 3.7, then the auction ends with a Walrasian equilibrium allocation with respect to the remaining items in N^t and the remaining bidders in M^t . By definition of the Walrasian equilibrium we then have that $p_k^* = p_k^t = c(k)$ for any unassigned item k .

Condition (iiia) immediately follows from (6.1). Further, since there is a Walrasian equilibrium for the remaining items N^t and bidders M^t when in the final round t the auction stops in Step 5, it also follows from (6.1) that rationing only occurs for items that have been assigned in some Step 4 before the final round t . To show that the bidder who is assigned a rationed item pays his full budget for that item, again observe that, when for some item j we have that $\pi(h) = j$ and $R_j^{i*} = 0$ for some bidder $i \neq h$, then item j has been allocated in some Step 4 before the end of the auction. Let item j be allocated in some round t . Then item j was in a minimal overdemanded set O at p^{t-1} and for bidder h to which j is assigned it holds that (i) $h \in \{h' \in M^t \mid D^{h'}(p^t) \subseteq O\}$, (ii) $j \in D^h(p^{t-1})$ and (iii) $j \notin D^h(p^t)$. Since $p_k^t = p_k^{t-1} + 1$ for all $k \in O$ and $p_k^t = p_k^{t-1}$ for all $k \in N^t \setminus \{O\}$, it follows that $p_j^{t-1} = m^h$, otherwise j should still have been in the demand set of h at p^t . Hence $p_j^* = p_j^{t-1} = m^h$, which shows (iiib). \square

Theorem 6.1 shows that the auction finds an equilibrium under allotment

in a finite number of price adjustments. First, note that the associated allotment scheme is *endogenously generated*. Second, Theorem 6.1 immediately implies that the existence Theorem 3.2 of Section 3 is true: the assignment model with financially constrained bidders has an equilibrium under allotment. Since at an equilibrium under allotment trade takes place at non-Walrasian prices, the corresponding allocation is typically suboptimal.⁷ Given this suboptimality principle, Example 2 in Section 3 has shown that our ascending auction can realize both a high total value and high revenue for the seller. Property (iiib) of the equilibrium definition also stresses that the seller extracts all the money from the buyer of an item, when other bidders feel themselves rationed for that item.

So far we have considered the case that some or all bidders may confront financial constraints. We have shown that the proposed ascending auction can handle such a situation and always finds an equilibrium under allotment. One may naturally ask whether the proposed auction can find a Walrasian equilibrium when no bidder faces a budget constraint. The following theorem demonstrates that this is indeed the case. This shows that the current auction is indeed an appropriate generalization of the DGS auction to the more complex situation where bidders have budget constraints.

Theorem 6.2. *If $m^i \geq \max_{j \in N} V^i(j)$ for all $i \in M$, then the auction for markets with financially constrained bidders coincides with the DGS auction and finds a Walrasian equilibrium with a minimal equilibrium price vector p^* in finitely many rounds.*

Proof. It is sufficient to show that the ascending auction never generates an underdemanded set of items. It is true in round 1 because the ascending auction starts with the reservation price vector C . Suppose that in some round t , there is no underdemanded set of items and O is the minimal overdemanded set of items chosen by the auctioneer as described in Step 3. We show that there will be no underdemanded set of items in round $t + 1$.

We first prove that no subset S of the set O is underdemanded at p^{t+1} . Because $m^i \geq \max_{j \in N} V^i(j)$ and $0 \notin O$, every bidder $i \in D_O^-(p^t)$ who demands items from S at p^t will continue to demand the same items in S and may demand other items as well at p^{t+1} . It follows from Lemma 3.4 that the set S cannot

⁷ It is known from the literature on equilibria under price rigidities that equilibria with rationing are typically not Pareto efficient, see e.g., Böhm & Müller (1977) and Herings & Kononov (2009).

be underdemanded at p^{t+1} . Second, no subset S of $N^t \setminus O$ is underdemanded at p^{t+1} , because S is not underdemanded at p^t and the price of each item in $N^t \setminus O$ in round $t + 1$ is the same as in round t and the price of each item in O is increased by one in round $t + 1$. Combining the two reasonings for the case $S \subseteq O$ and $S \subseteq N^t \setminus O$, it follows that also any $S \subseteq N^t$ with $S \cap O \neq \emptyset$ and $S \cap (N^t \setminus O) \neq \emptyset$ is not underdemanded at p^{t+1} . So the ascending auction never goes to Step 4 and thus coincides exactly with the DGS auction. It is known that the DGS auction finds an equilibrium with the minimal equilibrium price vector. \square

