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STABLE AND ENVY-FREE LOTTERY ALLOCATIONS FOR AFFORDABLE HOUSING

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ABSTRACT

Affordable housing lotteries often enforce a rule preventing duplicate lottery entries that makes the model in Hylland & Zeckhauser (1979) (HZ) inapplicable. We revisit HZ and propose a new individually stable (IS) allocation that can be achieved by a Tickets algorithm and accommodate the rule. A strictly envy-free (SEF) allocation is shown to be the unique IS and Pareto-optimal allocation, the outcome of the unique strong Nash equilibrium of a congestion game, and the unique Pseudo market equilibrium allocation in HZ. The algorithm always obtains the unique SEF allocation (if any) and fixes a designed flaw of existing lotteries.

Keywords: Affordable housing, lottery allocations, Tickets algorithm

JEL Classification Numbers: H42, C78, C72, D63

1. INTRODUCTION

Affordable housing is a major source of housing for low income families, the elderly, and disabled households–not only in the United States (U.S.)

All authors declare there are no conflicts of interest. Any errors are our own.

but also in countries like China and India. It may even be the only source of housing for these households in major cities like Hong Kong, Manhattan, and Boston. As such, it is both legally[1] and ethically necessary that the allocation of these units be both fair and equitable.

Lotteries are often, but not always, used as the method of allocation. In a typical lottery for affordable housing, a developer (e.g. New York City (NYC)) advertises to the public the units for rent or sale and the eligibility requirements in income and family sizes. Eligible applicants file their applications before a deadline. Applicants are then pooled and randomly selected by a lottery. In such a practice, the Housing Authority or developer often prohibits duplicate lottery entries in order to give each eligible applicant an equal access and an equal opportunity to access affordable housing. This rule has been widely used in practice. An example attached in the supplemental materials is Sachem's Path at Cape Cod in 2015, which used a lottery to sell 36 newly built single affordable houses with three different sizes of 1 BR, 2 BR and 3 BR. An applicant may qualify for different sizes but he can choose one and only one size in his application, i.e., an applicant holds one and only one lottery ticket in one pool. This restriction provides a constraint on feasible lottery allocations.

A lottery is an allocation mechanism that assigns each profile of preferences an ex ante allocation of lottery tickets. Whether a lottery is fair or not depends on the ex ante allocation it produces for each given profile. An agent is said to envy the other if he prefers that agent's lottery tickets to his own. An ex ante allocation is thus envy-free (EF) if no agent envies any others. A stronger version of this is that an ex ante allocation is strictly EF (SEF) if every agent strictly prefers his own ticket to any different ticket held by the other. The EF notion has been extensively studied in the economic literature for economies with divisible goods (Foley, 1967; Pazner & Schmeidler, 1974; Varian, 1974; Crawford, 1977), however the closely related SEF notion plays a new and important role for the economy in this paper.

Affordable housing in the U.S.[2] has deeply discounted fixed rents or selling prices.[3] There are often far more applicants than there are units available. The

[1] As stated in the Fair Housing Act of 1968 and the Affirmatively Furthering Fair Housing Rule of 2015, see http://portal.hud.gov/hudportal/HUD?src=/program_offices/fair_housing_equal_opp/progdesc/title8.

[2] Affordable housing in the U.S. includes Section 8 Housing, Income Based Housing, Low Income Rentals, Public Housing, Subsidized Housing, Low Income Apartments and so forth. More than 1.2 million households in the U.S. are living in affordable housing.

[3] A number of papers have studied economies where rents or selling prices are flexible. Yet they

question now raised is whether or not these lotteries indeed produce an optimal and fair allocation among all the households that participate. The following summarizes our common understanding of an affordable housing lottery:

> Affordable housing lotteries exist to ensure fair and equitable distribution of housing to eligible applicants. [...]. The workings of a lottery are very simple. All qualified applicants are pooled and chosen randomly.[4]

We find, on the contrary, that the above noted restriction used in common practice causes serious issues with fairness not noted before in both theory and practice. For example, the affordable housing lotteries in NYC and the City of Chongqing in China may generate allocations that are not EF and Pareto optimal (PO), even if such an allocation exists. The same design flaw exists for the school choice lotteries that assign students of all grades to the KIPP Houston Public Charter Schools in the State of Texas. Thus, the flaw is not limited to one specific lottery. See Section II for detail. In fact, it is not that hard to understand why a rule imposed on a mechanism may affect its outcome (Crawford, 1979, 1980)

This motivates our study in this paper about how to design an ex ante fair and efficient lottery for affordable housing that can accommodate this rule and resolve the design flaw in existing lotteries. To achieve that goal, we revisit the seminal model in Hylland & Zeckhauser (1979) (HZ) by adding a restriction such that each agent only holds one lottery ticket in one lottery pool in an ex ante lottery allocation. Because an EF allocation may not exist due to the rule, we define a weaker notion of envy-freeness called individual stability.

A lottery allocation is individually stable (IS) if no agent has any incentive to give up his existing lottery ticket in exchange for a better lottery ticket while all other agents still keep theirs. The IS notion is quite natural, slightly different from the EF notion in the sense that an agent will act to change her lot if she does envy another agent and such an action will make her better off. This IS notion makes sense because fairness of a lottery is based on an ex ante allocation, in

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are still controlled, falling into an upper and a lower bound. How to determine these controlled rents or prices and ex post allocations are the major issues addressed therein, see, e.g., Talman & Yang (2008); Zhu & Zhang (2011); Andersson & Svensson (2014); Andersson et al. (2015, 2016).

[4] Housing Assistance Corporation at Cape Cod in MA. See `http://www.haconcapecod.org/programs-and-services/homeownership-lottery`

which eligible agents voluntarily file or withdraw their applications before a deadline. In fact, the stability thus defined follows that of a stable matching in Gale & Shapley (1962), where their noted deferred acceptance algorithm is a procedure to achieve a stable matching. Their algorithm has an optimization process that allows those who receive proposals to improving their positions, which are only finalized at the moment when there are no more new proposals to improve their positions. We use this idea in the design of our "Tickets" algorithm with a rejection-acceptance optimization process.

Our algorithm works as follows. A builder provides a finite number of pools of lotteries in a room with a single door for entrance. Each application is seen as a lottery ticket from one pool. Self-interested applicants form a queue to enter the room one at a time to pick up the best lottery ticket from one pool–the best for him at the moment he enters the market. The first agent in the queue enters the room and picks up the lottery he likes the most. Clearly, the market reaches a stability for a single agent economy. The stability in the market may be broken after a new entry enters the room to pick up the best ticket she likes. If that is the case, the Tickets algorithm lets an applicant, who has been made worse off due to the new entry, return his existing ticket, go back to the door and then reenter the market as if he is a new entry; He reenters to pick up the ticket he like the best–a ticket he likes better than his returned ticket. This chain-reaction process can continue until the stability in the room is restored. After that, the next new applicant in the queue enters the room and the process continues until there is no new agent in the queue. We show that this process will end within finite steps and the finalized lottery is stable.

