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A NEW EVALUATION CRITERION FOR ALLOCATION MECHANISMS WITH APPLICATION TO VEHICLE LICENSE ALLOCATIONS IN CHINA

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ABSTRACT

In this paper, we propose an equality measure for allocation mechanisms with budget constraints to describe the difference in object obtaining opportunities among buyers with different budget ranks. We evaluate allocation mechanisms not only from the perspective of efficiency and revenue, but also with the criterion of equality. As an application of this new evaluation criterion – the equality measure, we study the vehicle license allocation problem in China, introduce a class of hybrid auction-lottery mechanisms, and evaluate China's vehicle license allocation in a unified framework from the criteria of efficiency, equality, and revenue.

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1. INTRODUCTION

IN auction theory and market design, allocation mechanisms are usually evaluated solely by the criteria of efficiency and revenue. Nevertheless, in the allocation of public resources, equality should not be neglected for the sake of efficiency and revenue, since the conflict between efficiency and equality, as Okun (1975) points out, is an important tradeoff in economics and resource allocation. Indeed, in the practice of public resource allocation, equality is often taken into account, and non-price and hybrid mechanisms are often applied for the sake of equality. For example, public school admission and public rental housing are usually allocated through lotteries in China. Public health care is allocated by queues in countries such as UK and Canada. The rights of passage through the Panama Canal, hunting permits in several U.S. states, and U.S. immigration visas are allocated through a combination of price and non-price mechanisms. To sum up, the importance of equality in both economics and the practice of public resource allocation necessitates that we evaluate allocation mechanisms by the criterion of equality, in addition to efficiency and revenue.

Public opinion believes that, for some publicly-provided goods, equality of allocation requires that people with different wealth levels should have more or less equal chance of obtaining these goods.[1] People also believe that lottery, which allocates goods to different people with absolutely equal chance, is the most equal mechanism. Based on these common understandings of equality, we think that the equality of an allocation mechanism should reflect the difference in object obtaining opportunities among buyers with different wealth levels, and we shall propose one method to measure this difference. In this paper, we provide a class of general incentive compatible (IC) random direct mechanisms with budget constraints to describe the allocation of

[1] For example, Williams (2010) argues that "the notion of equality of opportunity...[is] that a limited good shall in fact be allocated on grounds which do not *a priori* exclude any section of those that desire it", and he believes that allocating some goods on grounds of wealth constitutes such an *a priori* exclusion. (Williams (2010), pp.243-244.)

publicly provided homogeneous goods, and propose a proper equality measure to evaluate such mechanisms. More specifically, for any IC random direct mechanism, we compute the expected object obtaining probabilities of buyers with different wealth levels (budgets), draw a Lorenz curve with these expected probabilities, and define a formal equality measure that is analogous to the Gini coefficient.[2] By proposing such a new evaluation criterion, we fill a gap in literature and make a contribution to auction theory.

It is worth emphasizing that we use a budget-constraint model to present our equality measure for two main reasons. As mentioned above, we believe that the equality of an allocation mechanism reflects the difference in object obtaining opportunities among buyers with different wealth levels. In this paper, we use a budget-constraint model to represent the heterogeneity in buyers' wealth levels. In addition, if the publicly-provided goods are large durable goods, the quasi-linear utility assumption is no longer valid for the allocation of those goods. In literature, there are usually two approaches that deal with large durable goods: budget constraints, or non-linear utility. In this study, we incorporate budget constraints into our analysis.

After establishing a proper equality measure for public resource allocation mechanisms, we proceed to an application of our equality measure, i.e., vehicle license allocation in China, because it is representative of allocation of publicly-provided goods, and is also important in China. To evaluate China's major vehicle license allocation mechanisms in a unified framework, and to provide new insights for improving license allocation, we propose a class of hybrid auction-lottery mechanisms. To conveniently compute the characteristics (efficiency, equality, and revenue) of the hybrid mechanisms, we further provide a continuum-mass hybrid mechanism for each discrete hybrid mechanism, present the formulas for its characteristics, and discuss the false-name bidding proof condition under the continuum-mass hybrid mechanism. Furthermore, to give a benchmark to compare with the hybrid mechanisms, we propose a probability allocation mechanism which relaxes the requirement of ex-post individual rationality. Finally, using simulation and numerical computation, we verify the robustness of approximating the characteristics of discrete hybrid mechanisms from those of continuum-mass hybrid mechanisms, depict the set of attainable characteristics of the hybrid mechanisms, and demonstrate that there is considerable room for improvement for some license allocation

[2] Of course, there may exist different methods to measure this difference, so the equality measure is not unique.

mechanisms in China.

Our study is connected with several strands of literature, which covers equality in public resource allocation, auctions with budget constraints, hybrid auction-lottery mechanisms, and vehicle license allocation. We will introduce them briefly.

Equality is an important issue in literature on public resource allocation. For example, Kahneman et al. (1986) discuss public standards of fairness for market allocations and indicate that some market anomalies can be explained by introducing fairness or equality. Taylor et al. (2003) argue that "lotteries are usually employed to resolve allocation problems in order to reflect a spirit of fairness and equality" (p.1316). Evans et al. (2009) observe that the choice between market and non-market mechanisms reflects the trade-off between efficiency and equity. For vehicle license allocations, Chen & Zhao (2013) demonstrate that in their survey, most respondents held negative attitudes toward the equity of the Shanghai auction. Although these articles mention the importance of equality, they neither clarify the meaning of equality, nor provide a formal equality measure of allocation mechanisms. Thus, they cannot compare different mechanisms in terms of equality. In fact, Taylor et al. (2003) and Evans et al. (2009) do not compare the equality of different mechanisms. Dworczak et al. (2019) define an equality measure and design optimal mechanism under the setting of non-linear utility, however their approach is not connected with the rich literature on mechanism design with budget constraints, and is unintuitive to apply. In this paper, we clarify the meaning of equality, and provide a formal equality measure of allocation mechanisms with budget constraints.

Auctions with budget constraints have been widely discussed in the literature. Laffont & Roberts (1996) examine an optimal sealed-bid single-item auction with budget constrained bidders. Che & Gale (1998) analyze bidding strategies, as well as efficiency and revenue in standard single-object auctions with private budget constraints, and find that first-price auctions yield higher revenue and social surplus than second-price auctions when buyers face absolute budget constraints. Maskin (2000) investigates second-price single-item auctions and all-pay single-item auctions under financial constraints. Che & Gale (2000) discuss the optimal mechanism of selling an object to a buyer, and Pai & Vohra (2014) further characterize optimal single-unit budget-constrained auctions. Talman & Yang (2015) propose an efficient dynamic auction for a market where there are multiple heterogenous items for sale and every bidder demands at most one item but faces a budget constraint. They show that their auction always finds a core allocation. Laan van der & Yang (2016) study a similar market and develop an ascending auction which locates a constrained equilibrium. Li (2017) provides a two-stage surplus-maximizing mechanism including cash subsidies, lotteries, and resale tax. In this paper, to apply the equality to broader classes of allocation mechanisms, we adopt a discrete multi-unit allocation model to discuss allocation mechanism with budget constraints.

Another related strand of literature focuses on hybrid auction-lottery mechanisms. Evans et al. (2009), in a paper similar to our study, examine the hybrid mechanisms with an auction implemented before a lottery and discuss buyers' bidding strategies in such mechanisms. Condorelli (2013) reveals that if buyers' values and willingness to pay do not align, a hybrid mechanism may be efficiency-optimal. Che et al. (2013) confirm this result and prove that an efficiency-optimal mechanism with budget constraints involves an in-kind subsidy and a cash incentive for discouraging low-valuation buyers from claiming the good. Our study adds a common reserve price to the hybrid mechanisms to raise both efficiency and revenue.

Several studies discuss vehicle license allocation in practice. Liao & Holt (2013) examine the modification of Shanghai license auction in 2008 and indicate through experiments that this modification, which aims to curb revenue, will result in loss of efficiency. Huang & Wen (2019) present buyers' bidding strategies under the Guangzhou mechanism. To our knowledge, no studies have yet considered the social planner's overall objectives in designing a vehicle license allocation mechanism and examined the existing license allocation mechanisms in a unified framework.

The remainder of the study is organized as follows. Section 2 describes the basic environment, proposes a class of random direct mechanisms to generalize public resource allocation mechanisms with budget constraints, and defines an equality measure for evaluating such mechanisms in terms of equality, in addition to efficiency and revenue. Section 3, 4, 5 and 6 study vehicle license allocation in China as an application of the equality measure. Specifically, Section 3 introduces vehicle license allocation in China, and proposes a class of hybrid mechanisms. Section 4 introduces the continuum-mass hybrid mechanisms, presents the formulas of its characteristics, and discusses buyers' incentives for false-name bidding. Section 5 presents the probability allocation mechanism as a benchmark mechanism. Section 6 employs numerical analysis to present the attainable characteristics of different mechanisms with figures, and evaluate mechanisms in terms of these characteristics. All proofs are provided in Appendix A.

2. EQUALITY MEASURE FOR ALLOCATION MECHANISMS

In this section, we shall propose an equality measure for allocation mechanisms. We first present the basic model: a multi-unit auction model with budget constraints. To characterize allocation mechanisms with budget constraints, we define a class of IC random direct mechanisms. Under such an IC random direct mechanism framework, we then provide a new evaluation criterion for allocation mechanisms besides the canonical criteria of efficiency and revenue.

2.1. Basic model

A social planner wishes to allocate m units of publicly-provided goods to n buyers. Every buyer $i \in \mathcal{N} = \{1, 2, \cdots, n\}$ is assumed to have unit demand, holds a value v_i, and is subject to a budget of w_i. Buyer i's private type $x_i = (v_i, w_i)$ is drawn from $\mathcal{X}_i = [0, \bar{v}] \times [0, \bar{w}]$ (here, $\bar{v}$ and $\bar{w}$ may be $+\infty$) according to a commonly known joint distribution function $\Phi(v, w)$ with density function $\phi(v, w)$. We assume that $\Phi(v, w)$ is strictly increasing in both v and w. Buyers' types are mutually independent. Each buyer knows her own type but not others' types. Every buyer $i \in \mathcal{N}$ has a utility function that is quasi-linear up to her budget constraint,

$$u_i(q_i, p_i, v_i, w_i) = \begin{cases} q_i v_i - p_i, & \text{if } p_i \leq w_i, \\ -\infty, & \text{if } p_i > w_i, \end{cases}$$

where q_i is buyer i's probability of obtaining an object and p_i is her required payment.

First, we introduce some notation. Let $\mathcal{X} = \times_{i=1}^{n} \mathcal{X}_i$ and $\mathcal{X}_{-i} = \times_{j \neq i} \mathcal{X}_j$ denote the space of all buyers' types, and the space of types of all buyers excluding i, respectively. Then, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{X}$ represents a profile of all buyers' types, and $\boldsymbol{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathcal{X}_{-i}$ represents a type profile of all buyers but i. Write $\psi(\boldsymbol{x}) = \Pi_{i=1}^{n} \phi(x_i)$ as the joint density function of all buyers' types, and $\psi_{-i}(\boldsymbol{x}_{-i}) = \Pi_{j \neq i} \phi(x_j)$ as the joint density function of the types of all buyers but i. Let

$$F(v) = \int_0^v \int_0^{\bar{w}} \phi(t,w)dwdt \quad \text{and} \quad G(w) = \int_0^{\bar{v}} \int_0^{w} \phi(v,t)dtdv$$

denote the marginal distribution functions of a buyer's value and budget, respectively. The marginal density functions of a buyer's value and budget are

$$f(v) = \int_0^{\bar{w}} \phi(v,w)dw \quad \text{and} \quad g(w) = \int_0^{\bar{v}} \phi(v,w)dv,$$

respectively. Write $F(v|w) = \int_0^v \frac{\phi(t,w)}{g(w)}dt$ as the conditional distribution function of a buyer's value when her budget is w. We impose the following assumption on the distribution of buyers' types.

Assumption 1. *For any pair $\{w, w'\}$ with $w \leq w'$, the conditional distribution function $F(v|w')$ first-order stochastically dominates $F(v|w)$. That is, $F(v|w') \leq F(v|w)$ for all $v \in [0, \bar{v}]$.*

Assumption 1 is standard in the literature, and it implies that a buyer with a higher budget is more likely to have a higher value.

2.2. Random direct mechanism

By the Revelation Principle, we can restrict attention to incentive compatible direct mechanisms (abbreviated as IC direct mechanisms) in which every buyer $i \in \mathcal{N}$ prefers to report her true type $x_i = (v_i, w_i) \in \mathcal{X}_i$. Usually, a direct mechanism is defined by the interim assignment and the expected payment rules. However, in the presence of budget constraints, interim assignment and payment rules are not adequate for characterizing a direct mechanism,[3] because buyers' incentive properties may depend on the ex-post assignment and payment. Therefore, in this study, we shall consider the random direct mechanisms, in which interim assignments and payments are implemented by random assignment and payment rules that never make a buyer pay over her budget report.

Before defining random direct mechanisms, we first define the ex-post allocation of an allocation mechanism. Each ex-post assignment of objects

[3] To better appreciate the necessity of introducing random direct mechanisms, one can refer to the probability allocation mechanism presented in Section 5.

can be described as a vector $\pi \in \mathbb{Z}^n$ with each $\pi_i = 0$ or 1 and $\sum_{i\in\mathcal{N}} \pi_i \leq m$, where $\pi_i = 1$ (or, 0) represents that buyer i obtains (or does not obtain) an object. Let $\Pi = \{\pi \in \mathbb{Z}^n | \pi_i = 0, 1, \text{ and } \sum_{i\in\mathcal{N}} \pi_i \leq m\}$ denote the set of all possible ex-post assignments. Then, the set of all possible ex post allocations (allocation = assignment + payment) can be written as $\Omega = \Pi \times [0, \bar{w}]^n$. Let $\mathcal{F}(\Pi)$ and $\mathcal{F}(\Omega)$ be the spaces of all random vectors defined on Π and Ω, respectively.

We define a *random direct mechanism* for public resource allocation as a mapping

$$\mathbf{\Gamma} = (\Gamma_{11}, \Gamma_{21}, \ldots, \Gamma_{n1}, \Gamma_{12}, \Gamma_{22}, \ldots, \Gamma_{n2}) \ : \ \mathcal{X} \ \rightarrow \ \mathcal{F}(\Omega)$$

such that $\text{Prob}\{\Gamma_{i2}(\hat{\boldsymbol{x}}) \leq \hat{w}_i\} = 1$ for all $i \in \mathcal{N}$ and $\hat{\boldsymbol{x}} \in \mathcal{X}$.

For a report profile $\hat{\boldsymbol{x}}$, $\mathbf{\Gamma}(\hat{\boldsymbol{x}}) = (\Gamma_{11}(\hat{\boldsymbol{x}}), \ldots, \Gamma_{n1}(\hat{\boldsymbol{x}}), \Gamma_{12}(\hat{\boldsymbol{x}}), \ldots, \Gamma_{n2}(\hat{\boldsymbol{x}}))$ denotes a random allocation. The social planner assigns an object to each buyer i according to the random variable $\Gamma_{i1}(\hat{\boldsymbol{x}})$, and extracts payment from buyer i according to the random variable $\Gamma_{i2}(\hat{\boldsymbol{x}})$. Each realization $(\pi(\hat{\boldsymbol{x}}), p(\hat{\boldsymbol{x}}))$ of $\mathbf{\Gamma}(\hat{\boldsymbol{x}})$ denotes an ex-post outcome of the random allocation $\mathbf{\Gamma}(\hat{\boldsymbol{x}})$, where each buyer i gets $\pi_i(\hat{\boldsymbol{x}})$ object and pays $p_i(\hat{\boldsymbol{x}})$. The condition $\text{Prob}\{\Gamma_{i2}(\hat{\boldsymbol{x}}) \leq \hat{w}_i\} = 1$ ensures that no matter what other buyers report, each buyer i's ex-post payment can never exceed her reported budget. Note that these $2n$ random variables $\Gamma_{ij}(\hat{\boldsymbol{x}})$ $(i \in \mathcal{N}, j = 1, 2)$ may be inter-dependent.

For a given random direct mechanism $\mathbf{\Gamma}$ and an arbitrary $\hat{\boldsymbol{x}} \in \mathcal{X}$, let

$$Q_i(\hat{\boldsymbol{x}}) = E[\Gamma_{i1}(\hat{\boldsymbol{x}})] \quad \text{and} \quad M_i(\hat{\boldsymbol{x}}) = E[\Gamma_{i2}(\hat{\boldsymbol{x}})], \quad \text{for each } i \in \mathcal{N},$$

represent buyer i's expected probability of obtaining an object and expected payment, respectively. Thus, we obtain a normal interim direct mechanism

$$(\mathrm{Q}, \mathrm{M}) = (Q_1, \ldots, Q_n, M_1, \ldots, M_n) \ : \ \mathcal{X} \rightarrow \Delta \times [0, \bar{w}]^n,$$

where $\Delta = \left\{q \in \mathbb{R}^n_+ \mid q_i \in [0,1], \sum_{i=1}^n q_i \leq m\right\}$. Henceforth, we refer to (Q, M) as the *associated direct mechanism* of $\mathbf{\Gamma}$. In addition, we say a random direct mechanism $\mathbf{\Gamma}$ is *standard* if its associated direct mechanism (Q, M) satisfies the following two properties.