7. EFFICIENCY AND STRATEGIC ISSUES

7.1. Efficiency

We have seen that under financial constraints a Walrasian equilibrium may not exist. In Section 2 we also show by example that under financial constraints even if a Walrasian equilibrium exists, it need not be socially efficient. However, we will show that under financial constraints every Walrasian equilibrium is Pareto efficient. To discuss Pareto efficiency we first need to give the utilities of all agents, seller and bidders, at an allocation. An *allocation* is a pair (π, x) with π a feasible assignment and $x \in \mathbf{R}_+^m$ a nonnegative vector of money, assigning amount $x_i \geq 0$ of money to bidder i , $i \in M$. Everything that is not assigned to the bidders at allocation (π, x) , is assigned to the seller. So at allocation (π, x) the seller receives the total amount of money $\sum_{i \in M} (m^i - x_i)$ from the bidders and keeps all unsold items for himself. It follows that allocation (π, x) yields utilities

$$U^i(\pi(i), x_i) = V^i(\pi(i)) + x_i - m^i, \quad i \in M,$$

to the bidders and utility

$$U^s(\pi, x) = \sum_{i \in M} (m^i - x_i) + \sum_{j \in N_\pi} c(j)$$

to the seller, i.e., the utility of the seller is equal to the total amount of money he receives plus the sum of his reservation values of the unassigned items. Following the standard definition, we say an allocation (π^*, x^*) is *Pareto efficient* if there does not exist another allocation (π, x) such that

$$U^i(\pi(i), x_i) \geq U^i(\pi^*(i), x_i^*), \text{ for all } i \in M \text{ and } U^s(\pi, x) \geq U^s(\pi^*, x^*)$$

with strict inequality for at least one of the agents.

If (p^*, π^*) is a WE, with p^* the (Walrasian) equilibrium price vector and π^* the (Walrasian) equilibrium assignment, the corresponding allocation (π^*, x^*) with $x_i^* = m^i - p_{\pi^*(i)}^*$ is called a (Walrasian) equilibrium allocation. It is well-known (see [Mas-Colell et al., 1995](#)) that for exchange economies with *divisible goods*, under certain conditions every Walrasian equilibrium allocation is Pareto efficient. However, the proof of the standard model does not apply to our model.

Theorem 7.1. *For the market model with financially constrained bidders, let (p^*, π^*) be an admissible pair. If (p^*, π^*) is a Walrasian equilibrium, then its corresponding equilibrium allocation (π^*, x^*) is Pareto efficient.*

Proof. Suppose that (π^*, x^*) is not Pareto efficient. Then there exists an allocation (π, x) such that

$$V^i(\pi(i)) + x_i - m^i \geq V^i(\pi^*(i)) + x_i^* - m^i = V^i(\pi^*(i)) - p_{\pi^*(i)}^*, \quad (7.2)$$

for every bidder $i \in M$, and for the seller

$$\sum_{i \in M} (m^i - x_i) + \sum_{j \in N_\pi} c(j) \geq \sum_{i \in M} (m^i - x_i^*) + \sum_{j \in N_{\pi^*}} c(j),$$

where at least one of these $m+1$ inequalities is strict. Define $q_j = c(j)$ for $j \in N_\pi$, $q_{\pi(i)} = m^i - x_i$ for every $i \in M$ with $\pi(i) \neq 0$ and $Q = \sum_{\{i \in M | \pi(i) = 0\}} (m^i - x_i)$. Since $p_j^* = c(j)$ when $j \in N_{\pi^*}$, the seller's condition becomes

$$Q + \sum_{j \in N} q_j \geq \sum_{j \in N} p_j^*. \quad (7.3)$$

Since $\pi^*(i) \in D^i(p^*)$, $0 \in B^i(p^*)$ and $p_0^* = 0$, we have for every $i \in M$ that