The proposed Tickets algorithm is similar to a dynamic rematching process that has been used by Roth & Vate (1990) and Ma (1994) to reach a stable matching for the two-sided marriage market. The rematching process there goes from one stability to the next with one agent entering the market to make a proposal to his or her best mate of the opposite sex. Such an algorithm is called procedurally fair in Klaus & Klijn (2006). In this paper, the IS notion is justified by the Nash equilibrium (NE) of a lottery congestion game along the line of Milchtaich (1996).

A lottery allocation is (ex ante) PO if there is no other allocation that makes at least one applicant better off and no applicant worse off. A lottery allocation is (ex ante) weakly PO if there is no other allocation that makes all applicants better off. We find that an EF lottery allocation may not be PO, as in Pazner & Schmeidler (1974); Varian (1974); Crawford (1977) for divisible goods.

However, there always exists an IS allocation that is weakly PO in our economy. Moreover, an SEF allocation is the unique one that is both IS and PO.

The uniqueness of the SEF allocation is somewhat a surprise. An even more interesting result is that this SEF allocation is also the unique Pseudo market equilibrium in HZ, with the equilibrium price of a house being equal to the reciprocal of the lottery drawing probability of that house under the SEF allocation. Moreover, the unique SEF allocation is also the unique strong NE (Aumann, 1959) outcome of the lottery congestion game. The existence of an SNE is a complicated matter. Even a potential game (e.g., a Prisoner's dilemma game) may not have an SNE. The existence of an SNE for an unweighted congestion game in Milchtaich (1996) is shown by Konishi et al. (1997), who provided a different framework than that in Milchtaich. It is known, however, from Voorneveld et al. (1999) that the two classes of games in Milchtaich (1996); Konishi et al. (1997) are equivalent. Milchtaich (1996) showed that his game may not have the finite improvement property (Monderer & Shapley, 1996) and even a best-reply improvement path or dynamics may also last infinitely; He showed the existence theorem by establishing the existence of a finite best-reply improvement path that ends up with a pure NE. Konishi et al. (1997) used a different strategy by showing that any NE that is not SNE is Pareto-dominated by another NE. Our paper provides a condition under which there is an unique SNE for their games in Milchtaich and Konishi et al. Our Tickets-TTC algorithm incorporates Gale's Top Trading Cycles (TTC) (Shapley & Scarf, 1974) in order to reach a stable allocation that is immune to coalitional deviations. The Tickets-TTC algorithm always ends with an SNE outcome.

Given so many appealing properties associated with the unique SEF allocation, one may wonder whether the NYC affordable housing lottery achieves it or not. We find out that even if an economy has the SEF allocation, the NYC lottery, like many others, fails to achieve it. This causes a major concern because it implies that the NYC lottery is neither EF nor PO and thus suffers from a design flaw.

2. RELATED LITERATURE

To the best of our knowledge, there is no paper in the literature that has addressed the question of whether an affordable housing lottery is fair or not from a theoretical aspect. There is a growing interest in theory about random allocations for the housing allocation model in Hylland & Zeckhauser

(1979). They studied a housing allocation economy with a finite number of indivisible houses and unit demand and showed that there always exists an ex ante EF and Pareto-optimal (PO) random allocation when agents have the von Neumann-Morgenstern (vNM) utility over houses. By means of a pseudo market where each agent is artificially assigned an equal budget, they showed that their EF and PO random allocation can be achieved by a market mechanism at a competitive equilibrium. The equilibrium allocation is a doubly stochastic matrix so that a centralized lottery can be found to implement it with a collection of deterministic allocations via the celebrated Birkhoff-von Neumann decomposition theorem. Bogomolnaia & Moulin (2001) (BM) studied a similar economy but with ordinal preferences and provided an existence theorem of EF and ordinal PO random allocations by designing a novel probabilistic serial mechanism using an "eating" algorithm, in which all agents start to eat houses at the same time at an equal rate in the order of their first, their second, etc., ranked house (of unit supply) until an agent's total eating shares sum up to at most 1. Katta & Sethuraman (2006) extended EF and ordinal PO results in BM to the case with weak preferences and found a conflict between EF and ordinal PO properties and (weak) strategy-proofness. Budish et al. (2013) (BCKM) offered several extensions of those results in HZ and BM to economies that allow multiple unit demands and two-sided matching. In particular, BCKM provided an extension of the Birkhoff-von Neumann theorem and found an important necessary and sufficient condition for the existence of a centralized lottery to implement a random allocation with a collection of deterministic ones. A salient feature of these random allocations in the literature is the fact that they all require multiple lottery entries and a broad domain of feasible deterministic allocations for their implementation. Thus, the results in HZ, BM and BCKM that apply to a broader domain are not applicable to affordable housing lotteries studied in this paper because an identified equilibrium in these papers may use allocations that are not feasible.

The Tickets algorithm is related to the serial dictatorship in Svensson (1994) and the random serial dictatorship (RSD) in Abdulkadiroğlu & Sönmez (1998). A main difference is the dynamic optimization process in the Tickets algorithm that RSD does not have. The main issue with RSD for affordable housing is that it is not ex ante efficient and does not produce an ex ante stable allocation, known from Zhou's impossibility theorem Zhou (1990). RSD is also different from the NYC lottery and the Tickets algorithm by the fact that a random allocation under RSD is due to a random drawing of orders of applicants, not a lottery drawing as in the NYC lottery or the Tickets algorithm.

There is a surging interest in the study of ex post efficient allocations for affordable housing. For example, Talman & Yang (2008) investigated a housing market with rents that are either fixed or limited to an interval. They proposed a dynamic auction that finds an allocation of houses and its supporting rents such that the allocation is ex post PO and lies in the core Andersson et al. (2015). Andersson & Svensson (2014, 2016) investigated a more general but closely related market and demonstrated the existence of an ex post efficient allocation achieved by a strategy-proof mechanism. For an additional study on an ex post allocation mechanism for the housing market, see Andersson et al. (2016). In our lottery model, rents or selling prices are fixed by the housing authority or the developer in the advertisement and are not subject to any change. Our aim and approach in this paper are totally different from theirs in that we focus on the ex ante allocations and investigate the issue whether an affordable housing lottery is ex ante fair and PO.

The rest of the paper is organized as follows. Section II discusses the NYC lottery in detail and its design flaw. We also include a brief introduction to some other lotteries. Section III presents the model and definitions. Section IV presents the main results. Section V discusses the Pseudo market and the random assignments in HZ. Section VI presents a strategic form lottery congestion game and makes a connection of an IS (SEF) allocation to a pure (strong) Nash equilibrium. Section VII concludes.