1. **Anonymity** For any $i, j \in \mathcal{N}$, it holds that $Q_i(\bar{\boldsymbol{x}}) = Q_j(\hat{\boldsymbol{x}})$ and $M_i(\bar{\boldsymbol{x}}) = M_j(\hat{\boldsymbol{x}})$ for all $\bar{\boldsymbol{x}}, \hat{\boldsymbol{x}} \in \mathcal{X}$ satisfying $\bar{x}_i = \hat{x}_j$, $\bar{x}_j = \hat{x}_i$, and $\bar{x}_l = \hat{x}_l$ for all $l \in \mathcal{N} \setminus \{i, j\}$.

2. **Monotonicity** For any $i \in \mathcal{N}$, it holds that $Q_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}) \geq Q_i(\hat{x}'_i, \hat{\boldsymbol{x}}_{-i})$ for all $\hat{\boldsymbol{x}}_{-i} \in \mathcal{X}_{-i}$, $\hat{x}_i$, and $\hat{x}'_i \in \mathcal{X}_i$ such that $\hat{x}_i \geq \hat{x}'_i$.[4]

We just consider standard random direct mechanisms in this study, and hence we shall omit the term "standard" without confusion.

For each buyer i and any type $\hat{x}_i \in \mathcal{X}_i$, let

$$\begin{aligned} q_i(\hat{x}_i) &= \int_{\mathcal{X}_{-i}} Q_i(\hat{x}_i, \boldsymbol{x}_{-i}) \psi_{-i}(\boldsymbol{x}_{-i}) d\boldsymbol{x}_{-i} \\ m_i(\hat{x}_i) &= \int_{\mathcal{X}_{-i}} M_i(\hat{x}_i, \boldsymbol{x}_{-i}) \psi_{-i}(\boldsymbol{x}_{-i}) d\boldsymbol{x}_{-i} \end{aligned} \tag{2.1}$$

denote her expected probability of winning a object and her expected payment when she reports $\hat{x}_i$ and all other buyers report their true types. By the anonymity of a standard mechanism and the identical distribution of buyers' types, we see that all buyers are symmetric from an ex-ante perspective. Therefore, we use $q(v, w)$ to represent $q_i(v, w)$ and use $m(v, w)$ to represent $m_i(v, w)$ for any $i \in \mathcal{N}$ and $(v, w) \in \mathcal{X}_i$.

We say a random direct mechanism $\boldsymbol{\Gamma}$ is *interim individually rational* if, for all $i \in \mathcal{N}$ and $\hat{\boldsymbol{x}}_{-i} \in \mathcal{X}_{-i}$, the following is satisfied:

$$u_i(Q_i(x_i, \hat{\boldsymbol{x}}_{-i}), M_i(x_i, \hat{\boldsymbol{x}}_{-i}), x_i) = Q_i(x_i, \hat{\boldsymbol{x}}_{-i}) v_i - M_i(x_i, \hat{\boldsymbol{x}}_{-i}) \geq 0,$$

for all $x_i \in \mathcal{X}_i$. We also say a random direct mechanism $\boldsymbol{\Gamma}$ is *ex-post individually rational* if, for all $i \in \mathcal{N}$, $x_i \in \mathcal{X}_i$ and $\hat{\boldsymbol{x}}_{-i} \in \mathcal{X}_{-i}$, it satisfies that

$$u_i(\pi_i(x_i, \hat{\boldsymbol{x}}_{-i}), p_i(x_i, \hat{\boldsymbol{x}}_{-i}), x_i) = \pi_i(x_i, \hat{\boldsymbol{x}}_{-i}) v_i - p_i(x_i, \hat{\boldsymbol{x}}_{-i}) \geq 0,$$

for any realization $(\pi_i(x_i, \hat{\boldsymbol{x}}_{-i}), p_i(x_i, \hat{\boldsymbol{x}}_{-i}))$ of $(\Gamma_{i1}(x_i, \hat{\boldsymbol{x}}_{-i}), \Gamma_{i2}(x_i, \hat{\boldsymbol{x}}_{-i}))$.

In addition, a random direct mechanism $\boldsymbol{\Gamma}$ is said to be *incentive compatible* if, for all $i \in \mathcal{N}$ and $x_i \in \mathcal{X}_i$, the following is satisfied:

$$\begin{aligned} &u_i(q(x_i), m(x_i), x_i) \\ =\ &q(x_i) v_i - m(x_i) \\ \geq\ &q(\hat{x}_i) v_i - m(\hat{x}_i) \\ =\ &u_i(q(\hat{x}_i), m(\hat{x}_i), x_i), \end{aligned}$$

[4] The property of "monotonicity" implies that given all other buyers' reports, a buyer's object obtaining probability is nondecreasing in her report type. In literature, a standard auction is an auction in which buyers who propose the highest bids always win the objects (Krishna, 2010), while in a standard mechanism defined here, a higher type reporting is always accompanied by a higher object obtaining probability.

for all $\hat{x}_i \in \mathcal{X}_i$ satisfying Prob$\{\Gamma_{i2}(\hat{x}_i, \boldsymbol{x}_{-i}) \leq w_i\} = 1$ for all $\boldsymbol{x}_{-i} \in \mathcal{X}_{-i}$. A random direct mechanism $\boldsymbol{\Gamma}$ is *weakly dominant strategy incentive compatible* if, for all $i \in \mathcal{N}$, $x_i \in \mathcal{X}_i$ and $\hat{\boldsymbol{x}}_{-i} \in \mathcal{X}_{-i}$, the following is satisfied:

$$\begin{aligned} & u_i(Q_i(x_i, \hat{\boldsymbol{x}}_{-i}), M_i(x_i, \hat{\boldsymbol{x}}_{-i}), x_i) \\ = \; & Q_i(x_i, \hat{\boldsymbol{x}}_{-i})v_i - M_i(x_i, \hat{\boldsymbol{x}}_{-i}) \\ \geq \; & Q_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i})v_i - M_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}) \\ = \; & u_i(Q_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}), M_i(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}), x_i), \end{aligned}$$

for all $\hat{x}_i \in \mathcal{X}_i$ satisfying Prob$\{\Gamma_{i2}(\hat{x}_i, \hat{\boldsymbol{x}}_{-i}) \leq w_i\} = 1$.

It is obvious that ex-post individual rationality implies interim individual rationality, and weakly dominant strategy incentive compatibility implies incentive compatibility.

2.3. Characteristics of IC random direct mechanisms

When designing a public resource allocation mechanism, the social planner considers not only efficiency and revenue, but also equality. Therefore, for each IC random direct mechanism $\boldsymbol{\Gamma}$, we shall define its (ex-ante) characteristics of efficiency, revenue, and equality. Since the ex-ante features of a random direct mechanism $\boldsymbol{\Gamma}$ are usually determined by its associated (interim) direct mechanism (Q, M), we shall use (Q, M) to define the characteristics of $\boldsymbol{\Gamma}$.

Efficiency and revenue

Efficiency of a public resource allocation mechanism describes whether those buyers with higher values are more likely to obtain objects. Given an IC random direct mechanism $\boldsymbol{\Gamma}$, its efficiency is usually defined as the aggregate realized values and its revenue is defined as all buyers' expected payments to the social planner. For convenience, in the setting of multi-unit item allocations, we adopt the expected realized values per object as the measure of efficiency and take the expected payments per object as the measure of revenue.[5] Formally, we have the following definitions.

Definition 1. *The efficiency measure of an IC random direct mechanism* $\boldsymbol{\Gamma}$ *is defined as*

$$Ef(\boldsymbol{\Gamma}) = \frac{1}{m} \int_{\mathcal{X}} \sum_{i=1}^{n} Q_i(\boldsymbol{x}) v_i \, \psi(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.2}$$

[5] Note that in the efficiency and revenue measures we divide the total amount of values and payments by the number of objects m no matter how many objects are eventually allocated.

Definition 2. *The revenue measure of an IC random direct mechanism* $\mathbf{\Gamma}$ *is defined as*

$$Re(\mathbf{\Gamma}) = \frac{1}{m} \int_{\mathcal{X}} \sum_{i=1}^{n} M_i(\boldsymbol{x}) \psi(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.3}$$

Equality

The equality of public resource allocation is an inescapable issue. For each IC random direct mechanism $\mathbf{\Gamma}$, we shall define its equality measure to measure the difference in winning opportunities among buyers with different budget ranks under $\mathbf{\Gamma}$.

For a given IC random direct mechanism $\mathbf{\Gamma}$ and a profile $\boldsymbol{x} \in \mathcal{X}$, $\boldsymbol{w} = (w_1, w_1, \ldots, w_n)$ and $\mathrm{Q}(\boldsymbol{x}) = (Q_1(\boldsymbol{x}), Q_2(\boldsymbol{x}), \ldots, Q_n(\boldsymbol{x}))$ represent the budgets and the interim assignments of all buyers, respectively. After sorting the vector $\boldsymbol{w}$ from low to high, we obtain a permutation $\sigma : \mathcal{N} \to \mathcal{N}$ such that

$$w_{\sigma(1)} \leq w_{\sigma(2)} \leq \ldots \leq w_{\sigma(n)},$$

where ties are broken randomly. For each $j = 1, 2, \ldots, n$, let $Q_{(j)}(\boldsymbol{x}) \equiv Q_{\sigma(j)}(\boldsymbol{x})$ denote the probability that the buyer with the j-th lowest budget obtains an object at profile $\boldsymbol{x}$,[6] and let $\hat{\boldsymbol{Q}}(\boldsymbol{x}) = (Q_{(1)}(\boldsymbol{x}), Q_{(2)}(\boldsymbol{x}), \ldots, Q_{(n)}(\boldsymbol{x}))$. Then, $\hat{\boldsymbol{Q}}(\boldsymbol{x})$ denotes the vector of buyers' winning probability ranked by their budgets from low to high in profile $\boldsymbol{x}$. For each $j = 1, 2, \ldots, n$, we further define

$$\overline{Q}_{(j)} = \int_{\mathcal{X}} Q_{(j)}(\boldsymbol{x}) \psi(\boldsymbol{x}) d\boldsymbol{x} \tag{2.4}$$

as the average probability of obtaining an object for buyers with the j-th lowest budgets. In other words, $\overline{Q}_{(j)}$ is a buyer's expected probability of obtaining an object when she only knows that her budget is the j-th lowest among all buyers. We use

$$p(w) = \int_0^{\bar{v}} q(v, w) dF(v|w)$$

[6] For example, suppose $\boldsymbol{w} = (3, 7, 6, 4, 5)$ and $\mathrm{Q}(\boldsymbol{x}) = (0.15, 0.3, 0.25, 0.1, 0.2)$. Then we get a permutation $\sigma = (1, 4, 5, 3, 2)$ by sorting all buyers' budgets from low to high. Reorder $\mathrm{Q}(\boldsymbol{x})$ by the permutation σ, i.e., rank all buyers' winning probabilities by their budgets from low to high. We then get a new vector $(Q_{(1)}, Q_{(2)}, Q_{(3)}, Q_{(4)}, Q_{(5)}) = (Q_1, Q_4, Q_5, Q_3, Q_2) = (0.15, 0.1, 0.2, 0.25, 0.3)$.

to denote a buyer's expected probability of obtaining an object when her budget is w. Then $\overline{Q}_{(j)}$ can be rewritten as

$$\begin{aligned}\overline{Q}_{(j)} &= \int_0^{\bar{w}} p(w) dG_{(j)}(w) \\ &= \int_0^{\bar{w}} \int_0^{\bar{v}} \int_{\mathcal{X}_{-i}} Q_i((v,w),\boldsymbol{x}_{-i})\psi_{-i}(\boldsymbol{x}_{-i})d\boldsymbol{x}_{-i}dF(v|w)dG_{(j)}(w),\end{aligned} \tag{2.5}$$

where $G_{(j)}(w) = \sum_{i=j}^{n} \binom{n}{i} G^i(w)[1-G(w)]^{n-i}$ is the distribution function of the j-th lowest budget.

We thus obtain a vector $\overline{\boldsymbol{Q}} = (\overline{Q}_{(1)}, \overline{Q}_{(2)}, \ldots \overline{Q}_{(n)})$, which contains all information about the difference in object obtaining opportunities among buyers with different budget ranks. Based on $\overline{\boldsymbol{Q}}$, we can draw a Lorenz curve that describes the proportion of objects assigned to the poorest fraction of buyers. We further present the condition under which this Lorenz curve lies below the $45°$ line.

Lemma 1. *In an IC (standard) random direct mechanism* $\boldsymbol{\Gamma}$*, if Assumption 1 holds, then* $\overline{Q}_{(j)}$ *is nondecreasing in* j*, and the Lorenz curve lies below the* $45°$ *line.*

With this Lorenz curve, we can define an equality measure that describes the difference in winning opportunities among buyers with different budget ranks.

Definition 3. *The equality measure of an IC random direct mechanism* $\boldsymbol{\Gamma}$*, is defined as*

$$Eq(\boldsymbol{\Gamma}) = \frac{2}{n+1} \cdot \frac{1}{\sum_{j=1}^{n} \overline{Q}_{(j)}} \sum_{k=1}^{n} \sum_{j=1}^{k} \overline{Q}_{(j)}. \tag{2.6}$$

In Figure 1, A denotes the shaded area and B denotes the area in green. The Gini coefficient is $\frac{A}{A+B}$ and the equality measure is $\frac{B}{A+B}$. When n is sufficiently large, the upper edges of the shaded area and the green area become the $45°$ line and a smooth Lorenz curve, respectively. Thus, the Gini coefficient becomes $2A$, and the equality measure becomes $2B$.

It is worth noting that this equality measure just captures a bit of information of $\overline{\boldsymbol{Q}}$, and serves as a rough assessment of an allocation mechanism in

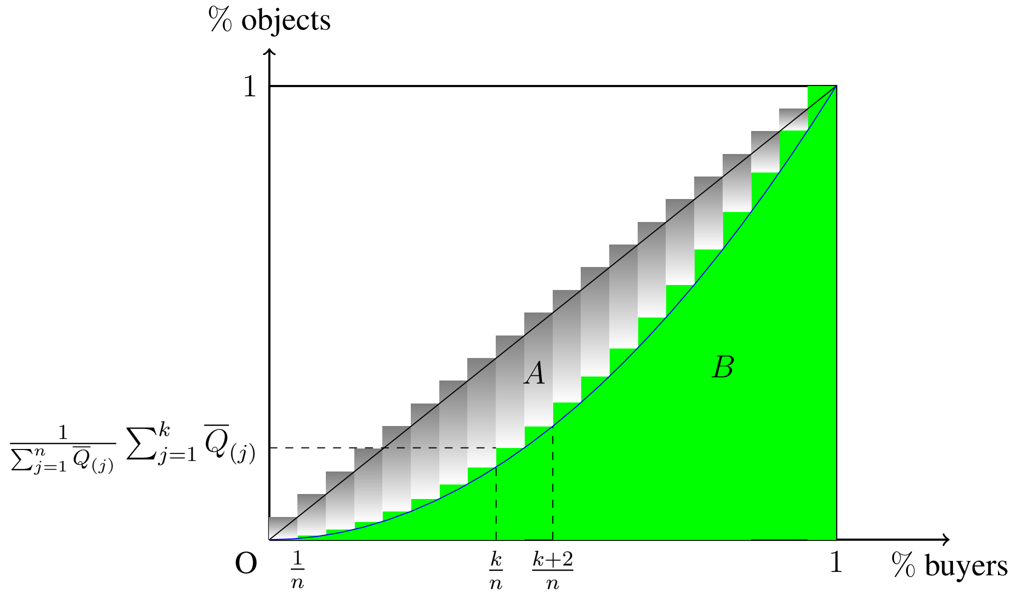

Figure 1: Lorenz curve of public resource allocation

terms of equality. To obtain more information about the impacts of allocation mechanisms on buyers from different social strata, one need to refer directly to the Lorenz curve.

For any IC random direct mechanism $\mathbf{\Gamma}$, we use

$$Ch(\mathbf{\Gamma}) = (Ef(\mathbf{\Gamma}), Eq(\mathbf{\Gamma}), Re(\mathbf{\Gamma}))$$

to denote its characteristics. Since concerning these characteristics, the social planner can be assumed to hold a social welfare function on the space of $Ch(\mathbf{\Gamma})$, and his objective is to choose an optimal allocation mechanism to maximize his social welfare function.

2.4. Continuum-mass IC random direct mechanism and its characteristics

In the above subsection, we present the characteristics of an IC direct mechanism, especially we define the equality measure for such a mechanism. Nevertheless, the equations (2.5) involved in the definition of the equality measure are rather complicated, making the computation of the equality measure difficult to be applied. Fortunately, if the sizes of buyers and objects are sufficiently large, we can describe the allocation problem with a continuum-mass model and simplify the computation of the equality measure. In this subsection, we will provide a continuum-mass version of the random direct mechanism, define its incentive properties, and present its characteristics of efficiency, equality, and revenue. We will first present the basic setting of the continuum-mass model.

A social planner wishes to allocate α mass of publicly-provided goods to a unit mass of buyers. Each buyer is assumed to have unit demand, holds a value v, and is subject to a budget of w. For any buyer, the distributions of her private type (v, w) and her utility function are identical to the settings in Subsection 2.1, and we will omit them here.

Continuum-mass random direct mechanism and its incentive properties
Let $\Omega' = \{0, 1\} \times [0, \bar{w}]$ represent the set of all possible ex-post allocations to a buyer, and let $\mathcal{F}(\Omega')$ be the space of all random vectors defined on Ω'. Following Che et al. (2013) and Richter (2019), we can define a continuum-mass random direct mechanism as a mapping

$$\mathbf{\Gamma} = (\Gamma_1, \Gamma_2) : [0, \bar{v}] \times [0, \bar{w}] \to \mathcal{F}(\Omega')$$

such that $\text{Prob}\{\Gamma_2(\hat{v}, \hat{w}) \leq \hat{w}\} = 1$ for all $(\hat{v}, \hat{w}) \in [0, \bar{v}] \times [0, \bar{w}]$.