$$V^i(\pi^*(i)) - p_{\pi^*(i)}^* \geq V^i(0) - p_0^* = 0, \quad i \in M.$$

So, for every $i \in M$ with $\pi(i) = 0$ it follows from (7.2) that $x_i \geq m^i$ and thus $Q \leq 0$. Suppose $q_j > p_j^*$ for some $j \in N$. Since $q_j > p_j^* \geq c(j)$, and $q_h = c(h)$ when $h \in N_\pi$, we must have that $\pi(i) = j$ for some $i \in M$. So, in (π, x) , bidder i receives item j and money amount $x_i \geq 0$. The latter inequality implies that $q_j \leq m^i$. So $p_j^* < q_j \leq m^i$, i.e., item j is in the budget set $B^i(p^*)$ of i at p^* . On the other hand

$$U^i(j, x_i) = V^i(j) + x_i - m^i = V^i(j) - q_j \geq V^i(\pi^*(i)) - p_{\pi^*(i)}^*$$

and thus

$$V^i(j) - p_j^* > V^i(j) - q_j \geq V^i(\pi^*(i)) - p_{\pi^*(i)}^*,$$

which contradicts that $\pi^*(i) \in D^i(p^*)$. Hence $q_j \leq p_j^*$ for all j . With $Q \leq 0$, it follows from inequality (7.3) that $Q = 0$ (and thus $x_i = m^i$ for all i with $\pi(i) = 0$) and $q_j = p_j^*$ for all $j \in N_\pi$. So, the seller's inequality holds with equality.

Suppose that there is a bidder i with strict inequality, thus

$$V^i(\pi(i)) + x_i - m^i > V^i(\pi^*(i)) - p_{\pi^*(i)}^*. \quad (7.4)$$

Since $x_i = m^i$ and thus $V^i(\pi(i)) + x_i - m^i = 0$ if $\pi(i) = 0$, we must have that $\pi(i) \neq 0$. Then $m^i - x_i = q_{\pi(i)} = p_{\pi(i)}^*$ and the inequality (7.4) becomes

$$V^i(\pi(i)) - p_{\pi(i)}^* > V^i(\pi^*(i)) - p_{\pi^*(i)}^*.$$

Since $x_i \geq m^i - p_{\pi(i)}^* \geq 0$ and thus $p_{\pi(i)}^* = q_{\pi(i)} \leq m^i$, this again contradicts that $\pi^*(i) \in D(p^*)$. \square

7.2. Strategic Issues

When the auction results in a Walrasian equilibrium, it also preserves the strategic properties of the DGS auction and thus truthful bidding is optimal for the bidders; see Leonard (1983). It should be noted, however, that without financial constraints in the DGS auction bidders only drop out for their bidding on an item when another item (maybe the dummy item) becomes more preferred. Under financial constraints it might also happen that a bidder drops out for an item because the price of the item rises above his budget. However, this does not affect the strategic properties of the auction as long as there is no underdemand. In conclusion, if underdemand never appears in Step 2, the auction behaves as the DGS auction, and no bidder has incentive to manipulate the auction.

In general, due to budget constraints a Walrasian equilibrium does not exist and our auction generates an equilibrium under allotment at which some bidders are rationed on their demands. Borgs et al. (2005) have demonstrated that it may be impossible to design truthful bidding multi-unit auctions with budget-constrained bidders. Indeed, it could be possible for a rationed bidder to attain a better outcome by misreporting his demands if this bidder knew

all valuations and budgets of all other bidders and convinced that all other bidders would bid honestly. On the other hand, truthful bidding is optimal when the auction terminates with a Walrasian equilibrium. Observe in this case that at the outcome of the auction no bidder is rationed on his demand. We conjecture that this is still true in case of financially constrained bidders: for every unrationed bidder at the outcome of the auction it is in his best interest to bid truthfully. We prove this conjecture for the case of at most two real items.