3. LOTTERIES IN PRACTICE

3.1. The NYC Lottery and Its Design Flaw

We use NYC as example to show how a lottery for affordable housing has been used to create a waitlist. Our focus is on the design of a lottery as a mechanism, not on a particular waitlist generated. Once there is a development available in NYC, the developer/market agent starts to advertise the availability of the development and the eligibility requirements of household income and family sizes. Rents for these apartments are fixed and heavily discounted. Discounted rents create excess demand. In NYC, there are about 1,000 applicants for every 1 available affordable rental apartment. A developer or market agent runs the lottery for that development, independently of the other, and the developer manages the waitlist after the lottery. After a waitlist is finalized, the developer allocates the apartments according to the list, a list of log numbers ranked from the low to the high. Because many applications that are opened in a lottery do not qualify and the developer has no obligation to fill all positions in the list ex post, the applications that are open in a lottery are at least 20 times the total number of available units. Such a long waitlist is helpful in protecting the developer's interest in an ex post allocation of the units and for the future to fill vacancies. This practice is not without a cost. Applicants who get a very high log number may easily get depressed. A high log number gives an applicant a hope but it may remain a hope indefinite time. See the following blog posted by **q41apartments** on [5]:

> I was recently awarded a middle income lottery and i wanted to share my entire experience form A to Z with everyone. I constantly see someone asking the same question everyday on this forum and i hope my post helps. This is to give anyone applying an insight to the process and what to expect. Now, I am going to be up front and let everyone know that it was a very hard and painful process. I cried many nights.

To model an ex ante lottery for a development in theory, we need to assume that all applications in the lottery are eligible for the units to which they have been assigned[6] and each application in the same pool has an equal right to be drawn.

A development in NYC may offer different unit sizes, called "communities" in this paper. Each size has a quota, the maximum number of available units of the same size. For example, a development may offer 50 1-bed and 30 2-bed apartments. Thus, we say that this developer offers two types of communities with 50 of the first size and 30 of the second size. Some applicant may qualify for both sizes but applies to one single desired size for which she is qualified.

[5] http://www.city-data.com/forum/new-york-city/2241280-how-housing-lottery-process-works-z.html

[6] In NYC, it is the officer who makes a single choice of the size of a single rental unit on behalf of an applicant by the end of each application form. Here we have to assume that the choice made by the officer is indeed in the best interest of the applicant. In particular, the officer knows the number of applications or applicants that have been filed for a unit. Application forms, online or by mail, are the same across all developments. For the sake of the reader, one application form is attached in our supplemental materials. Because no duplicate applications are allowed by the developer and the officer makes a single choice about the size of a rental unit, duplicate lottery entries are prohibited in the NYC lottery.

Some applicant may be just qualified for one size only. Under such a case, we assign zero utility to an applicant for those apartments for which she does not qualify.

Applicants are encouraged to register and apply for lotteries through the NYC Housing Connect[7]. An advantage of such a centralized application system is that an applicant can apply for multiple developments at the same time but each applicant must submit one application for each lottery, because each development must conduct its lottery independently. Once an applicant is awarded an apartment, his applications for others must be voided. Thus, the applications for lotteries in NYC have been centralized while each lottery is run by a market agent or developer in a decentralized manner. There are several important rules for the application of a lottery in NYC[8]:

> R1. Submit only one entry per household. You will be disqualified if more than one entry is received per lottery for your household. If you submit an entry online, you may NOT submit an entry via mail. If you submit an entry via mail, you may NOT submit an entry online.
> R2. Entries are selected randomly through a lottery. Depending on the volume received, it may not be possible for all entries to be included in the waiting list [in an ex post allocation process].
> R3. The entry must be submitted no later than the deadline indicated for each development.
> R4. Once the application has been submitted, it cannot be withdrawn. The process must be the same for both paper and online applications, and people are not able to withdraw a paper application once it has been mailed in.

The lottery is randomly drawn against the total pool of applicants for the available units of the development. Because an applicant has been assigned to a single unit by the officer, the probability of an applicant that has been drawn is the same as if each unit size has a pool of lottery tickets of a color. Assume that all drawn applicants are eligible for the units they were drawn and

[7] https://a806-housingconnect.nyc.gov/nyclottery/lottery.html. Paper applications in mail are still allowed, with one application for one lottery.

[8] See the application instructions from http://www1.nyc.gov/site/hpd/renters/housing-connect.page. We do not provide the instructions on the eligibility requirements, which are not relevant to our study.

they will choose ex post those units that were ex ante assigned to them. Under these cases, the extra number of drawn applicants beyond the quota in practice becomes irrelavent in theory.

The reason we need to design the Tickets algorithm is that the NYC lottery as a mechanism can generate an ex ante lottery that is not stable. Consider such an example in which there are two communities, 1 and 2, each of which has one single rental unit. There are six agents, $1, 2, \cdots, 6$. Agents 1 and 2 have the same vNM utilities $(2, 3)$. That is, both agents prefer an apartment in community two to an apartment in community one under ordinal preferences. Agents 3 to 6 have the same vNM utility $(0, 5)$. Consider the order $\sigma = (1, 2, 3, 4, 5, 6)$ of applications, where applicant 1 applies the first, applicant 2 applies the second, etc. Under the NYC lottery, with self-interested and myopically sincere agents[9], applicant 1 enters the market and chooses community two. When applicant 2 files his application, he does not choose community two because his expected utility by choosing two equals 1.5, smaller than 2. Note that by R4, applicants 1 and 2 will never change their choices no matter who comes later in the order. The ex ante lottery (ticket) allocation is given under the NYC lottery by $\mathcal{L}^\sigma$ that induces a random lottery allocation or probability distribution matrix $P(\mathcal{L}^\sigma)$ (the last column is the "null" object):

$$\mathcal{L}^\sigma = \begin{pmatrix} \mathcal{L}_1^\sigma \\ \mathcal{L}_2^\sigma \\ \mathcal{L}_3^\sigma \\ \mathcal{L}_4^\sigma \\ \mathcal{L}_5^\sigma \\ \mathcal{L}_6^\sigma \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad P(\mathcal{L}^\sigma) = \begin{pmatrix} 0 & \frac{1}{5} & \frac{4}{5} \\ 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & \frac{1}{5} & \frac{4}{5} \end{pmatrix}.$$

Thus, the total expected utility under $\mathcal{L}^\sigma$ equals $6\frac{3}{5}$. Applicant 1 has an incentive to withdraw his lottery in community 2 in exchange for a lottery in community 1 under $\mathcal{L}^\sigma$. It does not satisfy R4 voluntarily because it gives an agent the incentive to withdraw his lottery. In contrast, our Tickets algorithm

[9] This is the same assumption used in the noted deferred acceptance algorithm in Gale & Shapley (1962). We thank Vince Crawford for pointing out the importance of this assumption under the NYC lottery and the Tickets algorithm.

will provide a stable lottery allocation given by

$$\mathcal{L}' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P(\mathcal{L}') = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix},$$

which yields total expected utility of 7, $\frac{2}{5}$ higher than the lottery $\mathcal{L}^{\sigma}$. Under different orders σ, the NYC lottery may provide different allocations. On the other hand, our Tickets algorithm yields the same lottery $\mathcal{L}'$ for any order σ, out of 720 in total, and it is not Pareto dominated by any other lottery allocation because $\mathcal{L}'$ is SEF.