For a given continuum-mass random direct mechanism $\mathbf{\Gamma}$ and an arbitrary report $(\hat{v}, \hat{w})$ of the buyer, let $(\pi(\hat{v}, \hat{w}), p(\hat{v}, \hat{w}))$ be a realization of $(\Gamma_1(\hat{v}, \hat{w}), \Gamma_1(\hat{v}, \hat{w}))$, and let

$$q(\hat{v}, \hat{w}) = E[\Gamma_1(\hat{v}, \hat{w})] \quad \text{and} \quad m(\hat{v}, \hat{w}) = E[\Gamma_2(\hat{v}, \hat{w})]$$

represent the buyer's probability of obtaining an object, and the expected payment she must take, respectively. We say a continuum-mass random direct mechanism $\mathbf{\Gamma}$ is *interim individually rational* if, for all $v \in [0, \bar{v}]$ and $w \in [0, \bar{w}]$, the following is satisfied:

$$vq(v, w) - m(v, w) \geq 0.$$

We also say $\mathbf{\Gamma}$ is *ex-post individually rational* if, for all $v \in [0, \bar{v}]$ and $w \in [0, \bar{w}]$, it satisfies that

$$\pi(v, w)v - p(v, w) \geq 0$$

for any realization $(\pi(v,w), p(v,w))$ of $(\Gamma_1(v,w), \Gamma_2(v,w))$, and $\mathbf{\Gamma}$ is said to be *incentive compatible* if, for all $v \in [0, \bar{v}]$ and $w \in [0, \bar{w}]$, the following is satisfied:

$$vq(v,w) - m(v,w) \geq vq(v',w') - m(v',w'),$$

for all (v', w') such that Prob$\{\Gamma_2(v',w') \leq w\} = 1$.

Characteristics of a continuum-mass IC random direct mechanism

We still adopt the expected realized values per unit of good as the measure of efficiency and consider the expected payments per unit of good as the measure of revenue. Formally, we have the efficiency measure and the revenue measure defined as:

$$Ef(\mathbf{\Gamma}) = \frac{1}{\alpha} \int_0^{\bar{w}} \int_0^{\bar{v}} q(v,w) v \phi(v,t) dv dt, \tag{2.7}$$

and

$$Re(\mathbf{\Gamma}) = \frac{1}{\alpha} \int_0^{\bar{w}} \int_0^{\bar{v}} m(v,w) \phi(v,t) dv dt. \tag{2.8}$$

If a buyer's budget is $w \in [0, \bar{w}]$, $p(w) = \int_0^{\bar{v}} q(v,w) dF(v|w)$ denoted her expected probability of obtaining an object, and

$$P(w) = \int_0^{w} \int_0^{\bar{v}} q(v,t) \phi(v,t) dv dt$$

denotes the cumulative mass of objects won by buyers with budgets no greater than w. Let $s \in [0,1]$ represent the mass of the buyers whose budgets are less than $w = G^{-1}(s)$. Thus, $L(s) = \frac{1}{\alpha} P(G^{-1}(s))$ denotes the fraction of objects allocated to the s fraction of buyers with the lowest budgets. The function $L(s) : [0,1] \to [0,1]$ have the properties that $L(0) = 0$, $L(1) = 1$ and $L'(s) = \frac{1}{\alpha} p(G^{-1}(s))$. Thus, when $q(v,w)$ is nondecreasing, it follows from Assumption 1 that $L'(s)$ is nondecreasing in s and the function $L(s)$ represents a well-defined Lorenz curve. Consequently, the equality measure is well-defined, and can be expressed as

$$\begin{aligned} Eq(\mathbf{\Gamma}) &= 2 \int_0^1 L(s)\, ds = \frac{2}{\alpha} \int_0^{\bar{w}} P(w) dG(w) \\ &= \frac{2}{\alpha} \int_0^{\bar{w}} \int_0^{w} \int_0^{\bar{v}} q(v,t) \phi(v,t) dv dt dG(w). \end{aligned} \tag{2.9}$$

3. THE HYBRID MECHANISM FOR VEHICLE LICENSE ALLOCATION

From this section on, as an application of the new evaluation criterion — the equality measure, we shall study the vehicle license allocations in China.[7] Specifically, we will use a class of hybrid auction-lottery mechanisms to generalize several license allocation mechanisms in China in a unified framework, and to evaluate and improve upon these license allocation mechanisms in terms of equality, in addition to efficiency and revenue. In this section, we will first briefly introduce China's vehicle license allocation mechanisms in practice. Then, we provide a new class of discrete hybrid mechanisms to incorporate these mechanisms.

3.1. Vehicle license allocation in China

The rapid growth in private vehicle ownership in China has led to traffic jams and air pollution in big cities. To alleviate these problems, several cities have placed limits on vehicle license quota, and instituted different mechanisms to allocate the limited supply of vehicle licenses.

Since 2002, Shanghai has used a multi-unit, discriminatory price auction to allocate vehicle licenses. In July 2013, Shanghai modified the auction rule, and introduced a "warning price" that essentially serves as a price ceiling. The current Shanghai auction comprises two phases and each lasts for 30 minutes. In the first phase of the auction, each bidder submits a bid which cannot be higher than the warning price. In the second phase, up to two bid revisions are allowed for each bidder, and the revised bids are restricted to some interval (approximately 300 RMB) above the lowest bid at that moment. The narrow price window, the limited bidding chance, and the increasing lowest accepted bid together incentivize bidders to "snipe" in the last ten or twenty seconds of the auction. Therefore, the final trading prices are just slightly higher than the warning price. Indeed, during 2014, the warning price was fixed at 72,600 RMB and the auction prices remain between 73,000 and 74,000 RMB. The warning price is adjusted gradually in the following years, but during each

[7] Vehicle license allocation has been a heated topic in Shanghai since 2013. We have in several occasions, including in Rong & Sun (2015), briefly discussed vehicle license allocation in China and recommended the hybrid mechanism introduced below as a solution to Shanghai's vehicle license allocation. Formal definition of this mechanism and detailed discussion of its characteristics are given in this paper.

year, the warning price is roughly fixed. For example, in 2018, the warning price is set as 86,300 RMB, which is 5,400 RMB short of the warning price in 2017, and is far below the potential equilibrium price. Therefore, we state that the current Shanghai auction is a price ceiling mechanism. Nevertheless, the "price ceiling" of the Shanghai auction is implemented by buyers' sniping behavior, instead of an open and fair lottery. Therefore, many buyers, hoping to win licenses on time, either pay a large amount (about 20,000 RMB) to auction intermediaries with better internet connection, or try different bidding methods. The strategy complexity and redundant intermediaries remain severe problems faced by the current Shanghai auction.

Beijing and Guiyang have been implementing a vehicle license lottery since 2011 to ensure equal allocation. In 2018, the Beijing lottery featured a very low winning probability (roughly 0.05%) for participating buyers. Guangzhou, Tianjin, Hangzhou, and Shenzhen adopted a hybrid lottery-auction mechanism in 2012, 2013, 2014, and 2015 to allocate vehicle licenses. In the discussion that follows, we name this hybrid mechanism "the Guangzhou mechanism." In this mechanism, roughly half of the license quota are allocated by a discriminatory-price auction, and the remaining licenses are allocated by a lottery in which the winners need not pay anything for licenses. The auction and lottery are totally separate, namely, every buyer has to choose between entering the auction or lottery. Therefore, it is difficult for buyers to choose between these options.[8] In fact, the auction losers always regret not having chosen the lottery. Several other cities are also planning to institute a license quota. How to allocate the given license quota more efficiently and equally remains a challenge in China.

3.2. A general hybrid mechanism for vehicle license allocation in China

We propose a new class of hybrid mechanisms for three concerns: (1)to simplify buyers' bidding strategies, (2) to incorporate several China's vehicle license allocation mechanisms in a unified framework, (3) to provide the social planner with more flexible policy tools to improve license allocation. In our hybrid mechanisms, every buyer has the opportunity to enter both the auction and lottery. All buyers enter the auction first, with the auction losers then

[8] Buyers' bidding strategies in the Guangzhou mechanism may be rather complex. Huang & Wen (2019) study buyers' bidding behaviors under the Guangzhou mechanism, and provide an equilibrium bidding strategy.

entering the lottery. The lottery winners, unlike in the Guangzhou mechanism, are also required to pay a reserve price to the social planner. Formally, our hybrid mechanism is defined as follows:

Hybrid auction-lottery mechanism for vehicle licenses

Step 1 **Announcement** The social planner announces the total license quota m, the auction quota m_1, the lottery quota $m - m_1$, and a reserve price r.

Step 2 **Registration** All buyers decide whether to register for the mechanism. At the end of registration, the social planner announces the number of all registered buyers n_1.

Step 3 **Auction** All registered buyers submit their bids, which should be no less than r.[9] A buyer whose bid is among the m_1 highest bids wins the auction, obtains a license, and pays the $(m_1 + 1)$-th highest bid.[10]

Step 4 **Lottery** Those registered buyers who lose the auction enter the lottery in which $m - m_1$ licenses are allocated. Each lottery winner obtains a license and pays r.

Since the total license quota m is exogenously given and the social planner can only adjust r and m_1, we shall use $H(r, m_1)$ to denote such a hybrid mechanism. It is obvious that: (i) if $m_1 = 0$ and $r = 0$, then $H(r, m_1)$ becomes the Beijing mechanism; (ii) if $m_1 = m$ and $r = 0$, then $H(r, m_1)$ is roughly the Shanghai auction before July 2013. The current Shanghai mechanism with a warning price c is theoretically equivalent to the hybrid mechanism $H(c, 0)$ because in practice, the number of buyers willing to bid no less than the warning price c (denoted by m_c) is much more than the license quota m.[11]

For a hybrid mechanism $H(r, m_1)$, if $n_1 < m$, i.e., r is set so high that the number of registered buyers is less than the license quota, then $H(r, m_1)$

[9] If a registered buyer does not submit a bid, her bid is set as r by default.

[10] When the number of registered buyers n_1 is no greater than m_1, all registered buyers win the auction and pay r.

[11] If the warning price c is set so high that $m_c \leq m$, then the warning price does not take effect, and the current Shanghai mechanism with such a warning price reduces to the Shanghai auction before July 2013. Therefore, for any warning price c, the current Shanghai mechanism $M(c)$, essentially a price ceiling mechanism, can be induced from our hybrid mechanisms.

reduces to a posted price selling mechanism. Usually, for the allocation of scarce resource such as vehicle licenses, the reserve price is not set so high, thus registered buyers are more than objects allocated, i.e., $n_1 > m$. Let

$$\lambda = \begin{cases} \frac{m-m_1}{n_1-m_1}, & \text{if } n_1 > m, \\ 1, & \text{if } n_1 \leq m, \end{cases}$$

denote the probability of winning a license in the lottery. All buyers know λ because, according to the rules of $H(r, m_1)$, they can calculate it from n_1, m_1, and m.

In a hybrid mechanism $H(r, m_1)$, we say buyer i with type $x_i = (v_i, w_i)$ *bids sincerely* whenever

1. she registers for the mechanism, if and only if, $\min\{v_i, w_i\} \geq r$;
2. when she is a registered buyer, she bids $\min\{\, v_i - \lambda(v_i - r),\, w_i \,\}$.

We say a hybrid mechanism $H(r, m_1)$ is *ex-post individually rational* if, no matter how other buyers bid, each buyer i gets a non-negative ex-post utility when bidding sincerely. Clearly, by the rule of the hybrid mechanisms, each hybrid mechanism is ex-post individually rational. On sincere bidding strategy, we further have the following result.

Theorem 1. *Each hybrid mechanism $H(r, m_1)$ is ex-post individually rational. Moreover, in a hybrid mechanism $H(r, m_1)$, for every buyer i sincere bidding is a weakly dominant strategy.*

Theorem 1 implies that the hybrid mechanism is detail-free, namely, a buyer need not know the distributions of buyers' types when bidding. In addition, in a hybrid mechanism, every buyer is just required to report a bid instead of her whole type. In this sense, our hybrid mechanism is not a direct mechanism and is privacy preserving compared with direct mechanisms. Nevertheless, for every hybrid mechanism $H(r, m_1)$, by the Revelation Principle and Theorem 1, we can easily construct a relative IC random direct mechanism

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Formally, we have

$$M(c) = \begin{cases} H(c, 0), & \text{if } m_c > m, \\ H(0, m), & \text{if } m_c \leq m. \end{cases}$$

$\mathbf{\Gamma}$ to implement the same outcome of $H(r, m_1)$.[12] By Theorem 1, we can further show this random direct mechanism $\mathbf{\Gamma}$ is ex-post individually rational and weakly dominant strategy incentive compatible. For a hybrid mechanism $H(r, m_1)$, we use the characteristics $Ch(\mathbf{\Gamma})$ of its relative IC random direct mechanism to represent its characteristics. In particular, we write the characteristics of $H(r, m_1)$ as $Ch(r, m_1) = (Ef(r, m_1), Eq(r, m_1), Re(r, m_1))$. The social planner chooses the proper hybrid mechanism $H(r, m_1)$ to maximize his social welfare function.

Finally, it is worth noting that Theorem 1 is based on an implicit assumption that every buyer bids with her true identity. In practice, some buyers may have incentives to register for the mechanism under false names and submit multiple bids to increase their expected payoff. Indeed, it is widely reported that some buyers participate in the Beijing mechanism with multiple identities to increase their probability of winning. We shall discuss this issue in Subsection 4.3.

[12] The relative IC random direct mechanism can be roughly defined as follows. Each buyer $i \in \mathcal{N}$ reports a type $\hat{x}_i = (\hat{v}_i, \hat{w}_i)$. The social planner first computes each buyer i's "bid" $b_i(\hat{v}_i, \hat{w}_i)$ by

$$b_i(\hat{v}_i, \hat{w}_i) = \begin{cases} \min\{\hat{v}_i - \lambda(\hat{v}_i - r),\ \hat{w}_i\}, & \text{if } \min\{\hat{v}_i, \hat{w}_i\} \geq r, \\ 0, & \text{if } \min\{\hat{v}_i, \hat{w}_i\} < r. \end{cases}$$

Let $b^{(m_1+1)}$ denote the (m_1+1)-th highest bid in all bids $\{b_i(\hat{x}_i) : i \in \mathcal{N}\}$. Thus, according to the Birkhoff-von Neumann theorem (Budish et al., 2013), there is a random assignment $\mathbf{\Gamma}_1(\hat{\boldsymbol{x}}) = (\Gamma_{11}(\hat{\boldsymbol{x}}), \ldots, \Gamma_{n1}(\hat{\boldsymbol{x}})) \in \mathcal{F}(\Pi)$ satisfying the following: (i) for each buyer i with $b_i(\hat{v}_i, \hat{w}_i) \geq b^{(m_1+1)}$, $\Gamma_{i1}(\hat{\boldsymbol{x}})$ reduces to a deterministic assignment 1; (ii) for each buyer i with $b_i(\hat{v}_i, \hat{w}_i) = 0$, $\Gamma_{i1}(\hat{\boldsymbol{x}})$ reduces to a deterministic assignment 0; and (iii) for each buyer i with $r \leq b_i(\hat{v}_i, \hat{w}_i) \leq b^{(m_1+1)}$, $\Gamma_{i1}(\hat{\boldsymbol{x}})$ satisfies $Q_i(\hat{\boldsymbol{x}}) \equiv E[\Gamma_{i1}(\hat{\boldsymbol{x}})] = \lambda$.

The random payment rule is constructed as follows: (i) for each buyer i with $b_i(\hat{v}_i, \hat{w}_i) \geq b^{(m_1+1)}$, $\Gamma_{i2}(\hat{\boldsymbol{x}})$ is a deterministic payment $b^{(m_1+1)}$; (ii) for each buyer i with $b_i(\hat{v}_i, \hat{w}_i) = 0$, $\Gamma_{i2}(\hat{\boldsymbol{x}})$ is a deterministic payment 0; and (iii) for each buyer i with $r \leq b_i(\hat{v}_i, \hat{w}_i) \leq b^{(m_1+1)}$, $\Gamma_{i2}(\hat{\boldsymbol{x}})$ is defined by

$$\Gamma_{i2}(\hat{\boldsymbol{x}}) = \begin{cases} r, & \text{if } \Gamma_{i1}(\hat{\boldsymbol{x}}) = 1, \\ 0, & \text{if } \Gamma_{i1}(\hat{\boldsymbol{x}}) = 0. \end{cases}$$

By the random payment rule, it holds that $\text{Prob}\{\Gamma_{i2}(\hat{\boldsymbol{x}}) \leq \hat{w}_i\} = 1$ for each i. Thus, we have constructed a well-defined random direct mechanism $\mathbf{\Gamma} = (\Gamma_{11}, \ldots, \Gamma_{n1}, \Gamma_{12}, \ldots, \Gamma_{n2})$. Obviously, the associated direct mechanism of $\mathbf{\Gamma}$ satisfies anonymity and monotonicity, and thus $\mathbf{\Gamma}$ is a standard random direct mechanism.

4. CONTINUUM-MASS HYBRID MECHANISM

Section 3 proposed a class of discrete hybrid mechanisms and defined their characteristics. However, it is usually difficult to compute these characteristics. Fortunately, the number of buyers and licenses are both large in practice and the discrete problem is close to the version with a continuum of mass. In addition, in a setting of continuum-mass buyers and items, it is relatively simple to compute the characteristics of a mechanism, because the mass of buyers with certain types and accordingly the auction price are usually determined. In this section, with the aim of helping the social planner to choose an optimal mechanism, we shall provide a continuum-mass mechanism for each hybrid mechanism and use its characteristics to approximate those of the hybrid mechanism.