Theorem 7.2. *For the market model with at most two items and many financially constrained bidders, let (p^*, π^*) be the outcome of the auction when bidders report truthfully, and let i be a bidder that does not find himself rationed in (p^*, π^*) . Then there do not exist values $W^i(j)$, $j = 1, 2$, and outcome (q, ρ) when i reports his demands according to W^i , such that $U^i(\rho(i), m^i - q_{\rho(i)}) > U^i(\pi^*(i), m^i - p_{\pi^*(i)}^*)$.*

Proof. We prove the case of two items, i.e., $N = \{1, 2\}$. The case of one item can be shown similarly. Suppose that there exist $W^i(j)$, $j = 1, 2$, and (q, ρ) such that

$$U^i(\rho(i), m^i - q_{\rho(i)}) > U^i(\pi^*(i), m^i - p_{\pi^*(i)}^*). \quad (7.5)$$

For ease of notation, denote $\pi^*(i) = j$ and $\rho(i) = k$. If $p_k^* > m^i$, then $q_k < p_k^*$. When $p_k^* \leq m^i$, then either $k = j$ and thus $q_k < p_k^*$ (because of inequality (7.5)) or $k \neq j$. In the latter case

$$U^i(k, m^i - q_k) = V^i(k) - q_k > U^i(j, m^i - p_j^*) = V^i(j) - p_j^* \geq V^i(k) - p_k^*,$$

because bidder i is rationed, and thus also in this case $q_k < p_k^*$. So, $q_k < p_k^*$ must hold.

Since $V^i(k) - q_k > V^i(j) - p_j^*$, we must have that $k \neq 0$. So, when reporting according to W^i , bidder i gets a real item. Without loss of generality, take $k = 1$ and thus $q_1 < p_1^*$. Suppose $q_2 < p_2^*$. Then, by the feature of the ascending auction, the number of bidders h satisfying

$$D^h(q) \subseteq \{1, 2\}$$

is at least equal to 3, because otherwise there are at most two bidders that demand a real item at q and the auction cannot reach an outcome in which both prices are higher. So, also when bidder i misreports his demands there are at least two other bidders that demand a real item from $\{1, 2\}$. Since also at

least two bidders can afford the prices p_1^* and p_2^* , the auction cannot terminate with price system q and assigning item 1 to bidder i .

It remains to consider the case that $q_2 \geq p_2^*$. Then under the true valuations, there has been some round t with p^t such that $p_1^t = q_1 < p_1^*$, $p_2^t \leq p_2^* \leq q_2$ and $p_1^\tau > p_1^t$ for all $\tau > t$ (thus the price of item 1 was higher in every round after t). Then at p^t either $\{1\}$ or $\{1, 2\}$ was a minimal overdemanded set. In the first case there was at least one bidder $h \neq i$ that preferred item 1 to any other item. Since $q_2 \geq p_2^t$, also at q all these bidders prefer item 1 to any other item. Since item j was sold at p_1^* and thus at least one bidder has item 1 in his demand set at p^* and could afford p_1^* , also under W^i the auction cannot terminate with price system q and assigning item 1 to bidder i . Finally, in case $\{1, 2\}$ was a minimal overdemanded set, then there were at least two bidders $h \neq i$ with $D^h(p^t) = \{1, 2\}$. Then for all these bidders also $D^h(q) = \{1, 2\}$ when $q_2 = p_2^t$ and $D^h(q) = \{1\}$ if $q_2 > p_2^t$. Since again a bidder paid $p_1^* > q_1$, also in this case the auction cannot terminate under W^i with price system q and assigning item 1 to bidder i . \square

8. CONCLUDING REMARKS

In this paper we investigated a general and practical market model in which an auctioneer wants to sell a number of items to a group of financially constrained bidders. Every bidder demands at most one item and knows his valuation of the items and his budget information privately. The auctioneer does not know this private information unless bidders tell her. When bidders face budget constraints, a Walrasian equilibrium typically fails to exist. We proposed the notion of equilibrium under allotment to remedy the nonexistence of Walrasian equilibrium. An ascending auction has been developed which, starting with the seller's reservation price of each item, always finds an equilibrium under allotment in finite steps. This auction provides an effective allocation mechanism for handling situations with financially constrained bidders, generating high revenues for the seller and arguably efficient assignment of items. Another interesting feature of the auction is that it can extract all the money from those bidders who receive an item on which some other bidder is rationed. We have further shown that when no bidder is financially constrained, the proposed auction reduces to the auction of Demange et al. (1986) and thus preserves the strategic properties of the DGS auction. We have also examined the strategic and efficiency properties of the proposed auction and its outcome.

Finally it is worth noting that that Ausubel (2006), Gul & Stacchetti (2000), Kelso & Crawford (1982), Milgrom (2000), Sun & Yang (2009, 2014) have proposed dynamic auctions for more general environments in which each bidder may consume several items but has no budget constraint. It will be interesting but also significantly more difficult to study this more general situation with financially constrained bidders. Another important question is whether it is possible to design an efficient and strategy-proof dynamic multi-item auction with budget-constrained bidders.