The fact that the NYC lottery fails to achieve the unique SEF allocation has a serious consequence. It means that the NYC lottery is neither EF nor PO. This problem has been caused by the rule R4, which is designed for convenience in the administration of the lottery with the mail-in applications. In our design of the Tickets algorithm, we remove R4 from the NYC lottery temporarily by adding an iterated optimization process to it so that the finalized lottery satisfies both R1 and R4 voluntarily. The main idea is to allow an applicant to reject herself in her existing pool and then accept herself to the other better pool temporarily, similar to those processes in Roth & Vate (1990) and Ma (1996).

Consider the above example without agents 3 to 6. Using the noted eating algorithm in BM and BCKM, we obtain the random assignment given by $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, under which each agent's assignment (row vector) involves two lottery entries. Because x is also the random assignment in HZ for prices $(0, 2)$ when each agent is assigned a unit budget, a lottery outcome identified in HZ for cardinal utilities may also need multiple lottery entries, a clear violation of R1.

Let us go back to the above example with the six agents. Now applying the eating algorithm to it again, assuming agents eat nothing for those houses with zero utility, we get the random assignment under the BM algorithm as follows:

$$P_{BM} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & \frac{1}{6} & \frac{5}{6} \end{pmatrix},$$

which can be implemented by a centralized lottery over six deterministic assignments (BCKM):

$$P_{BM} = \tfrac{1}{6}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \tfrac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \tfrac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \tfrac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \tfrac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \tfrac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Total expected utility under the random assignment P_{BM} equals $4\frac{5}{6}$, substantially less than under the random assignments $\mathcal{L}^{\sigma}$ and $\mathcal{L}'$. More importantly, any pure assignment in the above implementation that assigns an agent to the null object will give her the incentive to unilaterally deviate from such an assigned choice. Thus, such an assignment is not IS and cannot be at a Nash equilibrium of our lottery game defined in Section VI. Our Tickets algorithm successfully resolves this issue because its outcome is always at a Nash equilibrium for any order of applicants.

3.2. The Chongqing Lottery

In the City of Chongqing in China, public rental apartments are assigned to qualified applicants with a lottery managed by the City Public Rental Housing Authority[10]. Unlike NYC, the City of Chongqing asks each applicant to choose one and only one size of rental units in her application form. The lottery is randomly drawn against each pool of applicants of the same size. Because this lottery system, as in NYC, lacks the rejection-acceptance optimization process in the Tickets algorithm, it has the same design flaw of generating an assignment that is not EF and PO.

[10] See http://www.cqgzfglj.gov.cn/flfg/201102/t20110211_164260.html, written in Chinese.

3.3. The KIPP Houston School Lottery

The KIPP Houston Public Schools are public charter schools that are free and open to students of all grades in the State of Texas. Students are assigned to the KIPP Houston Public Charter Schools by a lottery system.[11] Each student fills a single application for one and only one school. If the number of applications for a school is less than its capacity, all applicants are admitted to that school. If the number of applications is more than the capacity of a school, a lottery is drawn to assign applicants to that school. Interestingly, the lottery system leaves room for an applicant to file a duplicate application, say, by a mistake. In such a case, the new application becomes effective and the old one will be removed. Thus, the KIPP Houston School Lottery allows an applicant to make a mistake by rejecting an old application and accepting a new one, a key feature in our Tickets algorithm. The major difference is that applicants in the Kipp Houston School Lottery do not know the number of applicants that have been applying for a school when they file a new application by an "error". Thus, under the KIPP Houston School Lottery, a new application is filed based on the schools not on the schools and the number of filed applications, as in the Tickets algorithm. Thus, the Kipp Houston School Lottery has the same design defect as the NYC lottery. Nonetheless, the Kipp Houston School Lottery can be implemented similarly to our Tickets algorithm if it encourages applicants to file new applications, one at a time, to replace their old application, and the number of applicants for a school at any moment of time in the process is provided to the public.

4. MODEL AND DEFINITIONS

Because each development runs a lottery independently of the others, we model each development as an economy with indivisible objects and unit demand, as in HZ. Let $C = \{1, 2, \cdots, m\}$ denote the set of communities (i.e., objects or sizes) and community $j \in C$ has q_j number of idential apartments, where $q_j \geq 1$ is a finite integer. Thus, community j is indexed with the number $j = 1, 2, \cdots, m$. Let $q = (q_1, q_2, \cdots, q_m)$. For example, if a development provides 3 units of 1-bedroom and 5 units of 2-bedroom rental apartments, then $C = \{1, 2\}$ and $q = (3, 5)$. Let $A = \{1, 2, \cdots, n\}$ denote the set of agents

[11] See http://kipphouston.org/lottery-faqs for a detailed explanation of the lottery system.

or applicants and u_{ij} denote the vNM utility of agent $i \in A$ for renting an apartment in a community $j \in C$. Let $u_i = (u_{ij})$ be a given vNM utility vector for agent $i \in A$ such that $u_{ij} \geq 0$ for all $j \in C$ and $u_{ij} > 0$ for at least one $j \in C$. Let $u = (u_1, u_2, \cdots, u_n)'$ be the utility matrix. A lottery allocation problem for affordable housing is denoted (A, C, q, u).

The total capacity in the economy equals $\sum_{j=1}^{m} q_j$, which can vary from a few to several hundreds in NYC. We consider an economy where n is much larger than the total capacity. For examples, in 2014, there were 58,832 lottery applications for 105 affordable units in a new building in Greenpoint, Brooklyn; 98 rental apartments in the Sugar Hill development in Upper Manhattan attracted more than 48,000 applicants.[12] Note that it will not be easy to use a Pseudo market in HZ to find an equilibrium outcome for such a large scale economy.