4.1. Description of continuum-mass hybrid mechanisms

For each hybrid mechanism $H(r, m_1)$, we define its continuum-mass version hybrid mechanism as follows. A social planner wishes to assign a mass $\alpha = \frac{m}{n} \in (0, 1)$ of vehicle licenses to a unit mass of buyers, in which $\alpha_1 = \frac{m_1}{n}$ mass and $\alpha - \alpha_1 = \frac{m-m_1}{n}$ mass of licenses are allocated by auction and lottery, respectively. All assumptions on buyers' values, budgets, and utility functions are similar as in Section 2. The continuum-mass hybrid mechanism is defined as follows.

Continuum-mass hybrid auction lottery mechanism for vehicle licenses

Step 1 **Announcement** The social planner announces the total license quota α, the auction quota $\alpha_1 \in [0, \alpha]$, the lottery quota $\alpha - \alpha_1$, and a reserve price r.

Step 2 **Registration** All buyers decide whether to register for the mechanism. At the end of registration, the social planner announces the mass of registered buyers β.

Step 3 **Auction** All registered buyers submit their bids, which should not be less than r.[13] A buyer wins the auction, obtains a license, and pays the equilibrium price p^e if her bid is no less than p^e.[14]

[13] If a registered buyer does not submit a bid, her bid is set as r by default.

[14] Let $D(p)$ be the mass of those buyers who bid above p. Then, the equilibrium price p^e is the price that satisfies $D(p^e) = \alpha_1$.

Step 4 **Lottery** Registered buyers who lose the auction enter the lottery in which $\alpha - \alpha_1$ mass of licenses is allocated. Each winner in lottery obtains a license and pays r.

Since the total license quota α is exogenously given, we shall use $H(r, \alpha_1)$ to denote such a continuum-mass hybrid mechanism and write its characteristics as $Ch(r, \alpha_1) = (Eq(r, \alpha_1), Ef(r, \alpha_1), Re(r, \alpha_1))$. We also use

$$\lambda = \begin{cases} \frac{\alpha - \alpha_1}{\beta - \alpha_1}, & \text{if } \beta > \alpha, \\ 1, & \text{if } \beta \leq \alpha, \end{cases}$$

to denote the probability of winning a license in the lottery.

In a continuum-mass hybrid mechanism $H(r, \alpha_1)$, we say a buyer with type $x = (v, w)$ *bids sincerely* whenever

1. she registers for the mechanism, if and only if, $\min\{v_i, w_i\} \geq r$;
2. when she is a registered buyer, she bids $\min\{\, v_i - \lambda(v_i - r),\ w_i \,\}$.

We have the following result on sincere bidding strategy as the counterpart of Theorem 1.

Theorem 2. *Each continuum-mass hybrid mechanism $H(r, \alpha_1)$ is ex-post individually rational. Moreover, in a continuum-mass hybrid mechanism $H(r, \alpha_1)$, for every buyer, it is a weakly dominant strategy to bid sincerely.*

From the result of Theorem 2, we assume in the following that every buyer bids sincerely in $H(r, \alpha_1)$. Then, only those buyers whose values and budgets are both no less than r would register for the hybrid mechanism, and thus the mass of registered buyers would be

$$\beta = \int_r^{\bar{w}} \int_r^{\bar{v}} \phi(v, w) dv dw.$$

Thus, there is a unique critical reserve price $r^* \in [0, \min\{\bar{v}, \bar{w}\}]$ such that

$$\alpha = \int_{r^*}^{\bar{w}} \int_{r^*}^{\bar{v}} \phi(v, w) dv dw,$$

because $\Phi(v, w)$ is assumed to be strictly increasing in v and w. Note that in the case that $\alpha_1 = \alpha$ and $r = 0$, r^* is also the auction price. Obviously, for a mechanism $H(r,\alpha_1)$ with $r < r^*$ it holds that $\beta > \alpha$, and for a mechanism $H(r,\alpha_1)$ with $r \geq r^*$ it satisfies $\beta \leq \alpha$. We further see that, for a mechanism $H(r,\alpha_1)$ with $r < r^*$ and a (possible) price $p \in [r, \min\{(1-\lambda)\bar{v} + \lambda r,\ \bar{w}\}]$, only those buyers with budgets no less than p and values no less than $\frac{p-\lambda r}{1-\lambda}$ would bid above p. Therefore, the mass of buyers bidding above p would be

$$D(p) = \int_p^{\bar{w}} \int_{v_p}^{\bar{v}} \phi(v,w)dvdw,$$

where $v_p = \frac{p-\lambda r}{1-\lambda}$. Thus, there also exists a unique equilibrium price $p^e \in [r, \min\{(1-\lambda)\bar{v} + \lambda r, \bar{w}\}]$ such that

$$\alpha_1 = \int_{p^e}^{\bar{w}} \int_{v^p}^{\bar{v}} \phi(v,w)dvdw,$$

where $v^p = \frac{1}{1-\lambda}(p^e - \lambda r)$. In summary, we obtain the following result.

Corollary 1. *If every buyer bids sincerely in $H(r,\alpha_1)$, then it holds that*

(1) The mass of registered buyers β satisfies $\beta = \int_r^{\bar{w}} \int_r^{\bar{v}} \phi(v,w)dvdw$.

(2) If $r < r^*$, an equilibrium price $p^e \in [r,\ \min\{(1-\lambda)\bar{v} + \lambda r,\ \bar{w}\}]$ exists that satisfies $\alpha_1 = \int_{p^e}^{\bar{w}} \int_{v^p}^{\bar{v}} \phi(v,w)dvdw$, where $v^p = \frac{1}{1-\lambda}(p^e - \lambda r)$ is the lowest value of all winning bidders in auction.

4.2. Characteristics of continuum-mass hybrid mechanisms

In this subsection, we shall present the formulas of the characteristics for continuum-mass hybrid mechanisms. We first consider the characteristics $Ch(r,\alpha_1)$ of a hybrid mechanism $H(r,\alpha_1)$ with $r < r^*$.

For such a hybrid mechanism, we have $\beta > \alpha$. Let $A = [r,\bar{v}] \times [r, p^e]$, $B = [r, v^p) \times [p^e, \bar{w}]$, $C = [v^p, \bar{v}] \times [p^e, \bar{w}]$ and $D = A \cup B$. We can show that, if all buyers bid sincerely, buyers in C win the auction,[15] and buyers in D enter the lottery. Figure 2 demonstrates such different winning patterns of buyers.

In the following, we consider efficiency, revenue, and equality of $H(r,\alpha_1)$ sequentially.

Efficiency $Ef(r,\alpha_1)$

[15] Here, "buyers in C" means that buyers whose types are in C, and the same below.

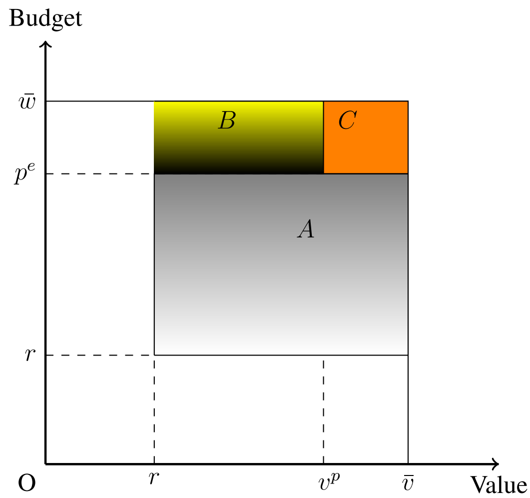


Figure 2: Winning patterns of buyers with different types

Let P_A, P_B, and P_C denote the masses of the buyers in A, B, and C, respectively, and let EV_A, EV_B, and EV_C represent the mean value of the buyers in A, B, and C, respectively. We then have that

$$P_A = \int_r^{p^e} \int_r^{\bar{v}} \phi(v,w)dvdw, \qquad EV_A = \frac{1}{P_A} \int_r^{p^e} \int_r^{\bar{v}} v\phi(v,w)dvdw,$$

$$P_B = \int_{p^e}^{\bar{w}} \int_r^{v^p} \phi(v,w)dvdw, \qquad EV_B = \frac{1}{P_B} \int_{p^e}^{\bar{w}} \int_r^{v^p} v\phi(v,w)dvdw,$$

and

$$P_C = \int_{p^e}^{\bar{w}} \int_{v^p}^{\bar{v}} \phi(v,w)dvdw, \qquad EV_C = \frac{1}{P_C} \int_{p^e}^{\bar{w}} \int_{v^p}^{\bar{v}} vd\,\Phi(v,w).$$

Thus, the mean value of winners in lottery EV_D can be written as

$$\begin{aligned} EV_D &= \frac{P_A}{P_A+P_B} \cdot EV_A + \frac{P_B}{P_A+P_B} \cdot EV_B \\ &= \frac{\lambda}{\alpha-\alpha_1} \cdot \Big(\int_r^{p^e} \int_r^{\bar{v}} vd\,\Phi(v,w) + \int_{p^e}^{\bar{w}} \int_r^{v^p} vd\,\Phi(v,w) \Big). \end{aligned}$$

Efficiency is measured by the mean value of all winning buyers. Therefore, it is the mean value of winners in auction (EV_C) weighted by the share of auction quota $\frac{\alpha_1}{\alpha}$ plus the mean value of lottery winners (EV_D) weighted by the lottery quota share $\frac{\alpha-\alpha_1}{\alpha}$. Formally, the efficiency measure can be written as

$$\begin{aligned} Ef(r,\alpha_1) &= \frac{\alpha_1}{\alpha}\cdot EV_C + \frac{\alpha-\alpha_1}{\alpha}\cdot EV_D \\ &= \frac{1}{\alpha}\cdot \int_{p^e}^{\bar{w}}\int_{v^p}^{\bar{v}} v d\,\Phi(v,w) \\ &\quad + \frac{\lambda}{\alpha}\cdot \Big(\int_r^{p^e}\int_r^{\bar{v}} v d\,\Phi(v,w) + \int_{p^e}^{\bar{w}}\int_r^{v^p} v d\,\Phi(v,w)\Big). \end{aligned} \tag{4.1}$$

Revenue $Re(r,\alpha_1)$

The revenue of the hybrid mechanism is induced from the price p^e paid by those winning buyers in the auction and the reserve price r paid by winners in the lottery. Therefore, the revenue measure, i.e., the mean payment for a license, can be written as

$$Re(r,\alpha_1) = \frac{\alpha_1}{\alpha}\cdot p^e + \frac{\alpha-\alpha_1}{\alpha}\cdot r. \tag{4.2}$$

Equality $Eq(r,\alpha_1)$

For a budget level $w \in [0,\bar{w}]$, recall that $p(w)$ represents the probability of winning a license for a buyer whose budget is w, and $P(w)$ represents the cumulative mass of licenses won by buyers with budgets no greater than w. First, it is clear that a buyer with budget $w \in [0,r)$ has no chance of winning a license. Second, for a buyer with $w \in [r,p^e)$, the probability that she is positioned in area A and enters the lottery is $1-F(r|w)$, and thus she can win a license with the probability $\lambda\big(1-F(r|w)\big)$. Third, for a buyer with budget $w \in [p^e,\bar{w}]$, the probability that she is positioned in area B and enters the lottery is $F(v^p|w)-F(r|w)$, whereas the probability that she is positioned in area C and wins the auction is $1-F(v^p|w)$, accordingly, her probability of winning a license is $\lambda\big(1-F(r|w)\big)+(1-\lambda)\big(1-F(v^p|w)\big)$. As a result, we can write $p(w)$ and $P(w)$ as

$$p(w) = \begin{cases} 0, & \text{if } w < r, \\ \lambda\big(1-F(r|w)\big), & \text{if } r \le w < p^e, \\ \lambda\big(1-F(r|w)\big)+(1-\lambda)\big(1-F(v^p|w)\big), & \text{if } p^e \le w \le \bar{w}, \end{cases}$$

and

$$P(w) = \begin{cases} 0, & \text{if } w < r, \\ \lambda \int_r^w \big(1 - F(r|t)\big) dG(t), & \text{if } r \leq w < p^e, \\ \lambda \int_r^w \big(1 - F(r|t)\big) dG(t) & \\ +(1-\lambda) \int_{p^e}^w \big(1 - F(v^p|t)\big) dG(t), & \text{if } p^e \leq w \leq \bar{w}. \end{cases}$$

Consequently, according to equation (2.9), the equality measure can be expressed as

$$\begin{aligned} Eq(r, \alpha_1) &= 2\int_0^1 L(s)\, ds = \frac{2}{\alpha} \cdot \int_0^{\bar{w}} P(w) dG(w) \\ &= \frac{2\lambda}{\alpha} \cdot \int_r^{\bar{w}} \int_r^w \big(1 - F(r|t)\big) dG(t) dG(w) \\ &\quad + \frac{2(1-\lambda)}{\alpha} \cdot \int_{p^e}^{\bar{w}} \int_{p^e}^w \big(1 - F(v^p|t)\big) dG(t) dG(w). \end{aligned} \tag{4.3}$$

Finally, let us consider the characteristics $Ch(r, \alpha_1)$ of a hybrid mechanism $H(r, \alpha_1)$ with $r \geq r^*$. Such a mechanism satisfies $\beta \leq \alpha$. This means that every buyer whose value and budget are both no less than r would register for the mechanism and win a license for sure. Therefore, the characteristics $Ch(r, \alpha_1)$ can be expressed as

$$\begin{aligned} Ef(r, \alpha_1) &= \frac{1}{\alpha} \cdot \int_r^{\bar{w}} \int_r^{\bar{v}} v d\Phi(v, w), \\ Re(r, \alpha_1) &= \frac{\beta}{\alpha} \cdot r, \qquad \text{and} \\ Eq(r, \alpha_1) &= \frac{2}{\beta} \cdot \int_r^{\bar{w}} \int_r^w \big(1 - F(r|t)\big) dG(t) dG(w). \end{aligned} \tag{4.4}$$

4.3. False-name bidding in continuum-mass hybrid mechanisms

In this subsection, we shall examine buyers' incentives to engage in false-name bidding in the continuum-mass hybrid mechanisms, and then provide a sufficient and necessary condition to prevent false-name bidding. Specifically, we shall analyze, for a given auction quota α_1, how the social planner chooses the proper reserve price r to curb false-name bidding. We first consider a simple case where a buyer bids under one false name.

Note that in any hybrid mechanism $H(r, \alpha_1)$, a buyer with budget $w < 2r$ has no incentive to bid under a false name because of the absolute budget constraint. Therefore, if $\bar{w} \neq +\infty$, the social planner can always set $r > \frac{1}{2}\bar{w}$ to prevent any buyer from bidding under a false name. However, this method cannot work if $\bar{w} = +\infty$. Moreover, if $r > \frac{1}{2}\bar{w} \geq r^*$, the characteristics $Ch(r, \alpha_1)$ will be impaired. Therefore, in the following, we shall consider some other methods of preventing false-name bidding, and thus assume that $r \leq \frac{1}{2}\bar{w}$.

Pick a hybrid mechanism $H(r, \alpha_1)$ with $r \leq \frac{1}{2}\bar{w}$. As in the previous subsection, we use C and D to denote the set of buyers who win the auction and the set of buyers who enter the lottery, when all buyers bid sincerely. Then, we see that when a buyer in C bids under her true identity, her expected payoff is $v - p^e$, and when a buyer in D bids under her true identity, her expected payoff is $\lambda(v-r)$. In addition, for a buyer with value $v \geq r$ and budget $w \geq 2r$, her expected payoff from bidding under a false name is $v - 2\lambda r - (1-\lambda)^2 v$.

When the hybrid mechanism $H(r, \alpha_1)$ satisfies $p^e > (1+\lambda)r$, we see that $\frac{r}{1-\lambda} < v^p < \bar{v}$ and $r < \frac{p^e - 2\lambda r}{(1-\lambda)^2}$, thus the set

$$E = D \bigcap \left(\frac{r}{1-\lambda}, \bar{v}\right] \times [2r, \bar{w}]$$

and the set

$$F = C \bigcap \left[r, \frac{p^e - 2\lambda r}{(1-\lambda)^2}\right) \times [2r, \bar{w}]$$

are both well-defined. We can further show that: (i) the sets E and F are both non-empty; (ii) for every buyer in E, her net surplus from bidding under a false name is

$$[v - 2\lambda r - (1-\lambda)^2 v] - \lambda(v - r) \;=\; \lambda(1-\lambda)v - \lambda r \;>\; 0;$$

and (iii) for every buyer in F, her net surplus from bidding under a false name is

$$[v - 2\lambda r - (1-\lambda)^2 r] - (v - p^e) \;=\; p^e - 2\lambda r - (1-\lambda)^2 v \;>\; 0.$$

Therefore, all buyers in $E \cup F$ have incentives to bid under a false name. Specifically, those buyers in E prefer bidding under two identities compared to their original strategy of participating in the lottery with one identity, and those buyers in F prefer bidding under two identities compared to their original

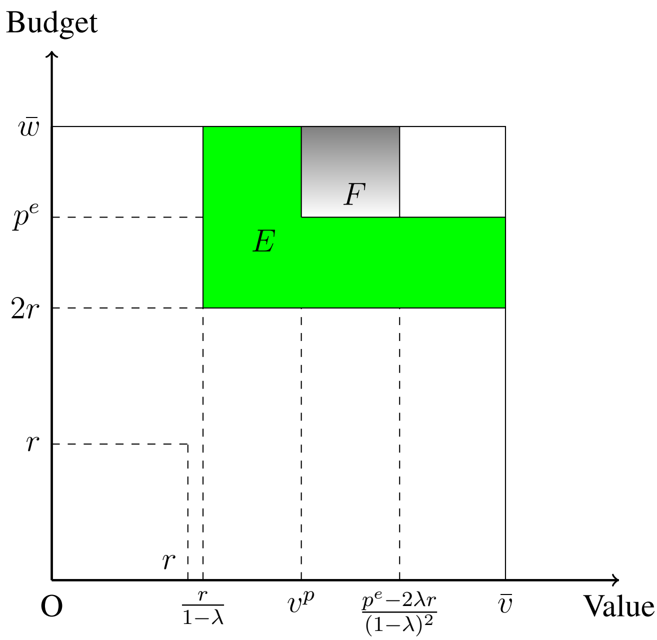


Figure 3: Buyers who have incentives to engage in false-name bidding in a hybrid mechanism $H(r, \alpha_1)$ with $p^e > 2r > (1+\lambda)r$

strategy of winning in the auction. Figure 3 demonstrates such different patterns of buyers who have incentives to bid under a false name in a hybrid mechanism with $p^e > 2r > (1+\lambda)r$.