9. APPENDIX

9.1. Proofs of the Lemmas of Section 3

Proof of Lemma 3.4. Since O is overdemanded at p , the constant $d = |D_O^-(p)| - |O|$ must be a positive integer. By definition the lemma holds (with equality) for $S = O$. For any nonempty strict subset S of O , define $D_S = \{i \in D_O^-(p) \mid D^i(p) \cap S \neq \emptyset\}$. Then we have

$$D_O^-(p) \setminus D_S = \{i \in D_O^-(p) \mid D^i(p) \subseteq O \setminus S\}.$$

Suppose to the contrary that $|D_S| < |S| + d$. Since $0 < |S| \leq |O| - 1$, we have that

$$\begin{aligned} |D_O^-(p) \setminus D_S| &= |D_O^-(p)| - |D_S| > |D_O^-(p)| - (|S| + d) = \\ &= |D_O^-(p)| - |S| - (|D_O^-(p)| - |O|) = |O| - |S| = |O \setminus S|. \end{aligned}$$

This means that the set $O \setminus S$ is overdemanded, contradicting the fact that O is a minimal overdemanded set. Hence, $|D_S| \geq |S| + d = |S| + |D_O^-(p)| - |O|$. \square

Proof of Lemma 3.6. If $|U| = 1$, then U consists of an overpriced item and $|D_U^+(p)| = 0$. So, both statements are true.

For $|U| \geq 2$, denote $T = D_U^+(p)$. To prove the first part, suppose $|T| \leq |U| - 2$. Then take any element k of U and denote $T' = D_{U \setminus \{k\}}^+(p)$. Clearly, $T' \subseteq T$ and thus $|T'| \leq |T|$. Hence

$$|T'| \leq |T| \leq |U| - 2 = |U \setminus \{k\}| - 1$$

and thus $U \setminus \{k\}$ is underdemanded, contradicting the assumption that U is a minimal underdemanded set.

To prove the second part, suppose there is a bidder i having only one element of U in his demand set. Let k be this element. Then T' does not contain bidder $i \in T$. Hence $|T'| \leq |T| - 1$ and thus

$$|T'| \leq |T| - 1 = |U| - 2 = |U \setminus \{k\}| - 1,$$

showing that $U \setminus \{k\}$ is underdemanded. Again this contradicts the fact that U is a minimal underdemanded set. \square

Proof of Lemma 3.7.⁸ First, let (p, π) be a Walrasian equilibrium (p, π) . Clearly, at p no set of items is overdemanded and no set of items is underdemanded.

To prove the other direction, let $M^1 = \{i \in M \mid 0 \notin D^i(p)\}$ and $N^1 = \{j \in N \mid p_j > c(j)\}$. First, consider any $T \subseteq M^1$ and let $D^T = \cup_{i \in T} D^i(p)$. Because D^T is not overdemanded, $|D^T| \geq |T|$. By the well-known theorem of Hall (1935), there exists a one-to-one mapping $\tau: M^1 \rightarrow N$ such that $\tau(i) \in D^i(p)$ for all $i \in M^1$. We can extend τ to a mapping from M to $N \cup \{0\}$ by setting $\tau(i) = 0$ for all $i \notin M^1$. Next, consider any $S \subseteq N^1$. Because S is not underdemanded, $|D_S^-(p)| \geq |S|$. Again by Hall's Theorem, there exists a one-to-one mapping $\rho: N^1 \rightarrow M$ such that $j \in D^{\rho(j)}(p)$ for all $j \in N^1$.

With respect to τ and ρ , denote $K = \{i \mid \tau(i) \in N^1\}$, $L = \{\tau(i) \mid i \in K\}$ and $Q = \{\rho(j) \mid j \in N^1 \setminus L\}$ and define the mapping $\pi: M \rightarrow N \cup \{0\}$ by

$$\pi(i) = \begin{cases} \tau(i), & \text{for } i \in M \setminus Q, \\ \rho^{-1}(i), & \text{for } i \in Q. \end{cases}$$

Clearly, $\pi(i) \in D^i(p)$ for all $i \in M$, and no real item is assigned by π to two different bidders, and for every item $j \in N^1$, there is a bidder i who demands the item at p and is assigned the item. This shows that (p, π) is a Walrasian equilibrium. \square

9.2. Proofs of the lemmas of Section 5

In proofs of this subsection it should be noted that the sets $D_S^-(p^\tau)$ and $D_S^+(p^\tau)$ are defined with respect to the current set of bidders M^τ , for any set $S \subset N^\tau$ and for any $\tau = t - 1, t$.