A lottery assignment or allocation is an $n \times m$ matrix x such that $x_{ij} = 1$ if agent i is assigned an apartment in community j and $x_{ij} = 0$ otherwise. The set of feasible lottery assignments is defined by $X = \{x | \sum_{j \in C} x_{ij} = 1, \forall i \in A\}$, because each agent can only rent one apartment ex post and no agent can submit duplicate lottery entries ex ante by the rule. Thus, x_i is a vector with one j such that $x_{ij} = 1$ for $x \in X$. That is, each community has a pool of lottery tickets and an agent holds one lottery ticket in one single pool, since each qualified or eligible agent must have an equal access to affordable housing. By the equal opportunity, we assume that each lottery ticket has an equal chance to be drawn against the quota each community has. We also use x_i for the component j such that $x_{ij} = 1$ for notational convenience. This will not cause any confusion from the context. It is this feasibility constraint on X that makes the Pseudo market in HZ in the general domain and the eating algorithm in BM and BCKM inapplicable to the problem (A, C, q, u). Note that x itself is not a random allocation. There is no restriction on the number of applicants for a community $j \in C$, i.e., $0 \leq \sum_{i \in A} x_{ij} \leq n$. Let

$$n_j(x) = |\{i \in A | x_{ij} = 1\}| \tag{4.1}$$

denote the number of agents who have been assigned to community j under $x \in X$. If we consider each community to have a lottery pool, $n_j(x)$ is the number of lottery tickets in the pool of community j under allocation x. The

[12] Mireya Navarrojan: "Long Lines, and Odds, for New York's Subsidized Housing Lotteries," New York Times, Jan. 29, 2015.

probability that an agent i is allocated with an apartment in community j under a feasible lottery assignment $x \in X$ is given by $z_j(x)$, where

$$z_j(x) = min\{1, \frac{q_j}{n_j(x)}\}. \tag{4.2}$$

An agent i's expected utility $EU_i(x)$ under $x \in X$ depends on the apartment in community j assigned to i and the total number of agents who have been assigned to community j, where $EU_i(x)$ is defined by

$$EU_i(x) = \sum_{j \in C} x_{ij} u_{ij} z_j(x). \tag{4.3}$$

Equivalently, $EU_i(x) = u_{ij} z_j(x)$ for j such that $x_{ij} = 1$ or $x_i = j$.

Remark. In HZ, the Pseudo market operates with vNM utility functions. This assumption of vNM utility functions in this paper is for expository convenience. Our main results still hold even if the expected utility EUs are not the vNM utility functions. For example, the utility functions may be defined by $EU_i(x) = u_{ij} f_i(z_j(x))$ for $x_{ij} = 1$ and $EU_i(x) = 0$ otherwise, where f_i is monotonically nonincreasing or nondecreasing in $z_j(x)$ for j such that $x_{ij} = 1$.

Definition 1. A feasible allocation $x \in X$ is envy-free (EF) if for any agent i, community $l \in C$, it holds that $EU_i(x) \geq u_{il} z_l(x)$.

A feasible allocation is EF if no agent prefers others' to his own lottery assignment (Foley, 1967; Pazner & Schmeidler, 1974; Varian, 1974; Crawford, 1977). Note that others' assignments may include those apartments that are not assigned to any agent because public rental units are initially owned by the public, not by the agents or applicants.

Example 1. An EF allocation may not exist. Let $A = \{1, 2\}$, $C = \{1, 2\}$, $(q_1, q_2) = (1, 1)$, and $u_1 = u_2 = (2, 3)$. Consider all four feasible allocations in X:

$$x1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \; x2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; x3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \text{and } x4 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Under allocation $x1$, agents 1 and 2 each hold a ticket in community one and no one holds a ticket in community two. Thus, allocation $x1$ induces a probability matrix $P(x1) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$, where the last column is for the "null" object.

Thus, the expected utility under $x1$ equals $EU(x1) = (EU_1, EU_2)(x1) = (1, 1)$. Similarly, the expected utilities under allocations $x2$ and $x3$ are given by $EU(x2) = (2, 3)$ and $EU(x3) = (3, 2)$, respectively. The expected utility under $x4$ is given by $EU(x4) = (1.5, 1.5)$. Clearly, $x2$ and $x3$ are not EF. Under $x1$, agent 1 or 2 prefers the unassigned apartment in C_2 to the assigned lottery over the apartment in C_1. Under $x4$, agent 1 or 2 prefers the unassigned apartment in C_1 to the assigned lottery for the apartment in C_2. So, no allocation in X is EF.

5. MAIN RESULTS

Example 1 has shown that an EF allocation may not exist for (A,C,q,u). Our resolution to this problem is the definition of an individually stable (IS) allocation. We provide an algorithm whose outcome is an IS allocation. However, an EF or IS allocation may not be PO. To find a PO allocation, we define a strictly envy-free (SEF) allocation and discuss a number of promising properties associated with it.

5.1. Tickets Algorithm and Stable Lottery Allocations

Let (x_{-i}, k) denote a feasible allocation in X such that only agent i's assignment j under x, i.e., $x_{ij} = 1$, is replaced by $k \in C$, $k \neq j$, and everyone else's assignment remains the same.

Definition 2. A feasible allocation $x \in X$ is an individually stable (IS) allocation if $EU_i(x) \geq EU_i((x_{-i}, k))$ for all $i \in A$ and $k \in C$.

The IS notion is quite natural, slightly different from the EF notion in the sense that an agent will act if they envy another agent, if such an action makes them better off. Thus, a lottery allocation is IS if no agent has any such incentive to move to another lottery lot. It is a notion that follows that of a stable matching in Gale and Shapley. It is also related to the notion of a Nash equilibrium, but the two are not the same because there are no strategies or games defined with the IS notion.

Theorem 1. *For any affordable housing economy (A, C, q, u), there always exists an IS lottery allocation.*

The proof is in Appendix IX.B. We prove the theorem by designing an algorithm. Now imagine that each community has a unique ticket machine j. Each machine j, $j = 1, 2, \cdots, m$, prints n identical lottery tickets. Each ticket gives its holder an equal right to be drawn to rent an apartment in that community. Different machines print different tickets so that no two machines print the same tickets, using distinctive colors or other means. We can put all the ticket machines in a room with a single entrance. Agents or applicants form an arbitrary random queue or order to enter the room to get tickets, one ticket for one agent.

Tickets Algorithm. Without loss of generality, let $(1,\ 2,\ \cdots,\ n)$ be the order of agents. That is, agent 1 is the first in the queue to enter the room to get a ticket. We now specify in steps where each agent gets his ticket. The algorithm operates similarly for other orders of agents. Note that agents are self-interested and myopically sincere, as in the deferred acceptance algorithm. The lottery allocation part of the algorithm consists of Step 1 to Step n:

Step 1. Let agent 1 enter the room first and choose a lottery ticket she likes the most. In case there are ties, agent 1 may use any tie-breaker. Let x^1 denote the allocation by the end of this step.

Step 2. Let agent 2 enter the room to choose the lottery ticket he likes the most. If agent 2 chooses a different lottery ticket from agent 1's, then the process moves to the next step. If 2's choice is the same as agent 1's, his choice reduces 1's expected utility. Thus, agent 1 may like some other lottery tickets better than what she held in Step 1. If that is the case, let agent 1 return her ticket and go out of the door, and then reenter, before any other agent in the order, to re-choose a lottery ticket mostly preferred by agent 1. The process moves to the next step. Agents 1 and 2 may use any tie-breaker in case there are ties. Let x^2 denote the allocation by the end of this step.