As argued above, the social planner must set r such that $p^e \leq (1+\lambda)r$ to prevent buyers from bidding under a false name. Indeed, there always exists some $r \in (0, r^*)$ such that $p^e \leq (1+\lambda)r$ because $p^e < (1+\lambda)r$ for $r \to r^{*-}$. In fact, we can further show that $p^e \leq (1+\lambda)r$ is also a sufficient condition to prevent buyers from bidding under any number of false names. Formally, we have the following result.

Theorem 3. *In a continuum-mass hybrid mechanism $H(r, \alpha_1)$ with $2r \leq \bar{w}$, no buyer has an incentive to bid under any number of false names if and only if $p^e \leq (1+\lambda)r$.*

The above analysis is based on the assumption that a buyer's cost for holding an extra license is just the reserve price r. In practice, besides adjusting

r, the social planner can implement other policies to increase buyers' cost of holding extra licenses, and further curb false-name bidding. For instance, Beijing has adopted resale prohibitions and identity regulations, in addition to a stipulation that winners have to buy vehicles within a limited time period after the lottery.

5. PROBABILITY ALLOCATION MECHANISM

We have provided a class of hybrid mechanisms that are ex-post individually rational. In this section, we shall relax the requirement of ex-post individual rationality, and explore an allocation mechanism that can improve upon the hybrid mechanisms in all three factors of efficiency, equality, and revenue. In the following, we shall refer to it as *the probability allocation mechanism*. This mechanism's associated interim allocation mechanism is essentially similar to the allocation rule of Ausubel (2004),[16] and its random payment rule is similar to randomized extraction by Bhattacharya et al. (2010). Before defining this mechanism, we first introduce some notations.

Let $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in \mathcal{X}$, where $\hat{x}_i = (\hat{v}_i, \hat{w}_i)$ for each $i \in \mathcal{N}$, be a report profile of buyers. Then every buyer i can be viewed as holding a left-continuous demand function

$$d_i(p) = \begin{cases} 1, & \text{if } p \leq \hat{v}_i \text{ and } p \leq \hat{w}_i, \\ \frac{\hat{w}_i}{p}, & \text{if } p \leq \hat{v}_i \text{ and } p > \hat{w}_i, \\ 0, & \text{if } p > \hat{v}_i. \end{cases} \tag{5.1}$$

Thus, there is a left-continuous total demand function $D(p) = \sum_{i=1}^{n} d_i(p)$, and every buyer i faces a left-continuous **residual supply function** $s^{-i}(p) = \max\left\{m - \sum_{j\neq i} d_j(p), 0\right\}$, where m is the license quota. Since $D(p)$ is a nonincreasing function with $D(0) = n > m$ and $D(\bar{v}) = 0 < m$, there always exists a critical price $p^* = p(\hat{\boldsymbol{x}}) \in [0, \bar{v}]$ such that $p^* = \max\{p : D(p) \geq m\}$. Let $D(p^{*+}) = \lim\limits_{p\to p^{*+}} D(p)$ be the right-sided limit of $D(p)$ with p approaching p^* from the right.

[16] Dobzinski et al. (2012) and Bhattacharya et al. (2010) propose an adaptive clinching auction allocating the probability of winning objects based on Ausubel (2004). Their mechanism exhausts most winning buyers' budgets but is far more complex than our probability allocation mechanism.

Having prepared these notation, we define the probability allocation mechanism as follows. For each report profile $\hat{\boldsymbol{x}}$ and each buyer $i \in \mathcal{N}$, the interim winning probability $Q_i(\hat{\boldsymbol{x}})$ and the expected payment $M_i(\hat{\boldsymbol{x}})$ of buyer i are given by

$$Q_i(\hat{\boldsymbol{x}}) = \begin{cases} \frac{m-D(p^{*+})}{D(p^*)-D(p^{*+})} \cdot d_i(p^*), & \text{if } v_i = p^*, \\ d_i(p^*), & \text{if } v_i \neq p^*, \end{cases}$$

and[17]

$$M_i(\hat{\boldsymbol{x}}) = \int_0^{p^*} \big(Q_i(\hat{\boldsymbol{x}}) - s^{-i}(p)\big)\, dp.$$

According to the Birkhoff-von Neumann theorem (Budish et al., 2013), there is a random assignment $\boldsymbol{\Gamma}_1(\hat{\boldsymbol{x}}) = (\Gamma_{11}(\hat{\boldsymbol{x}}), \Gamma_{21}(\hat{\boldsymbol{x}}), \ldots, \Gamma_{n1}(\hat{\boldsymbol{x}})) \in \mathcal{F}(\Pi)$ such that $E[\Gamma_{i1}(\hat{\boldsymbol{x}})] = Q_i(\hat{\boldsymbol{x}})$ for all $i \in \mathcal{N}$.

Note that for some buyer i, it may hold that $0 < Q_i(\hat{\boldsymbol{x}}) < 1$ and $M_i(\hat{\boldsymbol{x}}) < \hat{w}_i$. Therefore, if we take buyer i's expected payment $M_i(\hat{\boldsymbol{x}})$ as a deterministic payment, then she has an incentive to over-report her budget. To prevent such over-reportings, we, similar to Bhattacharya et al. (2010), need a random payment rule with a positive probability that buyer i will be required to pay the full amount of her reported budget. We construct a random payment for each buyer i satisfying $E[\Gamma_{i2}(\hat{\boldsymbol{x}})] = M_i(\hat{\boldsymbol{x}})$, Prob$\{\Gamma_{i2}(\hat{\boldsymbol{x}}) = \hat{w}_i\} > 0$ and Prob$\{\Gamma_{i2}(\hat{\boldsymbol{x}}) \leq \hat{w}_i\} = 1$ as

$$\Gamma_{i2}(\hat{\boldsymbol{x}}) = \begin{cases} \hat{w}_i, & \text{with probability } \frac{M_i(\hat{\boldsymbol{x}})}{\hat{w}_i}, \\ 0, & \text{with probability } 1 - \frac{M_i(\hat{\boldsymbol{x}})}{\hat{w}_i}. \end{cases}$$

We thus obtain a well-defined random direct mechanism:

$$\boldsymbol{\Gamma} = (\Gamma_{11}, \ldots, \Gamma_{n1}, \Gamma_{12}, \ldots, \Gamma_{n2}),$$

which is referred to as the probability allocation mechanism. Note that since its associated direct mechanism (Q, M) satisfies anonymity and monotonicity, $\boldsymbol{\Gamma}$ is a standard random direct mechanism. In this mechanism, each buyer i's random payment $\Gamma_{i2}(\hat{\boldsymbol{x}})$ is independent of her random assignment $\Gamma_{i1}(\hat{\boldsymbol{x}})$, and

[17] We can use an example to illustrate the allocation rule. Suppose $m = 10000$, $D(p^*) = 10000.25$ and $D(p^{*+}) = 9999.5$, and suppose that buyer i is the "jumping" buyer with value p^* and budget $0.75p^*$. Then, buyer i is assigned a license with the probability $\frac{10000-9999.5}{10000.25-9999.5} \cdot \frac{0.75p^*}{p^*} = 0.5$.

thus $\mathbf{\Gamma}$ is not ex-post individually rational. Unlike in our hybrid mechanisms, for some report profile $\hat{\boldsymbol{x}}$ and some buyer i, her payment probability $\frac{M_i(\hat{\boldsymbol{x}})}{\hat{w}_i}$ may be higher than her interim winning probability $Q_i(\hat{\boldsymbol{x}})$ in $\mathbf{\Gamma}$. Thus, we cannot construct a random allocation rule such that only those buyers who eventually obtain licenses are required to pay. Thus, $\mathbf{\Gamma}$ cannot be modified to be ex-post individually rational. Fortunately, $\mathbf{\Gamma}$ satisfies some other desirable properties as the following theorem states.

Theorem 4. *The probability allocation mechanism* $\mathbf{\Gamma}$ *is interim individually rational and weakly dominant strategy incentive compatible.*

In the probability allocation mechanism, indivisible licenses are treated as divisible goods. Namely, this mechanism sells license winning probabilities to buyers. Therefore, the probability allocation mechanism further relaxes buyers' budget constraints, and thus it can achieve better characteristics than our hybrid mechanisms. In fact, the numerical analysis in the next section confirms these better characteristics. Moreover, it is worth mentioning that the probability allocation mechanism is not ex-post individually rational, thus it is not covered in Che et al. (2013)'s discussion of the efficiency-optimal mechanism.

6. NUMERICAL ANALYSIS OF THE CHARACTERISTICS

Section 4 defined the continuum-mass mechanism $H(r, \alpha_1)$ for every hybrid mechanism $H(r, m_1)$, and presented the formulas for its characteristics $Ch(r, \alpha_1)$. In this section, we shall first use simulation and numerical computation to test if $Ch(r, \alpha_1)$ is a good approximation of $Ch(r, m_1)$. Second, to enable the social planner to choose, we shall plot the attainable characteristics of the hybrid mechanisms through numerical computation. Third, using these figures, we shall compare the attainable characteristics of the hybrid mechanisms with those of the existing license allocation mechanisms in China, and with those of the probability allocation mechanism.

Environment of numerical analysis

In the following, we set the number of potential buyers as $n = 100000$ and the license quota as $m = 10000$, and thus the mass of license quota $\alpha = \frac{1}{10}$. For simplicity, the value and budget of each buyer is set to be mutually independently and identically distributed. We indeed conduct the numerical analysis under two distributions: the uniform distribution on $[0, 10000]$ and the exponential distribution with a mean value of 5000, respectively.

Since Theorem 1 indicates that all buyers have incentives to bid sincerely, we can use the simulation method to examine the characteristics of discrete hybrid mechanisms. Specifically, for a hybrid mechanism $H(r, m_1)$, we first draw 1000 profiles $\{\boldsymbol{x}^{(k)} \mid k = 1, \cdots, 1000\}$ according to the uniform distribution (or, the exponential distribution). For each profile $\boldsymbol{x}^{(k)}$, according to Theorem 1 and the rule of $H(r, m_1)$ we can obtain an interim allocation $(\mathrm{Q}(\boldsymbol{x}^{(k)}), \mathrm{M}(\boldsymbol{x}^{(k)}))$. Accordingly, we can further obtain the interim efficiency $Ef(\boldsymbol{x}^{(k)})$, the interim revenue $Re(\boldsymbol{x}^{(k)})$, and the vector $\hat{\boldsymbol{Q}}(\boldsymbol{x}^{(k)})$ of interim winning probability for all buyers ranked by their budgets from low to high. We then use $\frac{1}{1000}\sum_{k=1}^{1000} Ef(\boldsymbol{x}^{(k)})$ to represent efficiency $Ef(r, m_1)$, and use $\frac{1}{1000}\sum_{k=1}^{1000} Re(\boldsymbol{x}^{(k)})$ to represent revenue $Re(r, m_1)$. By formula (1.6), we calculate the equality measure $Eq(r, m_1)$ from

$$\overline{\boldsymbol{Q}} = \frac{1}{1000} \cdot \sum_{k=1}^{1000} \hat{\boldsymbol{Q}}(\boldsymbol{x}^{(k)}).$$

To test whether it is robust to approximate $Ch(r, m_1)$ with $Ch(r, \frac{m_1}{n})$, we uniformly draw 121 parameters from

$$P_1 = \Big\{(r, m_1) = 1000(s, t) \mid s, t = 0, 1, \ldots, 10\Big\}$$

and compare each $Ch(r, m_1)$ with its continuum-mass counterpart $Ch(r, \frac{m_1}{n})$. The characteristics $Ch(r, m_1)$ are computed through the simulation method described above, and the characteristics $Ch(r, \frac{m_1}{n})$ are computed according to the formulas in Subsection 4.2. The relative difference between efficiency $Ef(r, m_1)$ and $Ef(r, \frac{m_1}{n})$ is calculated as

$$\frac{Ef(r, \frac{m_1}{n}) - Ef(r, m_1)}{Ef(r, m_1)},$$ [18]

and the relative differences in equality and revenue are similarly computed. We find that the absolute values of the relative differences between these characteristics $Ch(r, m_1)$ and $Ch(r, \frac{m_1}{n})$ with economically meaningful $r <$

[18] In the trival case of $r \geq \min\{\bar{v}, \bar{w}\}$, it holds that $Ef(r, m_1) = 0$. In this case, we set the relative difference between $Ef(r, m_1)$ and $Ef(r, \frac{m_1}{n})$ as 0.

r^* are usually less than 0.01%, while the largest absolute values of relative differences are those between $Eq(r, m_1)$ and $Eq(r, \frac{m_1}{n})$ with $r \geq r^*$, and the scale of these largest absolute values is approximately 0.1%. The comparison results are listed in Appendix B.

To plot the attainable characteristics of hybrid mechanisms, we uniformly draw 10201 parameters from

$$P_2 = \Big\{(r, m_1) = 100(s, t) \mid s, t = 0, 1, \ldots, 100\Big\}.$$

Since we have shown that the characteristics of continuum-mass mechanisms can approximate the characteristics of discrete mechanism well, we can substitute $Ch(r, m)$ with $Ch(r, \frac{m_1}{n})$. The characteristics of the hybrid mechanisms with parameters selected from P_2 are computed numerically and are plotted in blue in Figures 4, 5, and 6. In addition, the characteristics achievable by those hybrid mechanisms that are false-name-bidding-proof, i.e., satisfy $p^e \leq (1 + \lambda)r$, are plotted in yellow in those figures.

Recall that the current Shanghai mechanism $M(c)$ with a warning price c can be induced from our hybrid mechanisms. Formally, we have

$$M(c) = \begin{cases} H(c, 0), & \text{if } c \leq p^e, \\ H(0, \frac{m}{n}), & \text{if } c > p^e. \end{cases}$$

Let $Ch(c)$ denote the characteristics of $M(c)$. Then we have

$$Ch(c) = \begin{cases} Ch(c, 0), & \text{if } c \leq p^e, \\ Ch(0, \frac{m}{n}), & \text{if } c > p^e. \end{cases}$$

We uniformly draw the parameter c from

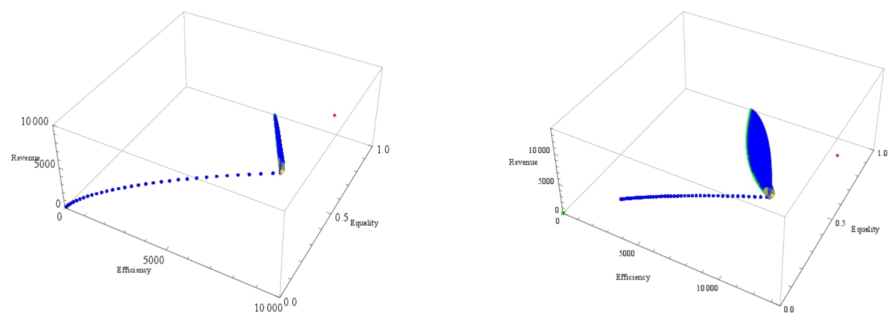
$$P_3 = \Big\{100s \mid s = 0, 1, \ldots, 100\Big\},$$

and plot the attainable characteristics of the current Shanghai mechanisms $Ch(c)$ in green in Figures 4, 5, and 6.

According to Theorem 4, the probability allocation mechanism is an IC mechanism, and thus we can compute its characteristics through simulation.[19]

[19] Since every buyer's winning probability is nondecreasing in her reported budget and value in the probability allocation mechanism, we see that the equality measure of the probability allocation mechanism is well-defined according to Theorem 4 and Assumption 1.

The computed characteristics are plotted in red in Figures 4, 5, and 6. Figure 4 presents the 3-D characteristics of different mechanisms. Since these 3-D graphs are difficult to comprehend intuitively, we plot the 2-D efficiency-equality characteristics of different mechanisms with uniformly distributed buyers' types in Figure 5, and with exponentially distributed buyers' types in Figure 6. Since wealth of buyers are usually supposed to be subject to Pareto distribution in literature, and that the Pareto distributions are closer to exponential distributions than uniform distributions, buyers' values and budgets are more likely to be exponentially distributed in practice. Therefore, in the following, we shall focus our discussion on Figure 6.



(a) 3-D characteristics of different mechanisms with uniform distribution

(b) 3-D characteristics of different mechanisms with exponential distribution

Figure 4: 3-D characteristics of different mechanisms

In Figure 6, we highlight the 2-D characteristics of several special mechanisms: (i) b — the Beijing mechanism, i.e., a pure lottery with no reserve price $H(0, 0)$; (ii) s — the Shanghai auction before July 2013, i.e., a pure auction with no reserve price $H(0, 0.1)$; (iii) e — the hybrid mechanism with the highest efficiency;[20] (iv) p — the probability allocation mechanism. These special mechanisms and their characteristics are listed in Table 1.