Proof of Lemma 5.1. Suppose to the contrary that $U \cap O = \emptyset$. Since U is underdemanded at round t , we have that $p_j^t > c(j)$ for any $j \in U$. Further, since $U \cap O = \emptyset$, we have for any $j \in U$ that $j \notin O$. Hence $p_j^t = p_j^{t-1}$ and thus also $p_j^{t-1} > c(j)$ for all $j \in U$. Since there is no underdemand in round $t - 1$, it follows that $|D_U^+(p^{t-1})| \geq |U|$. Moreover, any bidder that demands some item $j \in U$ at p^{t-1} , also demands this item at p^t , because only prices of the items in O are increased. Hence $|D_U^+(p^t)| \geq |D_U^+(p^{t-1})| \geq |U|$ and thus U is not underdemanded at p^t , yielding a contradiction. Hence $U \cap O \neq \emptyset$. \square

Proof of Lemma 5.2. Since O is overdemanded at p^{t-1} , we have $|D_O^-(p^{t-1})| > |O|$.

⁸ This proof is much simpler than the original one given by Mishra and Talman (2006).

Now, consider the set $S = U \cap O$. By Lemma 5.1 this set is not empty. When $U = O$ and thus $S = O$, then by Lemma 3.6 there are $|U| - 1 = |O| - 1$ bidders demanding at least one item from U at p^t , because U is underdemanded at p^t . So, in this case there are at least two bidders in $D_O^-(p^{t-1})$ not demanding any item from $U = O$ anymore at price p^t . Select h from this set of bidders and select k from the set $D^h(p^{t-1})$ (recall that this set is never empty and does not contain any dummy item). Since $D^h(p^{t-1}) \subseteq O$ and for each bidder $h \in D_O^-(p^{t-1})$, this item k and this bidder h satisfy the requirements.

Next, consider the case that S is a strict subset of O . Denote $H = \{i \in D_O^-(p^{t-1}) \mid D^i(p^{t-1}) \cap S \neq \emptyset\}$. From Lemma 3.4 we have that

$$|H| \geq |S| + |D_O^-(p^{t-1})| - |O| \geq |S| + 1,$$

i.e., the number of bidders in $D_O^-(p^{t-1})$ that demand an item of S at p^{t-1} is at least one more than the number of items in S . Next, consider the set $T = U \setminus O$. Since there is no underdemand at p^{t-1} we have that

$$|D_T^+(p^{t-1})| \geq |T|.$$

Since $p_j^t = p_j^{t-1}$ for all $j \in T = U \setminus O$, any bidder that demands an item from T at p^{t-1} , is still demanding this item at p^t , so $D_T^+(p^{t-1}) \subseteq D_T^+(p^t)$. On the other hand, U is underdemanded at p^t , so

$$|D_U^+(p^t)| < |U|.$$

Further, observe that $H \cap D_T^+(p^{t-1}) = \emptyset$, since $H \subseteq D_O^-(p^{t-1})$ and the members of $D_O^-(p^{t-1})$ demand only items in O , whereas the members of $D_T^+(p^{t-1})$ demand at least one item from $T = U \setminus O$ at p^{t-1} . Therefore, the number of bidders in H that still demand items in S at p^t can be at most $|S| - 1$. Suppose not, i.e., the number is at least $|S|$. Then the number of bidders in $D_U^+(p^t)$ (demanding at least one item of U at p^t) is at least equal to $|S|$ plus the number of bidders in $D_T^+(p^{t-1})$, i.e.

$$|D_U^+(p^t)| \geq |S| + |T| = |U \cap O| + |U \setminus O| = |U|,$$

contradicting the fact that U is underdemanded. Hence there are at least two bidders in H that are no longer demanding items in $U \cap O$ at p^t . Select h as one of these bidders and k as one of the elements in the non-empty set $D^h(p^{t-1}) \cap S$. Then item k and bidder h satisfy the requirements. \square