Step i. Let agent i enter the room to choose the lottery ticket he likes the most. The choice made by agent i reduces the expected utility for those who have been in the room and held the same lottery tickets as agent i does. Some of those agents may want to return their tickets in exchange for some other tickets they like better. If there are such agents, then let one and only one agent h among them return his lottery ticket and go out of the door, and then reenter the room, to re-choose a lottery ticket mostly preferred by agent h. Repeat this process by treating agent h as agent i, until no agent in the room wants to return his ticket. Let x^i denote the allocation by the end of this step.

Repeat **Step i** above until all agents in the order are exhausted.

This procedure generates a sequence of allocations $x^1, x^2, \cdots, x^i, \cdots, x^n$, each of which is IS with respect to the agents in the room. Note this sequence of allocations is order-dependent because different orders may have different sequences.

NYC Lottery. In the NYC lottery, each applicant enters the room to pick up a lottery ticket that she likes the most and holds her lottery ticket to the close. That is, the NYC lottery is equivalent to the serial dictatorship for lottery tickets, not for the actual apartments. Then these lottery tickets are drawn to generate a waitlist in an ex post allocation. Because different drawings create different waitlists, the NYC lottery is not quite the same as RSD.

The Tickets algorithm incorporates a rejection-acceptance optimization process into the NYC lottery. A main change comes from those steps after Step 1. We have to make the change because the NYC lottery is not stable so that R4 cannot hold voluntarily, as shown in Section II.A.

A key observation in our algorithm is that any agent in the room who rejects and accepts herself due to the new entry only does that once in order to restore the stability in the market. Thus, the process of the adjustment must be finite. It is critical to let just one agent from the order enter the market each time. If one lets two or more agents enter the market at the same time, there is no guarantee the process will converge. In fact, if two agents are allowed to exchange their lots at the same time, it is easy to construct an example where the two agents can form a cycle by moving back and forth between two communities, no convergence to stability.

Next we use an example to illustrate how the Tickets algorithm reaches a stable lottery allocation for every order of applicants and point out some general properties that will be studied in the next sections.

Example 2. Let $A = (1, 2, 3, 4)$, $C = (1, 2, 3)$ and $q = (q_1, q_2, q_3) = (1, 1, 1)$.

Table 1: Tickets Algorithm for Example 2

Step \ Order	(1,2,3,4)	(1,2,4,3)	(1,3,2,4)
1	$(C_2, , ,)$	$(C_2, , ,)$	$(C_2, , ,)$
2	$(C_2 , C_2, ,)$	$(C_2, C_2, ,)$	$(C_2, , C_2,)$
Agent 1 re-assigned	$(C_3, C_2, ,)$	$(C_3, C_2, ,)$	$(C_3, , C_2,)$
3	$(C_3, C_2, C_2,)$	$(C_3, C_2, , C_1)$	$(C_3, C_2, C_2,)$
4	(C_3, C_2, C_2, C_1)	(C_3, C_2, C_2, C_1)	(C_3, C_2, C_2, C_1)

The utility matrix is given by

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}.$$

There are 24 orders in total under the Tickets algorithm. We apply the Tickets algorithm to three orders to illustrate the algorithm here. We use C_j for community j, to avoid confusion.

Consider three orders $(1,2,3,4)$, $(1,2,4,3)$ and $(1,3,2,4)$, where agent 1 enters the room first (see Table 1) and chooses C_2 because C_2 is agent 1's best choice in **Step 1**. The allocation is simply $x^1 = (C_2, , ,)$, where an empty component indicates an agent is assigned with no apartment.

Step 2. Now C_2 is the community that agents 2 and 3 prefer the most. Therefore, they choose C_2 after they enter. After agents 2 and 3 enter, their choices reduce the expected utility for agent 1, who can find a better lottery by returning the existing one. Therefore, agent 1 returns her ticket and goes out of the door and reenters to choose C_3.[13] $x^2 = (C_3, C_2, ,)$ is the allocation in Table 2 with the order $(1,2,3,4)$ because no agent in $\{1,2\}$ would like to return her ticket. Similarly, we get x^2 for the other two orders.

Step 3. Now a new agent from the order enters the room. With the order $(1,2,3,4)$, agent 3 enters the room and chooses C_2. Because no agent in $\{1,2,3\}$ would like to return her ticket after agent 3 took a ticket from C_2, the algorithm at Step 3 ends with $x^3 = (C_3, C_2, C_2,)$. Similarly, we get x^3 for the other two orders.

[13] This is the key difference between the Tickets algorithm and the serial dictatorship or the NYC lottery.

Step 4. When agent 4 enters the room under the order $(1, 2, 3, 4)$, she chooses C_1. Because no agent in the room would like to return her ticket after agent 4 took a ticket from C_1, the algorithm at Step 4 ends with $x^4 = (C_3, C_2, C_2, C_1)$. Because there are no more agents in the order, this ends the algorithm with x^4.

There are several interesting properties that should be noted in this example. It has two IS allocations:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad and \quad x^* = \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The expected utilities are given by $EU(x) = (2, 1, 1, 2)$ and $EU(x^*) = (3, 1, 1, 3)$. Thus, x^* Pareto dominates x. Under all three orders, the final allocation achieved by the Tickets algorithm is the allocation x^*, not the less efficient x. We show in Theorem 6 that this is a general result. The Tickets algorithm will always reach this more efficient allocation x^*, if any, over all other less efficient IS allocations. Moreover, this conclusion does not depend on the orders (see Appendix IX.F for the rest orders).

Under the NYC lottery with the application order $(1, 2, 3, 4)$, applicants 1, 2 and 3 get their tickets in C_2 and applicant 4 gets his from C_1. Clearly, such an allocation leaves C_3 unfilled and is not stable.

5.2. IS and Weakly PO Allocations

We start with the definition of coalitionally stable lottery allocation.

Definition 3. An allocation $x \in X$ is coalitionally stable (CS) if there is no other allocation $x' \in X$ that satisfies for any agent $i \in A' = \{a | x_a \neq x'_a\}$, inequality $EU_i(x') > EU_i(x)$ holds.

An allocation that is CS must be IS but the converse is not right. If an allocation is not coalitionally stable, then a coalition of agents may jointly withdraw their lottery tickets in exchange for other different lottery tickets so that everyone in the coalition prefers their new lottery to the existing one under the allocation.

Definition 4. A feasible allocation $x \in X$ is weakly Pareto optimal if there is no other feasible allocation $y \in X$ such that $EU_i(y) > EU_i(x)$ for all agents $i \in A$.

A feasible allocation is weakly PO if there is no other feasible allocation that gives each agent a higher expected utility. Note that an IS allocation x may not be weakly PO but every CS allocation is weakly PO.

Thus, an interesting question is whether there always exists an IS and weakly PO allocation in the economy (A, C, q, u). The answer is not obvious. We need to construct a lottery housing market to trade lottery "houses". For example, it is possible to use TTC to achieve x^* from x in Example 2 (see Theorem 5 for a general result associated with SEF). Starting with the IS allocation x, we can construct a lottery housing market in which a top trading cycle between agents 1 and 4 is formed to obtain the unique SEF x^*. What happens if an economy does not have an SEF allocation? The next result answers this question.