Figure 6 illustrates that each current Shanghai mechanism $M(c)$ generates the lowest efficiency $Ef(c, 0)$ in all hybrid mechanisms that keep the same equality level $Eq(c, 0)$. For instance, the current Shanghai mechanism $M(2500) = H(2500, 0)$ yields the 2-D characteristics $d = (7500, 0.607)$.

[20] Here, the "highest" and "lowest" are among all hybrid mechanisms with parameters in P_2.

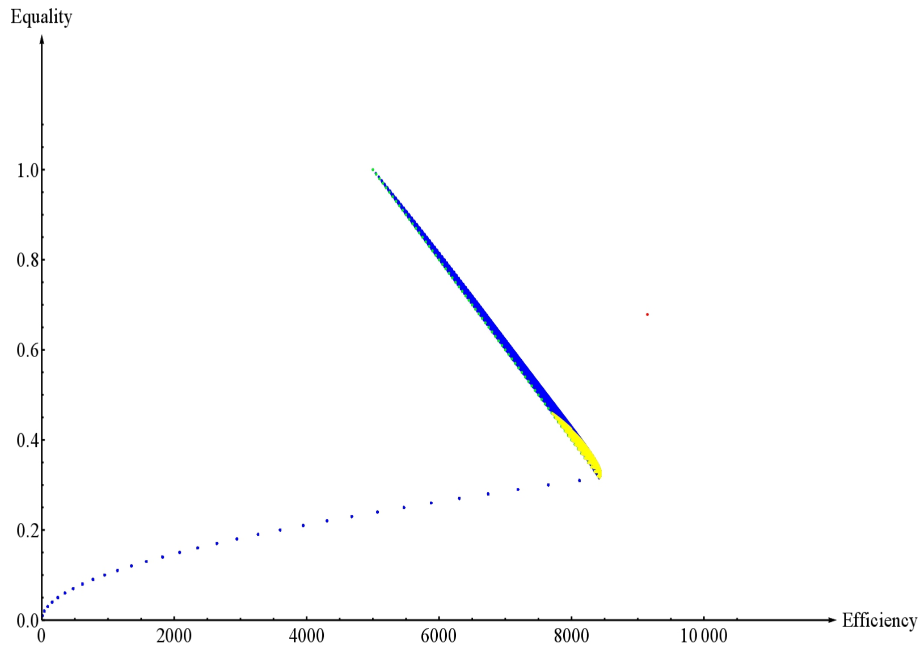


Figure 5: 2-D characteristics of different mechanisms with uniform distribution

However, compared with $M(2500)$, the hybrid mechanism $H(0,\ 0.055)$ brings the 2-D characteristics of $d' = (8890.8,\ 0.603)$, raising efficiency by 18.5%. The policy implication of our finding is profound: regardless of the social planner's objective, the current Shanghai mechanism, as a price ceiling mechanism, is not a wise choice.

Although the pure auction is usually viewed as the most efficient mechanism, Figure 6 demonstrates that it is less efficient than many other hybrid mechanisms. In theory, a hybrid mechanism provides an opportunity of lottery for buyers, and thus buyers will discount their bids in auction. Thus, buyers' budget constraints are relaxed as in the first-price auction with budget constraints; see e.g., Che & Gale (1998). Therefore, a hybrid mechanism may achieve higher efficiency than the pure auction. In fact, Che et al. (2013) have highlighted that in the presence of budget constraints, the efficiency-optimal mechanism always entails some random assignment. Our numerical analysis confirms this result. Specifically, among all hybrid mechanisms with parameters selected from P_2, the hybrid mechanism $H(5400,\ 0.076)$ achieves the highest efficiency. Comparing the characteristics of the pure auction $H(0, 0.1)$

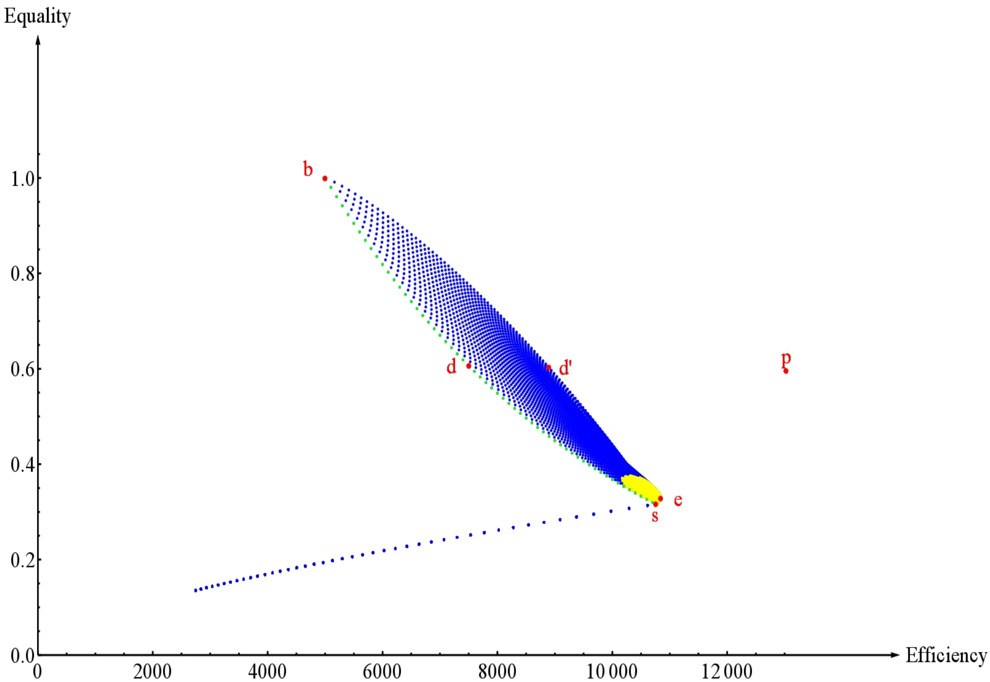


Figure 6: 2-D characteristics of different mechanisms with exponential distribution

Table 1: Characteristics of several special mechanisms with exponential distribution

$H(r, \alpha_1)$	(Ef, Eq)	Re	β	p^e	v^p	λ
$H(0,\ 0.000)$	$b = (\ 5000.0,\ 1.000)$	0.0	1.0000	-	-	0.1000
$H(0,\ 0.100)$	$s = (10756.5,\ 0.316)$	5756.5	1.0000	5756.5	5756.5	0.3162
$H(5400,\ 0.076)$	$e = (10844.4,\ 0.328)$	5844.4	0.1153	5984.7	6900.4	0.6103
PA mechanism	$p = (13024.7,\ 0.616)$	8027.0	-	-	-	-
$H(2500,\ 0.000)$	$d = (\ 7500.0,\ 0.607)$	2500.0	0.3679	-	-	0.2718
$H(0,\ 0.055)$	$d' = (\ 8890.8,\ 0.603)$	3890.8	1.0000	7074.2	7427.9	0.0476

with those of $H(5400,\ 0.076)$ provides a hint of how a hybrid mechanism yields higher efficiency than the pure auction.

Figure 7 demonstrates the different types of winning buyers in the pure auction mechanism $H(0, 0.1)$ and the hybrid mechanism $H(5400,\ 0.076)$. In $H(0, 0.1)$, the auction price is $r^* = 5756$, and buyers in $A \cup B$ win the auction. In $H(5400,\ 0.076)$, the reserve price is $r = 5400$, the equilibrium price is $p^e = 5985$, and the critical value level below which no buyer wins the auction is $v^p = 6900$. In this mechanism, buyers in A still win the auction, and buyers in $B \cup C$ participate in the lottery. Figure 7 illustrates that B mainly comprises buyers with relatively low values and high budgets, which results in a lower mean value for buyers in B than for buyers in C. Indeed, the mean value for buyers in C 7967.3 is much higher than the mean value for buyers in B 7134.2. As a result, the efficiency of $H(5400,\ 0.076)$ is higher than the efficiency of the pure auction $H(0, 0.1)$.

The above Figures 4, 5, and 6 also show that the probability allocation mechanism achieves much higher efficiency and revenue than the hybrid mechanisms. In fact, from Table 1 we see that the probability allocation mechanism increases efficiency by 20.11% and raises revenue by 37.35% compared with the most efficient hybrid mechanism $H(5400,\ 0.076)$. Figure 6 further shows that point p — the 2-D characteristics of the probability allocation mechanism, lies far beyond the attainable set of hybrid mechanisms' characteristics. Therefore, although the probability allocation mechanism is not ex-post individually rational and can hardly be put into practice, it may provide inspiration for finding other mechanisms that are ex-post individually rational and weakly dominant strategy incentive compatible, and yield better characteristics than our hybrid mechanisms.

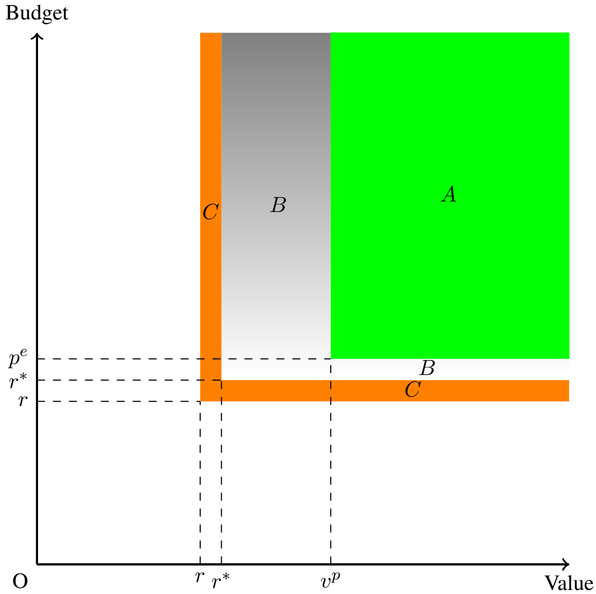


Figure 7: Buyers who can win licenses in $H(0, 0.1)$ and $H(5400, 0.076)$

7. CONCLUSION AND POLICY IMPLICATIONS

In this study, we examine the equality of public resource allocation mechanisms and introduce an equality measure as a new evaluation criterion for them in a multi-unit auction model with budget constraints. Our equality measure describes the difference in object obtaining opportunities among buyers with different wealth levels. As an application, we study vehicle license allocations in China. We especially propose a class of hybrid auction-lottery mechanisms to evaluate and improve upon China's vehicle license allocation mechanisms from the criteria of efficiency, equality and revenue, in a unified framework. For helping the social planner to evaluate hybrid mechanisms, we also provide a relative continuum-mass hybrid mechanism for each hybrid mechanism. In addition, we provide a probability allocation mechanism as a benchmark to compare with our hybrid mechanisms. Finally, using numerical analysis, we evaluate the characteristics of the hybrid mechanisms, compare different license allocation mechanisms in China, and provide useful insights into the improvement of vehicle license allocations in China. Our study can be widely applied to allocations of different public resources.

Several issues need to be addressed in future studies. First, it is both important and interesting to explore mechanisms that are no longer detail free, while achieve better characteristics than our hybrid mechanisms, and maintain simplicity of implementation, ex-post individual rationality, and incentive compatibility. Second, the possibility of extending our equality measure to the study of set-aside auctions and affirmative action in school choice can be discussed. Third, we just provide one equality measure in this paper. It is also an interesting question whether there are more reasonable equality measures based on the information about object obtaining opportunities of buyers with different budgets.

Appendix A: Proofs

A.1. Proof of Lemma 1

By the monotonicity of a standard direct mechanism and equation (1.1), we see that $q(v,w)$ is nondecreasing in both v and w. Therefore, by Assumption 1, for any $w,\ w' \in [0,\bar{w}]$ such that $w \leq w'$, we have

$$\begin{aligned} p(w) = \int_0^{\bar{v}} q(v,w)dF(v|w) &\leq \int_0^{\bar{v}} q(v,w)dF(v|w') \\ &\leq \int_0^{\bar{v}} q(v,w')dF(v|w') = p(w'). \end{aligned}$$

Thus, $p(w)$ is a nondecreasing function. Note that for any $j \leq j'$,

$$\begin{aligned} G_{(j)}(w) &= \sum_{i=j}^{n} \binom{n}{i} G^i(w)[1-G(w)]^{n-i} \\ &\geq \sum_{i=j'}^{n} \binom{n}{i} G^i(w)[1-G(w)]^{n-i} = G_{(j')}(w). \end{aligned}$$

Therefore, we further have

$$\overline{Q}_{(j)} = \int_0^{\bar{w}} p(w)dG_{(j)}(w) \leq \int_0^{\bar{w}} p(w)dG_{(j')}(w) = \overline{Q}_{(j')}.$$

$\square$

A.2. Proof of Theorem 1

We only prove the second part of the theorem. Suppose buyer i's type is $x_i = (v_i, w_i)$. According to the rule of the hybrid mechanisms, each buyer's license payment is always no less than r. Therefore, when $\min\{v_i, w_i\} < r$, a rational buyer i will not register for the mechanism no matter how other buyers bid. In the following, we assume $\min\{v_i, w_i\} \geq r$. Note that if buyer i registers for the hybrid mechanism, she can at least get an expected payoff $\lambda(v_i - r) \geq 0$ by bidding r. Therefore, we just consider that buyer i registers for the mechanism. Let $b_{-i}^{(m_1)}$ denote the m_1-th highest bid of all other buyers (if $n_1 < m_1$, set $b_{-i}^{(m_1)} = r$), and let $b_i(v_i, w_i) = \min\{\, v_i - \lambda(v_i - r),\, w_i \,\}$ be buyer i's sincere bid. In the following, we shall show that buyer i cannot improve her payoff by making a bid $b \geq r$ other than $b_i(v_i, w_i)$ in two cases.

Case 1, $b_i(v_i, w_i) \geq b_{-i}^{(m_1)}$. By submitting her sincere bid, buyer i wins the auction and receives a payoff $v_i - b_{-i}^{(m_1)} \geq \lambda(v_i - r)$. If she submits a bid $b \geq b_{-i}^{(m_1)}$, she still wins the auction and obtains the same payoff. If she bids $b < b_{-i}^{(m_1)}$, she enters the lottery and obtains a payoff $\lambda(v_i - r) \leq v_i - b_{-i}^{(m_1)}$. Therefore, buyer i receives the highest payoff by bidding $b_i(v_i, w_i)$ in this case.

Case 2, $b_i(v_i, w_i) < b_{-i}^{(m_1)}$. In this case, it satisfies that $w_i < b_{-i}^{(m_1)}$ or $v_i - b_{-i}^{(m_1)} < \lambda(v_i - r)$. By bidding sincerely, buyer i enters the lottery and obtains an expected payoff $\lambda(v_i - r)$. If buyer i bids $b \geq b_{-i}^{(m_1)}$, she wins the auction and obtains a payoff no greater than than $\lambda(v_i - r)$. If she makes a bid $b < b_{-i}^{(m_1)}$, she still enters the lottery and receives the same payoff. Therefore, buyer i cannot increase her expected payoff by any other bid.

To sum up, no matter how other buyers bid, buyer i obtains the highest expected payoff if she bids sincerely. □

A.3. Proof of Theorem 3

According to the argument in Subsection 4.3, when $p^e > (1+\lambda)r$, the sets E and F are both non-empty, and all buyers in E and F have incentives to bid under a false name. Therefore, $p^e \leq (1+\lambda)r$ is a necessary condition to prevent false-name bidding for a hybrid mechanism $H(r, \alpha_1)$ with $2r \leq \bar{w}$. In the following, we shall show that it is also a sufficient condition for preventing buyers from bidding under any number of false names.

Assume $p^e \leq (1+\lambda)r$. Note that only a buyer with value $v \geq r$ and budget $w \geq kr$ may have an incentive to bid under $k-1$ ($k \in \mathbb{Z}_+$ and $k \geq 2$) false names, and such a buyer's expected payoff from bidding under $k-1$ false names is $[1-(1-\lambda)^k]v - k\lambda r$. Thus, for such a buyer in D, her net surplus from bidding under $k-1$ false names is

$$\begin{aligned}
&[1-(1-\lambda)^k]v - k\lambda r - \lambda(v-r)\\
&= (1-\lambda)[1-(1-\lambda)^{k-1}]v - (k-1)\lambda r\\
&\leq (1-\lambda)[(k-1)\lambda]v - (k-1)\lambda r\\
&\leq (k-1)\lambda[(1-\lambda)\cdot\frac{p^e-\lambda r}{1-\lambda} - r]\\
&= (k-1)\lambda[p^e-(1+\lambda)r]\\
&\leq 0.
\end{aligned}$$

For such a buyer in C, her net surplus from bidding under $k-1$ false names is

$$\begin{aligned}
&[1-(1-\lambda)^k]v - k\lambda r - (v-p^e)\\
&= p^e - (1-\lambda)^k v - k\lambda r\\
&\leq p^e - (1-\lambda)^k\cdot\frac{p^e-\lambda r}{1-\lambda} - k\lambda r\\
&= [1-(1-\lambda)^{k-1}](p^e-\lambda r) - (k-1)\lambda r\\
&\leq [(k-1)\lambda](p^e-\lambda r) - (k-1)\lambda r\\
&= (k-1)\lambda[p^e-(1+\lambda)r]\\
&\leq 0.
\end{aligned}$$

We therefore see that all buyers in $C\cup D$ have no incentives to bid under any number of false names. In addition, it is clear that every buyer not in $C\cup D$ will not register for the mechanism, and has no incentive to bid under false names. Consequently, we have proved that $p^e \leq (1+\lambda)r$ is a sufficient condition for preventing buyers from bidding under any number of false names. □

A.4. Proof of Theorem 4

By the probability allocation mechanism rule, no matter what other buyers report, each buyer's unit expected payment for her assigned license never exceeds her reported value, and her ex-post payment never exceeds her reported budget. Therefore, the probability allocation mechanism is interim individually rational. We then proceed to prove that the probability allocation mechanism is weakly dominant strategy incentive compatible.