Proof of Lemma 5.3. First, observe that, by definition of the auction, $M^{t+1} = M^{t-1} \setminus \{h\}$, $N^{t+1} = N^{t-1} \setminus \{k\}$ and $N^{t+1} \neq \emptyset$ (otherwise the auction ends in Step 4). Further, $p_j^{t+1} = p_j^{t-1}$ for all $j \in N^{t+1}$. Denote $\tilde{O} = O \setminus \{k\}$. For $S \subseteq N^{t+1}$ we

consider two cases, namely $S \subseteq \tilde{O}$ and $S \setminus \tilde{O} \neq \emptyset$. In the first case we have by Lemma 3.4 that at least $|S| + 1$ members of the set $D_{\tilde{O}}^-(p^{t-1}) = \{i \in M^{t-1} \mid D^i(p^{t-1}) \subseteq O\}$ demanded at least one item of S in round $t - 1$. Since $p_j^{t+1} = p_j^{t-1}$ for all $j \in N^{t+1}$, for any bidder i in M^{t+1} it holds that

$$D^i(p^{t+1}) = D^i(p^{t-1}) \setminus \{k\}$$

and thus any bidder $i \in M^{t+1} \cap D_{\tilde{O}}^-(p^{t-1}) = D_{\tilde{O}}^-(p^{t-1}) \setminus \{h\}$ that demanded an item of S at round $t - 1$ is still demanding an item of S at round $t + 1$. So, when h demanded an item of S at round $t - 1$, the number of bidders of M^{t+1} demanding an item of S at round $t + 1$ is at least $|S|$, otherwise the number is at least $|S| + 1$. Hence S is not underdemanded.

For the second case $S \setminus \tilde{O} \neq \emptyset$ we consider the partition of S given by $S^1 = S \cap \tilde{O}$ and $S^2 = S \setminus \tilde{O}$. Denote

$$K^1 = \{i \in D_{\tilde{O}}^-(p^{t-1}) \mid D^i(p^{t-1}) \cap S^1 \neq \emptyset\}$$

and

$$K^2 = \{i \in M^{t-1} \mid D^i(p^{t-1}) \cap S^2 \neq \emptyset\}.$$

Since $D^i(p^{t-1}) \subseteq O$ for all $i \in D_{\tilde{O}}^-(p^{t-1})$ and $S^2 \subseteq N^{t-1} \setminus O$, it follows that $K^1 \cap K^2 = \emptyset$. Since O is a minimal overdemanded set in round $t - 1$ and there is no underdemand in round $t - 1$, we have that S^1 is neither overdemanded nor underdemanded at p^{t-1} , because it is a strict subset of O . By Lemma 3.4 we have that at least $|S^1| + 1$ members of $D_{\tilde{O}}^-(p^{t-1})$ demanded at least one item of S^1 in round $t - 1$ and similarly as above it follows that at least $|S^1|$ members of $D_{\tilde{O}}^-(p^{t-1}) \setminus \{h\}$ are still demanding an item of S^1 at round $t + 1$. Furthermore, none of these bidders belong to K^2 , because $D_{\tilde{O}}^-(p^{t-1}) \cap K^2 = \emptyset$. Further $|K^2| \geq |S^2|$, because there is no underdemand at round $t - 1$. Clearly, any member of K^2 is still demanding an item of S^2 at round $t + 1$, because all prices of the remaining items in N^{t+1} are equal to the prices in round $t - 1$. Therefore the number of bidders that demand at least one item of $S = S^1 \cup S^2$ is at least equal to

$$|S^1| + |K^2| \geq |S^1| + |S^2| = |S|$$

and thus S is not underdemanded in round $t + 1$. \square

Proof of Lemma 5.4. Each time when an item is assigned in Step 4, the number of items and the number of bidders decreases with one. Suppose that in some round t Step 4 is performed for the ℓ th time. As long as $\ell < |M| - 1$, we have that $M^{t+1} = |M| - \ell > 1$. Now, suppose that $\ell = |M| - 1$. Then $|M^{t+1}| = 1$ and the auction returns to Step 2. According to Lemma 5.3, there is no underdemand in Step 2 and thus the auction goes to Step 3. However, because only one bidder is left, also overdemand cannot occur and thus the auction goes to Step 5 and terminates. \square

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