Since $u_{ij} > 0$ for at least one j, this implies that each agent must have a lottery ticket $j \in C$ under an IS allocation x. Given an IS allocation x, we now construct a housing market to find the strict core allocation as follows. Each lottery ticket is seen as a "house". Let the lottery allocation x be the initial endowment in the lottery housing market. Because there are m communities, there are just m different houses. Some ties may exist.[14] An agent ranks n lottery houses according to his expected utilities $u_{ij} z_j(x)$ of those lottery tickets of j type, $j = 1, 2, \cdots, m$. In case there is a tie between his initial endowed lottery house and any other lottery houses of the same kind, he always prefers his own lottery house to others' lottery houses and uses an arbitrary tie-breaker to break other ties, if any. In case there is a tie between his initial endowed lottery house and any other lottery houses that are not the same type, he prefers others' lottery houses to his initial endowed lottery house and uses an arbitrary tie-breaker to break other ties, if any. This completes our construction of a lottery housing market. Then we can apply TTC to this lottery housing market to get the final lottery allocation, denoted x^*. We can use the TTC algorithm to find the allocation that is IS and in the strict core for the lottery housing market with the initial endowment x. The algorithm that uses the Tickets algorithm first and then the TTC algorithm is called the Tickets-TTC algorithm.

Theorem 2. *For any lottery allocation problem (A,C,q,u), the allocation* x^*

[14] How to break ties in a housing market turns out to be critical for TTC, see, e.g., Saban & Sethuraman (2013) and references therein.

that is obtained from the Tickets-TTC algorithm is CS, i.e., an IS and weakly PO allocation always exists.

Proof. Assume, on the contrary, that x^* is not coalitionally stable. That is, there is an allocation, denoted x', such that there is a subset of agents $A' = \{a | x_a \neq x'_a\}$, which is clearly nonempty, such that $x_i \neq x'_i$ and the inequality $EU_i(x') > EU_i(x^*)$ holds for all $i \in A'$.

We consider two cases: (a) for all $j \in C$, $n_j(x^*) = n_j(x')$; (b) there is a community $j \in C$ such that $n_j(x') > n_j(x^*)$.

We prove Case (a) first. By the assumption $n_j(x^*) = n_j(x')$ for all $j \in C$, this implies that x' is obtained from x^* by forming some trading cycles. But the fact that all agents in A' strictly prefer x' to x^* contradicts to the use of the TTC algorithm. This completes the proof of Case (a).

Next, we prove Case (b). Because there is a community j such that $n_j(x') > n_j(x^*)$, there must be at least one agent, denoted i', satisfying 1). $i' \in A'$ and 2). $x'_{i'j} = 1$.

By the assumption $n_j(x') > n_j(x^*)$, we have

$$n_j(x') \geq n_j(x^*_{-i'}, j), \tag{5.1}$$

which implies

$$EU_{i'}((x^*_{-i'}, j)) \geq EU_{i'}(x') \tag{5.2}$$

By Theorem 1, we know that x^* is IS. Thus,

$$EU_{i'}(x^*) \geq EU_{i'}((x^*_{-i'}, j)) \tag{5.3}$$

Thus, by inequalities (5.2) and (5.3), we have

$$EU_{i'}(x^*) \geq EU_{i'}(x'), \tag{5.4}$$

which is a contradiction to the assumption agent i' is in A'. This completes the proof. □

Our next example answers the question whether all CS allocations can be obtained by the Tickets-TTC algorithm. The answer is negative. Surprisingly the Tickets-TTC algorithm is selective among all CS allocations by choosing the ones that are the most efficient.

Example 3. A CS allocation may not be obtained by the Tickets-TTC algorithm. Let an affordable housing economy be given by $A = \{1, 2, 3, 4\}$, $C = \{1, 2\}$, $u_1 = u_2 = u_3 = (12, 7)$, and $u_4 = (5, 2)$. Each community has a unit capacity, i.e., $q = (1, 1)$.

There are four CS allocations:

$$x1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, x2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, x3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } x4 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The expected utility vectors for allocations from $x1$ to $x3$ are, respectively, $(4, 4, 7, \frac{5}{3})$, $(4, 7, 4, \frac{5}{3})$ and $(7, 4, 4, \frac{5}{3})$, with the same total expected utility of $14 + \frac{5}{3}$. The expected utility vector for allocation $x4$ is $(4, 4, 4, 2)$, with the total expected utility of 14, less efficient than others. To achieve $x4$, we need agent 4 in community C_2. But we find out that the Tickets-TTC algorithm can only achieve the three more efficient CS allocations from $x1$ to $x3$ but cannot achieve $x4$, even though $x4$ is CS.

Table 2: Outcomes of the Tickets-TTC Algorithm with All Orders

Agents' Orders	*Agents in Community C_1*	*Agents in Community C_2*
$(1, 2, 3, 4)$; $(1, 2, 4, 3)$; $(1, 4, 2, 3)$; $(3, 2, 1, 4)$ $(3, 2, 4, 1)$; $(3, 4, 2, 1)$; $(4, 2, 3, 1)$; $(4, 2, 1, 3)$	1, 3, 4	2
$(1, 3, 2, 4)$; $(1, 3, 4, 2)$; $(1, 4, 3, 2)$; $(2, 3, 1, 4)$ $(2, 4, 3, 1)$; $(3, 2, 4, 1)$; $(4, 3, 2, 1)$; $(4, 3, 1, 1)$	1, 2, 4	3
$(2, 1, 3, 4)$; $(2, 1, 4, 3)$; $(2, 4, 1, 3)$; $(3, 1, 2, 4)$ $(3, 1, 4, 2)$; $(3, 4, 1, 2)$; $(4, 1, 3, 2)$; $(4, 1, 2, 3)$	2, 3, 4	1

Table 2 provides the outcomes of the Tickets-TTC algorithm for all possible orders. As one can see from Table 2, agent 4 is not allocated to community C_2, no matter which order has been used. It is worthwhile noting that under the RSD algorithm, agent 4 will be assigned to community C_2 as long as she is ranked the second in an order. This example shows that there is a substantial difference between RSD and the Tickets-TTC algorithm.

5.3. Strictly Envy-free Allocations

Definition 5. A feasible allocation $x \in X$ is (ex ante) Pareto optimal (PO) if there exists no other feasible allocation $x' \in X$ such that $EU_i(x') \geq EU_i(x)$ for all $i \in A$ and the strict inequality holds for at least one $i \in A$.

If a feasible allocation x is PO, then there does not exist any feasible allocation y that makes at least one agent better off without making the rest worse off. It is easy to have an example such that an EF allocation is not PO.