Suppose the report profile of other buyers is $\hat{\boldsymbol{x}}_{-i}$. Then, buyer i faces a fixed nondecreasing residual supply function $s^{-i}(p)$. To simplify the notation, let $p^* = p(x_i, \hat{\boldsymbol{x}}_{-i})$, $q_i^* = Q_i(x_i, \hat{\boldsymbol{x}}_{-i})$ and $u_i^* = u_i(Q_i(x_i, \hat{\boldsymbol{x}}_{-i}), M_i(x_i, \hat{\boldsymbol{x}}_{-i}), x_i)$

denote the equilibrium price, the winning probability and expected utility of buyer i when she reports her true type x_i, respectively. And use $p' = p(x'_i, \hat{\boldsymbol{x}}_{-i})$, $q'_i = Q_i(x'_i, \hat{\boldsymbol{x}}_{-i})$ and $u'_i = u_i(Q_i(x'_i, \hat{\boldsymbol{x}}_{-i}), M_i(x'_i, \hat{\boldsymbol{x}}_{-i}), x_i)$ to denote the equilibrium price, the winning probability and expected utility of buyer i when she strategically reports her type $x'_i = (v'_i, w'_i)$, respectively. Since Prob$\{\Gamma_{i2}(x'_i, \hat{\boldsymbol{x}}_{-i}) = \hat{w}_i\} > 0$, buyer i cannot get a greater expected utility $u'_i > u^*_i$ by reporting a higher budget $w'_i > w_i$. To prove this theorem, it is sufficient to show $u'_i \leq u^*_i$ for all $x'_i = (v'_i, w'_i)$ with $w'_i \leq w_i$.

From the expected payment rule $M_i(\cdot)$, the expected utilities u^*_i and u'_i can be written as

$$u^*_i \;=\; q^*_i v_i - M_i(x_i, \hat{\boldsymbol{x}}_{-i}) \;=\; q^*_i(v_i - p^*) + \int_0^{p^*} s^{-i}(p)\,dp,$$

$$u'_i \;=\; q'_i v_i - M_i(x'_i, \hat{\boldsymbol{x}}_{-i}) \;=\; q'_i(v_i - p') + \int_0^{p'} s^{-i}(p)\,dp.$$

By the interim assignment rule $Q_i(\cdot)$, we can show that $s^{-i}(\bar{p}) \leq q'_i \leq s^{-i}(\hat{p})$ for all prices $\bar{p}$ and $\hat{p}$ such that $\bar{p} \leq p' < \hat{p}$. Therefore, it satisfies that

$$\begin{aligned} u'_i - u^*_i \;&=\; (q'_i - q^*_i)(v_i - p^*) - \int_{p^*}^{p'} [q'_i - s^{-i}(p)]\,dp \\ &\leq\; (q'_i - q^*_i)(v_i - p^*). \end{aligned}$$

Note that in the case of $v_i < p^*$, it holds that $q^*_i = 0$, and hence $u'_i - u^*_i \leq q'_i(v_i - p^*) \leq 0$. Next, in the case of $p^* < v_i$, it holds $q^*_i = \min\{1, \frac{w_i}{p^*}\}$. Thus, for any $x'_i = (v'_i, w'_i)$ with $w'_i \leq w_i$, if $p' \geq p^*$ then $q'_i \leq \min\{1, \frac{w'_i}{p'}\} \leq \min\{1, \frac{w_i}{p^*}\} = q^*_i$, if $p' < p^*$ then $q'_i < s^{-i}(p^*) \leq q^*_i$, and so it always holds $q'_i \leq q^*_i$. Hence, $u'_i - u^*_i \leq (q'_i - q^*_i)(v_i - p^*) \leq 0$. Consequently, it satisfies that $u'_i \leq u^*_i$ for all $x'_i = (v'_i, w'_i)$ with $w'_i \leq w_i$. □

Appendix B: Comparisons between characteristics of discrete and continuum-mass hybrid mechanisms

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In the following table, $Ef(r, m_1)$, $Eq(r, m_1)$, and $Re(r, m_1)$ represent the computed efficiency, equality, and revenue of the discrete hybrid mechanism

$H(r, m_1)$, respectively. $Ef(r, \frac{m_1}{n})$, $Eq(r, \frac{m_1}{n})$, and $Re(r, \frac{m_1}{n})$ represent the computed efficiency, equality, and revenue of the continuum-mass hybrid mechanism $H(r, \frac{m_1}{n})$, respectively. $Diff_{Ef}$, $Diff_{Eq}$, and $Diff_{Re}$ denote the relative difference between computed characteristics of $H(r, \frac{m_1}{n})$ and those of $H(r, m_1)$. Buyers' types are assumed to be uniformly distributed.

r	m_1	$Ef(r, m_1)$	$Eq(r, m_1)$	$Re(r, m_1)$	$Ef(r, \frac{m_1}{n})$	$Eq(r, \frac{m_1}{n})$	$Re(r, \frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
0	0	5000.09	0.999919	0.00	5000.00	1.000000	0.00	-0.002%	0.008%	0.000%
0	1000	5424.18	0.922825	848.90	5424.46	0.922826	848.92	0.005%	0.000%	0.002%
0	2000	5817.85	0.849801	1635.26	5817.64	0.849820	1635.29	-0.004%	0.002%	0.002%
0	3000	6189.43	0.779192	2379.07	6189.83	0.779206	2379.67	0.007%	0.002%	0.025%
0	4000	6545.41	0.710296	3090.01	6545.19	0.710277	3090.38	-0.003%	-0.003%	0.012%
0	5000	6886.16	0.642631	3771.87	6886.14	0.642626	3772.28	0.000%	-0.001%	0.011%
0	6000	7214.24	0.575980	4428.40	7214.30	0.575985	4428.60	0.001%	0.001%	0.005%
0	7000	7530.64	0.510111	5061.65	7530.85	0.510159	5061.69	0.003%	0.009%	0.001%
0	8000	7836.68	0.445099	5673.11	7836.67	0.444999	5673.34	0.000%	-0.022%	0.004%
0	9000	8132.69	0.380413	6264.49	8132.49	0.380388	6264.97	-0.002%	-0.007%	0.008%
0	10000	8418.61	0.316301	6838.14	8418.86	0.316228	6837.72	0.003%	-0.023%	-0.006%
1000	0	5499.90	0.899912	1000.00	5500.00	0.900000	1000.00	0.002%	0.010%	0.000%
1000	1000	5871.63	0.834156	1742.38	5871.21	0.834110	1742.43	-0.007%	-0.006%	0.003%
1000	2000	6212.79	0.771880	2425.41	6212.83	0.771871	2425.66	0.001%	-0.001%	0.010%
1000	3000	6533.99	0.711749	3068.22	6534.25	0.711713	3068.51	0.004%	-0.005%	0.009%
1000	4000	6839.48	0.653076	3678.91	6839.42	0.652990	3678.84	-0.001%	-0.013%	-0.002%
1000	5000	7131.03	0.595284	4260.90	7130.63	0.595330	4261.25	-0.006%	0.008%	0.008%
1000	6000	7409.43	0.538526	4819.10	7409.40	0.538487	4818.80	0.000%	-0.007%	-0.006%
1000	7000	7676.89	0.482306	5353.47	7676.86	0.482279	5353.71	0.000%	-0.006%	0.004%
1000	8000	7933.53	0.426662	5867.45	7933.83	0.426570	5867.66	0.004%	-0.022%	0.004%
1000	9000	8180.71	0.371219	6361.66	8180.99	0.371250	6361.98	0.003%	0.008%	0.005%
1000	10000	8418.94	0.316186	6837.71	8418.86	0.316228	6837.72	-0.001%	0.013%	0.000%
2000	0	5999.89	0.799943	2000.00	6000.00	0.800000	2000.00	0.002%	0.007%	0.000%
2000	1000	6317.02	0.745589	2633.97	6317.03	0.745652	2634.06	0.000%	0.008%	0.004%
2000	2000	6606.58	0.694336	3213.06	6606.67	0.694322	3213.34	0.001%	-0.002%	0.009%
2000	3000	6876.96	0.644733	3754.36	6877.19	0.644694	3754.38	0.003%	-0.006%	0.000%
2000	4000	7132.44	0.596267	4264.17	7132.19	0.596207	4264.37	-0.004%	-0.010%	0.005%
2000	5000	7374.18	0.548458	4747.26	7373.78	0.548529	4747.55	-0.005%	0.013%	0.006%
2000	6000	7602.87	0.501385	5206.70	7603.37	0.501441	5206.75	0.007%	0.011%	0.001%
2000	7000	7821.66	0.454824	5643.09	7821.99	0.454780	5643.99	0.004%	-0.010%	0.016%
2000	8000	8029.95	0.408480	6060.71	8030.40	0.408422	6060.81	0.006%	-0.014%	0.002%
2000	9000	8229.37	0.362164	6458.42	8229.20	0.362266	6458.40	-0.002%	0.028%	0.000%
2000	10000	8419.63	0.316188	6837.58	8418.86	0.316228	6837.72	-0.009%	0.013%	0.002%
3000	0	6499.60	0.700002	3000.00	6500.00	0.700000	3000.00	0.006%	0.000%	0.000%
3000	1000	6761.16	0.657535	3522.87	6761.44	0.657516	3522.88	0.004%	-0.003%	0.000%
3000	2000	6998.74	0.617236	3996.91	6998.49	0.617271	3996.99	-0.003%	0.006%	0.002%
3000	3000	7217.76	0.578274	4435.47	7217.90	0.578270	4435.79	0.002%	-0.001%	0.007%
3000	4000	7422.61	0.540086	4845.44	7422.77	0.540053	4845.54	0.002%	-0.006%	0.002%
3000	5000	7614.82	0.502336	5229.79	7614.96	0.502348	5229.92	0.002%	0.002%	0.003%
3000	6000	7795.89	0.464964	5590.76	7795.71	0.464965	5591.42	-0.002%	0.000%	0.012%
3000	7000	7965.68	0.427845	5931.38	7965.89	0.427763	5931.78	0.003%	-0.019%	0.007%
3000	8000	8125.78	0.390589	6252.51	8126.16	0.390632	6252.33	0.005%	0.011%	-0.003%
3000	9000	8277.21	0.353557	6553.84	8277.03	0.353480	6554.05	-0.002%	-0.022%	0.003%
3000	10000	8418.31	0.316304	6837.30	8418.86	0.316228	6837.72	0.007%	-0.024%	0.006%
4000	0	6999.62	0.600030	4000.00	7000.00	0.600000	4000.00	0.005%	-0.005%	0.000%
4000	1000	7203.54	0.569886	4407.22	7203.59	0.569752	4407.19	0.001%	-0.024%	-0.001%
4000	2000	7386.38	0.540814	4773.78	7387.02	0.540809	4774.03	0.009%	-0.001%	0.005%
4000	3000	7554.86	0.512541	5109.86	7554.96	0.512552	5109.92	0.001%	0.002%	0.001%
4000	4000	7709.42	0.484655	5419.61	7709.80	0.484657	5419.60	0.005%	0.001%	0.000%
4000	5000	7852.69	0.456924	5705.78	7852.98	0.456919	5705.96	0.004%	-0.001%	0.003%
4000	6000	7985.69	0.429217	5970.69	7985.45	0.429189	5970.90	-0.003%	-0.007%	0.004%
4000	7000	8107.78	0.401402	6215.46	8107.87	0.401348	6215.74	0.001%	-0.013%	0.005%

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r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
4000	8000	8220.54	0.373338	6440.94	8220.71	0.373298	6441.41	0.002%	-0.011%	0.007%
4000	9000	8324.41	0.344948	6648.20	8324.29	0.344951	6648.59	-0.001%	0.001%	0.006%
4000	10000	8418.80	0.316289	6836.96	8418.86	0.316228	6837.72	0.001%	-0.019%	0.011%
5000	0	7500.24	0.500027	5000.00	7500.00	0.500000	5000.00	-0.003%	-0.005%	0.000%
5000	1000	7641.92	0.482241	5283.64	7641.81	0.482274	5283.62	-0.002%	0.007%	-0.001%
5000	2000	7769.57	0.464808	5538.97	7769.50	0.464848	5539.00	-0.001%	0.009%	0.001%
5000	3000	7885.03	0.447403	5770.18	7885.20	0.447472	5770.41	0.002%	0.015%	0.004%
5000	4000	7990.44	0.429969	5980.25	7990.14	0.429981	5980.27	-0.004%	0.003%	0.000%
5000	5000	8084.78	0.412279	6169.83	8085.04	0.412244	6170.07	0.003%	-0.008%	0.004%
5000	6000	8169.94	0.394298	6340.66	8170.36	0.394154	6340.71	0.005%	-0.037%	0.001%
5000	7000	8246.38	0.375618	6492.96	8246.35	0.375608	6492.70	0.000%	-0.003%	-0.004%
5000	8000	8312.57	0.356517	6625.28	8313.12	0.356508	6626.25	0.007%	-0.002%	0.015%
5000	9000	8371.07	0.336760	6741.35	8370.67	0.336750	6741.33	-0.005%	-0.003%	0.000%
5000	10000	8418.77	0.316294	6837.97	8418.86	0.316228	6837.72	0.001%	-0.021%	-0.004%
6000	0	7999.87	0.400051	6000.00	8000.00	0.400000	6000.00	0.002%	-0.013%	0.000%
6000	1000	8071.84	0.394220	6144.34	8072.18	0.394226	6144.35	0.004%	0.002%	0.000%
6000	2000	8138.74	0.388067	6277.36	8138.63	0.388117	6277.27	-0.001%	0.013%	-0.001%
6000	3000	8198.62	0.381588	6398.39	8199.19	0.381613	6398.39	0.007%	0.006%	0.000%
6000	4000	8253.26	0.374600	6507.25	8253.59	0.374641	6507.18	0.004%	0.011%	-0.001%
6000	5000	8301.38	0.367143	6602.55	8301.44	0.367115	6602.89	0.001%	-0.008%	0.005%
6000	6000	8342.14	0.359055	6684.39	8342.26	0.358929	6684.52	0.002%	-0.035%	0.002%
6000	7000	8375.27	0.349958	6750.31	8375.39	0.349947	6750.79	0.001%	-0.003%	0.007%
6000	8000	8400.12	0.339994	6800.22	8400.00	0.340000	6800.00	-0.001%	0.002%	-0.003%
6000	9000	8415.14	0.328874	6830.14	8414.98	0.328861	6829.95	-0.002%	-0.004%	-0.003%
6000	10000	8418.90	0.316294	6837.51	8418.86	0.316228	6837.72	0.000%	-0.021%	0.003%
7000	0	7650.92	0.300057	6297.26	7650.00	0.300000	6300.00	-0.012%	-0.019%	0.044%
7000	1000	7652.60	0.299894	6299.86	7650.00	0.300000	6300.00	-0.034%	0.036%	0.002%
7000	2000	7648.85	0.300241	6303.28	7650.00	0.300000	6300.00	0.015%	-0.080%	-0.052%
7000	3000	7649.41	0.300138	6302.06	7650.00	0.300000	6300.00	0.008%	-0.046%	-0.033%
7000	4000	7656.82	0.299784	6297.43	7650.00	0.300000	6300.00	-0.089%	0.072%	0.041%
7000	5000	7648.99	0.299804	6297.59	7650.00	0.300000	6300.00	0.013%	0.065%	0.038%
7000	6000	7651.31	0.300165	6300.64	7650.00	0.300000	6300.00	-0.017%	-0.055%	-0.010%
7000	7000	7652.21	0.299963	6303.34	7650.00	0.300000	6300.00	-0.029%	0.012%	-0.053%
7000	8000	7647.05	0.300182	6296.54	7650.00	0.300000	6300.00	0.039%	-0.060%	0.055%
7000	9000	7652.63	0.299984	6301.19	7650.00	0.300000	6300.00	-0.034%	0.005%	-0.019%
7000	10000	7649.25	0.299956	6301.77	7650.00	0.300000	6300.00	0.010%	0.015%	-0.028%
8000	0	3598.52	0.199858	3201.02	3600.00	0.200000	3200.00	0.041%	0.071%	-0.032%
8000	1000	3598.64	0.200114	3200.66	3600.00	0.200000	3200.00	0.038%	-0.057%	-0.021%
8000	2000	3600.16	0.200018	3200.65	3600.00	0.200000	3200.00	-0.004%	-0.009%	-0.020%
8000	3000	3602.56	0.200099	3200.79	3600.00	0.200000	3200.00	-0.071%	-0.050%	-0.025%
8000	4000	3599.96	0.200021	3201.02	3600.00	0.200000	3200.00	0.001%	-0.011%	-0.032%
8000	5000	3599.36	0.199955	3198.02	3600.00	0.200000	3200.00	0.018%	0.022%	0.062%
8000	6000	3599.75	0.200281	3200.78	3600.00	0.200000	3200.00	0.007%	-0.140%	-0.024%
8000	7000	3603.34	0.199869	3200.63	3600.00	0.200000	3200.00	-0.093%	0.066%	-0.020%
8000	8000	3600.70	0.200214	3200.19	3600.00	0.200000	3200.00	-0.019%	-0.107%	-0.006%
8000	9000	3599.05	0.200057	3198.74	3600.00	0.200000	3200.00	0.026%	-0.029%	0.039%
8000	10000	3603.64	0.200052	3201.47	3600.00	0.200000	3200.00	-0.101%	-0.026%	-0.046%
9000	0	949.35	0.100101	900.01	950.00	0.100000	900.00	0.068%	-0.101%	-0.001%
9000	1000	948.80	0.099910	898.69	950.00	0.100000	900.00	0.126%	0.090%	0.146%
9000	2000	949.04	0.099885	899.51	950.00	0.100000	900.00	0.101%	0.115%	0.055%
9000	3000	949.61	0.100083	900.76	950.00	0.100000	900.00	0.041%	-0.083%	-0.084%
9000	4000	949.13	0.099985	901.41	950.00	0.100000	900.00	0.092%	0.015%	-0.156%
9000	5000	950.95	0.099973	898.67	950.00	0.100000	900.00	-0.100%	0.027%	0.148%
9000	6000	949.24	0.099976	900.78	950.00	0.100000	900.00	0.080%	0.024%	-0.086%
9000	7000	950.34	0.100060	900.14	950.00	0.100000	900.00	-0.035%	-0.060%	-0.015%
9000	8000	950.21	0.100012	898.89	950.00	0.100000	900.00	-0.022%	-0.012%	0.124%
9000	9000	949.76	0.100040	900.30	950.00	0.100000	900.00	0.025%	-0.040%	-0.033%
9000	10000	950.03	0.099958	899.95	950.00	0.100000	900.00	-0.003%	0.042%	0.006%
10000	0	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	1000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	2000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	3000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	4000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	5000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	6000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%

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r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
10000	7000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	8000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	9000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%
10000	10000	0.00	0.000000	0.00	0.00	0.000000	0.00	0.000%	0.000%	0.000%

The following table presents the similar results of comparisons with exponential distributed buyers' types.

r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
0	0	5000.08	0.999911	0.00	5000.00	1.000000	0.00	-0.002%	0.009%	0.000%
0	1000	6095.69	0.919248	1095.79	6096.47	0.919235	1096.47	0.013%	-0.001%	0.062%
0	2000	6871.71	0.844542	1872.62	6872.78	0.844556	1872.78	0.015%	0.002%	0.008%
0	3000	7530.19	0.773169	2531.26	7531.47	0.773132	2531.47	0.017%	-0.005%	0.009%
0	4000	8115.84	0.703995	3114.96	8115.04	0.703997	3115.04	-0.010%	0.000%	0.003%
0	5000	8642.34	0.636615	3642.77	8643.46	0.636611	3643.46	0.013%	-0.001%	0.019%
0	6000	9127.22	0.570680	4127.95	9128.37	0.570617	4128.37	0.013%	-0.011%	0.010%
0	7000	9578.16	0.505831	4576.73	9577.41	0.505759	4577.41	-0.008%	-0.014%	0.015%
0	8000	9994.27	0.441793	4996.25	9995.95	0.441840	4995.95	0.017%	0.011%	-0.006%
0	9000	10385.86	0.378719	5386.20	10387.94	0.378705	5387.94	0.020%	-0.004%	0.032%
0	10000	10754.89	0.316171	5756.09	10756.46	0.316228	5756.46	0.015%	0.018%	0.006%
1000	0	6000.47	0.818689	1000.00	6000.00	0.818731	1000.00	-0.008%	0.005%	0.000%
1000	1000	6974.12	0.758125	1973.85	6974.41	0.758089	1974.41	0.004%	-0.005%	0.028%
1000	2000	7641.02	0.703054	2640.65	7640.92	0.702958	2640.92	-0.001%	-0.014%	0.011%
1000	3000	8196.06	0.650569	3195.02	8195.20	0.650593	3195.20	-0.010%	0.004%	0.006%
1000	4000	8675.03	0.600140	3677.51	8678.01	0.600079	3678.01	0.034%	-0.010%	0.013%
1000	5000	9109.33	0.550960	4107.84	9108.41	0.550911	4108.41	-0.010%	-0.009%	0.014%
1000	6000	9497.30	0.502751	4495.95	9497.47	0.502756	4497.47	0.002%	0.001%	0.034%
1000	7000	9853.30	0.455422	4852.26	9852.38	0.455375	4852.38	-0.009%	-0.010%	0.003%
1000	8000	10176.00	0.408488	5178.09	10178.20	0.408586	5178.20	0.022%	0.024%	0.002%
1000	9000	10477.03	0.362303	5478.21	10478.61	0.362243	5478.61	0.015%	-0.017%	0.007%
1000	10000	10756.49	0.316201	5756.56	10756.46	0.316228	5756.46	0.000%	0.009%	-0.002%
2000	0	7000.75	0.670306	2000.00	7000.00	0.670320	2000.00	-0.011%	0.002%	0.000%
2000	1000	7842.75	0.626892	2841.92	7842.73	0.626900	2842.73	0.000%	0.001%	0.029%
2000	2000	8394.90	0.588345	3395.69	8396.15	0.588241	3396.15	0.015%	-0.018%	0.013%
2000	3000	8845.23	0.551788	3844.99	8845.06	0.551758	3845.06	-0.002%	-0.005%	0.002%
2000	4000	9228.92	0.516614	4227.71	9227.57	0.516621	4227.57	-0.015%	0.001%	-0.003%
2000	5000	9560.97	0.482321	4561.45	9561.36	0.482371	4561.36	0.004%	0.010%	-0.002%
2000	6000	9854.51	0.448767	4855.94	9856.57	0.448708	4856.57	0.021%	-0.013%	0.013%
2000	7000	10119.11	0.415365	5119.15	10119.74	0.415415	5119.74	0.006%	0.012%	0.012%
2000	8000	10353.08	0.382417	5356.11	10355.40	0.382326	5355.40	0.022%	-0.024%	-0.013%
2000	9000	10566.56	0.349254	5566.33	10566.83	0.349302	5566.83	0.003%	0.014%	0.009%
2000	10000	10757.23	0.316368	5756.03	10756.46	0.316228	5756.46	-0.007%	-0.044%	0.008%
3000	0	8000.47	0.548800	3000.00	8000.00	0.548812	3000.00	-0.006%	0.002%	0.000%
3000	1000	8694.65	0.520237	3695.53	8695.69	0.520324	3695.69	0.012%	0.017%	0.004%
3000	2000	9131.23	0.495674	4131.13	9131.13	0.495618	4131.13	-0.001%	-0.011%	0.000%
3000	3000	9472.23	0.472417	4473.35	9473.57	0.472399	4473.57	0.014%	-0.004%	0.005%
3000	4000	9757.11	0.449960	4756.34	9756.90	0.449962	4756.90	-0.002%	0.000%	0.012%
3000	5000	9997.23	0.427945	4996.87	9996.57	0.427917	4996.57	-0.007%	-0.006%	-0.006%
3000	6000	10200.98	0.406082	5201.22	10201.29	0.406006	5201.29	0.003%	-0.019%	0.001%
3000	7000	10372.08	0.384019	5375.24	10376.52	0.384033	5376.52	0.043%	0.004%	0.024%
3000	8000	10526.88	0.361778	5525.08	10525.91	0.361838	5525.91	-0.009%	0.016%	0.015%
3000	9000	10650.27	0.339309	5650.19	10651.97	0.339279	5651.97	0.016%	-0.009%	0.032%
3000	10000	10754.81	0.316292	5757.90	10756.46	0.316228	5756.46	0.015%	-0.020%	-0.025%
4000	0	8998.27	0.449324	4000.00	9000.00	0.449329	4000.00	0.019%	0.001%	0.000%
4000	1000	9520.69	0.433872	4520.68	9521.14	0.433884	4521.14	0.005%	0.003%	0.010%
4000	2000	9830.16	0.420935	4830.16	9830.14	0.420936	4830.14	0.000%	0.000%	-0.001%
4000	3000	10066.34	0.408699	5064.90	10064.33	0.408726	5064.33	-0.020%	0.007%	-0.011%
4000	4000	10248.23	0.396720	5249.90	10250.66	0.396737	5250.66	0.024%	0.004%	0.015%
4000	5000	10402.01	0.384565	5401.69	10400.95	0.384673	5400.95	-0.010%	0.028%	-0.014%
4000	6000	10521.75	0.372313	5521.01	10521.48	0.372316	5521.48	-0.003%	0.001%	0.009%
4000	7000	10615.56	0.359465	5615.62	10615.84	0.359479	5615.84	0.003%	0.004%	0.004%
4000	8000	10686.40	0.346031	5685.56	10686.00	0.345981	5686.00	-0.004%	-0.014%	0.008%
4000	9000	10732.96	0.331623	5732.86	10732.85	0.331634	5732.85	-0.001%	0.003%	0.000%

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r	m_1	$Ef(r,m_1)$	$Eq(r,m_1)$	$Re(r,m_1)$	$Ef(r,\frac{m_1}{n})$	$Eq(r,\frac{m_1}{n})$	$Re(r,\frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
4000	10000	10757.08	0.316275	5756.14	10756.46	0.316228	5756.46	-0.006%	-0.015%	0.006%
5000	0	9998.00	0.367911	5000.00	10000.00	0.367879	5000.00	0.020%	-0.009%	0.000%
5000	1000	10283.92	0.363458	5286.15	10286.47	0.363356	5286.47	0.025%	-0.028%	0.006%
5000	2000	10448.00	0.359801	5448.09	10448.41	0.359734	5448.41	0.004%	-0.019%	0.006%
5000	3000	10566.30	0.356250	5567.84	10567.65	0.356215	5567.65	0.013%	-0.010%	-0.003%
5000	4000	10657.60	0.352576	5658.53	10659.20	0.352565	5659.20	0.015%	-0.003%	0.012%
5000	5000	10728.43	0.348659	5729.96	10728.94	0.348616	5728.94	0.005%	-0.012%	-0.018%
5000	6000	10778.11	0.344223	5779.88	10779.13	0.344200	5779.13	0.009%	-0.007%	-0.013%
5000	7000	10811.47	0.339129	5809.86	10809.90	0.339109	5809.90	-0.015%	-0.006%	0.001%
5000	8000	10818.38	0.333138	5819.40	10819.53	0.333064	5819.53	0.011%	-0.022%	0.002%
5000	9000	10802.58	0.325599	5802.64	10804.10	0.325652	5804.10	0.014%	0.016%	0.025%
5000	10000	10755.42	0.316222	5754.93	10756.46	0.316228	5756.46	0.010%	0.002%	0.027%
6000	0	9978.27	0.301103	5443.72	9978.97	0.301194	5443.08	0.007%	0.030%	-0.012%
6000	1000	9978.50	0.301119	5443.73	9978.97	0.301194	5443.08	0.005%	0.025%	-0.012%
6000	2000	9980.90	0.301258	5442.88	9978.97	0.301194	5443.08	-0.019%	-0.021%	0.004%
6000	3000	9982.01	0.301352	5444.84	9978.97	0.301194	5443.08	-0.030%	-0.052%	-0.032%
6000	4000	9984.73	0.301194	5441.91	9978.97	0.301194	5443.08	-0.058%	0.000%	0.021%
6000	5000	9980.83	0.300867	5442.83	9978.97	0.301194	5443.08	-0.019%	0.109%	0.005%
6000	6000	9976.23	0.301284	5439.83	9978.97	0.301194	5443.08	0.028%	-0.030%	0.060%
6000	7000	9985.65	0.301350	5442.63	9978.97	0.301194	5443.08	-0.067%	-0.052%	0.008%
6000	8000	9977.38	0.301247	5441.49	9978.97	0.301194	5443.08	0.016%	-0.018%	0.029%
6000	9000	9978.29	0.301443	5444.26	9978.97	0.301194	5443.08	0.007%	-0.083%	-0.022%
6000	10000	9976.47	0.300933	5439.76	9978.97	0.301194	5443.08	0.025%	0.087%	0.061%
7000	0	7293.20	0.246530	4253.12	7297.21	0.246597	4256.70	0.055%	0.027%	0.084%
7000	1000	7298.56	0.246488	4257.01	7297.21	0.246597	4256.70	-0.018%	0.044%	-0.007%
7000	2000	7297.00	0.246669	4257.72	7297.21	0.246597	4256.70	0.003%	-0.029%	-0.024%
7000	3000	7298.33	0.246880	4255.08	7297.21	0.246597	4256.70	-0.015%	-0.115%	0.038%
7000	4000	7295.60	0.246347	4254.06	7297.21	0.246597	4256.70	0.022%	0.101%	0.062%
7000	5000	7296.20	0.246760	4257.12	7297.21	0.246597	4256.70	0.014%	-0.066%	-0.010%
7000	6000	7295.83	0.246751	4256.17	7297.21	0.246597	4256.70	0.019%	-0.062%	0.013%
7000	7000	7297.36	0.246464	4256.03	7297.21	0.246597	4256.70	-0.002%	0.054%	0.016%
7000	8000	7292.28	0.246641	4254.38	7297.21	0.246597	4256.70	0.068%	-0.018%	0.055%
7000	9000	7299.53	0.246548	4256.35	7297.21	0.246597	4256.70	-0.032%	0.020%	0.008%
7000	10000	7297.47	0.246605	4257.46	7297.21	0.246597	4256.70	-0.004%	-0.003%	-0.018%
8000	0	5300.57	0.201915	3257.00	5299.09	0.201897	3260.98	-0.028%	-0.009%	0.122%
8000	1000	5304.05	0.201816	3260.26	5299.09	0.201897	3260.98	-0.094%	0.040%	0.022%
8000	2000	5299.36	0.202042	3264.41	5299.09	0.201897	3260.98	-0.005%	-0.072%	-0.105%
8000	3000	5298.77	0.201896	3261.42	5299.09	0.201897	3260.98	0.006%	0.000%	-0.014%
8000	4000	5298.19	0.201912	3260.72	5299.09	0.201897	3260.98	0.017%	-0.008%	0.008%
8000	5000	5301.89	0.201865	3259.62	5299.09	0.201897	3260.98	-0.053%	0.016%	0.042%
8000	6000	5298.29	0.201995	3261.30	5299.09	0.201897	3260.98	0.015%	-0.049%	-0.010%
8000	7000	5294.70	0.201829	3259.46	5299.09	0.201897	3260.98	0.083%	0.033%	0.047%
8000	8000	5307.03	0.201951	3260.23	5299.09	0.201897	3260.98	-0.150%	-0.027%	0.023%
8000	9000	5298.80	0.201828	3259.75	5299.09	0.201897	3260.98	0.005%	0.034%	0.038%
8000	10000	5297.73	0.201940	3262.21	5299.09	0.201897	3260.98	0.026%	-0.021%	-0.038%
9000	0	3821.88	0.165397	2458.53	3825.32	0.165299	2459.14	0.090%	-0.059%	0.024%
9000	1000	3824.07	0.165210	2460.75	3825.32	0.165299	2459.14	0.033%	0.054%	-0.066%
9000	2000	3828.63	0.165300	2461.19	3825.32	0.165299	2459.14	-0.086%	-0.001%	-0.083%
9000	3000	3827.14	0.165284	2458.33	3825.32	0.165299	2459.14	-0.048%	0.009%	0.033%
9000	4000	3825.61	0.165300	2461.25	3825.32	0.165299	2459.14	-0.007%	-0.001%	-0.086%
9000	5000	3820.51	0.165254	2459.44	3825.32	0.165299	2459.14	0.126%	0.027%	-0.012%
9000	6000	3829.83	0.165284	2460.88	3825.32	0.165299	2459.14	-0.118%	0.009%	-0.071%
9000	7000	3827.34	0.165300	2460.72	3825.32	0.165299	2459.14	-0.053%	-0.001%	-0.064%
9000	8000	3821.39	0.165254	2459.83	3825.32	0.165299	2459.14	0.103%	0.027%	-0.028%
9000	9000	3823.16	0.165345	2457.97	3825.32	0.165299	2459.14	0.057%	-0.028%	0.047%
9000	10000	3825.98	0.165284	2457.09	3825.32	0.165299	2459.14	-0.017%	0.009%	0.083%
10000	0	2750.49	0.135363	1832.41	2747.35	0.135335	1831.56	-0.114%	-0.020%	-0.046%
10000	1000	2748.85	0.135431	1831.23	2747.35	0.135335	1831.56	-0.055%	-0.070%	0.018%
10000	2000	2742.58	0.135334	1832.21	2747.35	0.135335	1831.56	0.174%	0.001%	-0.035%
10000	3000	2746.74	0.135367	1829.56	2747.35	0.135335	1831.56	0.022%	-0.024%	0.109%
10000	4000	2748.50	0.135264	1832.44	2747.35	0.135335	1831.56	-0.042%	0.053%	-0.048%
10000	5000	2743.25	0.135337	1833.26	2747.35	0.135335	1831.56	0.149%	-0.001%	-0.093%
10000	6000	2748.85	0.135345	1832.11	2747.35	0.135335	1831.56	-0.055%	-0.007%	-0.030%
10000	7000	2748.14	0.135349	1834.10	2747.35	0.135335	1831.56	-0.029%	-0.010%	-0.138%
10000	8000	2745.06	0.135242	1832.60	2747.35	0.135335	1831.56	0.083%	0.069%	-0.057%

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r	m_1	$Ef(r, m_1)$	$Eq(r, m_1)$	$Re(r, m_1)$	$Ef(r, \frac{m_1}{n})$	$Eq(r, \frac{m_1}{n})$	$Re(r, \frac{m_1}{n})$	Dif_{Ef}	Dif_{Eq}	Dif_{Re}
10000	9000	2747.76	0.135350	1832.87	2747.35	0.135335	1831.56	-0.015%	-0.011%	-0.071%
10000	10000	2746.79	0.135289	1830.36	2747.35	0.135335	1831.56	0.020%	0.034%	0.066%

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