The following definition of a strictly envy-free (SEF) allocation is just slightly stronger than the EF allocation, i.e., inequality is strict for every agent in A and any community $l \neq j$ such that $x_{ij} = 1$ in the EF definition. This definition is equivalent to an EF allocation in an economy with strict ordinal preferences. As in the housing market in Shapley & Scarf (1974) where strict preferences are quite important to the model, the strict inequality in the definition of an SEF allocation has an enormous consequence for the problem (A, C, q, u). Recall our notation of x_i that also denotes the community j such that $x_{ij} = 1$.

Definition 6. A feasible allocation $x \in X$ is SEF if $EU_i(x) > u_{il} z_l(x)$ for all agents $i \in A$ and communities $l \in C$ such that $l \neq x_i$.

Theorem 3. *Every SEF allocation $x^* \in X$ is Pareto optimal.*

Proof. Let x^* be an SEF allocation. Assume, on the contrary, that x^* is not Pareto optimal. Thus, there must be a feasible allocation $x' \in X$ such that x' Pareto dominats x^*, denoted by $x'Px^*$. We consider two cases: (a) for all $j \in C$, $n_j(x^*) = n_j(x')$; and (b) there is a community j such that $n_j(x') > n_j(x^*)$.

We prove Case (a) first. Because x^* is SEF, we have $EU_i(x^*) > u_{ij} z_j(x^*)$ for all i such that $j \neq x_i^*$ and $EU_i(x^*) \geq u_{ij} z_j(x^*)$ for all i and j. By the assumption that $n_j(x^*) = n_j(x')$ for all j, we have that for all j, $z_j(x^*) = z_j(x')$. Thus, we obtain

$$EU_i(x') = u_{ix'_i} z_{x'_i}(x') = u_{ix'_i} z_{x'_i}(x^*) \leq EU_i(x^*)$$

for all i, which is a contradiction to the assumption $x'Px^*$. This completes the proof of Case (a).

Next we prove Case (b): $\exists j$ such that $n_j(x') > n_j(x^*)$. This implies that there must be at least one agent, denoted i^*, such that $x'_{i^*} = j$ and $x^*_{i^*} \neq j$. Moreover, the fact that $n_j(x') > n_j(x^*)$ implies $z_j(x') \leq z_j(x^*)$. Because x^*

is SEF, we have that for all i such that $j \neq x_i^*$, $EU_i(x^*) > u_{ij}z_j(x^*)$. Thus, it follows $EU_{i^*}(x^*) > u_{i^*j}z_j(x^*)$ and then

$$EU_{i^*}(x^*) > u_{i^*j}z_j(x^*) \geq u_{i^*j}z_j(x') = EU_{i^*}(x'),$$

which is a contradiction to the assumption $x'Px^*$. This completes the proof of Case (b) and the proof of the theorem. □

It is clear that an SEF allocation is also IS. The following result shows that an SEF allocation is the unique allocation that is both IS and PO. The uniqueness is somehow a surprise, given the fact that there are so many PO allocations. The proof of this result is given in Appendix IX.C.

Theorem 4. *An SEF allocation $x^* \in X$ is the unique allocation that is both individually stable and Pareto optimal.*

Example 4. An IS allocation may not be SEF or PO. Consider Example 1 with $u_1 = (2,3)$ and $u_2 = (3,2)$. Consider two allocations $x' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $x^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The expected utilities of the two allocations are given by $EU(x') = (2,2)$ and $EU(x^*) = (3,3)$. Note that both x' and x^* are IS. But x^* is the only SEF allocation. Also note that x^*Px'. So an IS allocation may be neither PO nor weakly PO.

Even though an IS allocation may not be SEF, starting with an IS allocation, we can always achieve the unique SEF allocation, if any exists, using the noted TTC.[15]

Theorem 5. *Starting with an IS allocation x that is not SEF, one can get the SEF allocation x^*, if any, by using the Tickets-TTC algorithm.*

Proof. Follow from Theorems 2 and 4. If the allocation x^* is not SEF, then x^* is not in the strict core of the housing market constructed, a contraction to the result in Roth & Postlewaite (1977) that shows that the strict core is a singleton consisting of x^*. □

[15] For another use of TTC, see Pápai (2000). Sen (2008) also provided a detailed discussion of the TTC algorithm.

A question of interest is how to know when the SEF allocation exists or does not exist for a given problem (A, C, q, u). We find that our Tickets algorithm can answer this question. There may exist many IS allocations, but there exists at most one SEF allocation. If an SEF allocation does not exist, our algorithm reaches an IS allocation. If an SEF allocation exists, then our algorithm only achieves that SEF allocation, no matter how many IS allocations are there, without using TTC. This result in Theorem 6 does not depend on the random orders under which agents enter the room. Thus, Theorem 6 is important because we can apply the Tickets algorithm to any single order. In practice such an order is naturally formed in an application process. Recall x^n denotes the allocation at the close of the Tickets algorithm.

Theorem 6. *For any lottery allocation problem* (A, C, q, u)*, an SEF allocation exists if and only if* x^n *is SEF.*

The proof is in Appendix IX.D. We have used Example 2 to illustrate how the Tickets algorithm has reached this more efficient SEF lottery allocation, if any, over those that are less efficient (see Appendix IX.F). Let us use a metaphor to illustrate the "peculiar" feature in Theorem 6. Imagine there are millions of fish in red (IS) or blue (PO) colors. There is one and only one fish in a purple color (SEF). For some reason we don't know, our Tickets algorithm, as a kind of "net", can catch the only purple fish, no matter which order has been used and how many fish are there in red or blue colors.

6. HYLLAND AND ZECKHAUSER PSEUDO MARKET

Hylland & Zeckhauser (1979)(HZ) studied an economy with a finite number of indivisible houses (or jobs) and agents with each agent consuming one house (or assigned to one job). In their original model each house may have multiple copies of the same kind with the total number of houses in the economy being equal to the number of agents so that each agent is assigned to at least one house (one job). Specifically, HZ studied a housing allocation problem (A, C, q, u), where $A = \{1, 2, \cdots, n\}$ is the set of agents, $C = \{1, 2, \cdots, m\}$ is the set of houses with a capacity $q = (q_1, q_2, \cdots, q_m)$ such that $q_j \geq 1$ for all $j \in C$ and $\sum_{j=1}^{m} q_j = n$. $u = (u_1, u_2, \cdots, u_n)'$ denotes the vNM utility matrix with $u_i = (u_{ij})$ as agent i's utility vector over house $j \in C$ for $i \in A$. A pure assignment μ for the economy is a matching $\mu : A \to C$ that assigns an agent a house, each copy of which is assigned to at most one agent. Each matching μ

can be represented by a matrix $[p_{ij}]$, each of whose components equals either 0 or 1, satisfying $\sum_{j=1}^{m} p_{ij} = 1$ and $\sum_{i=1}^{n} p_{ij} = q_j$ for all $i \in A$ and $j \in C$. If $q_j = 1$ for all $j \in C$, $[p_{ij}]$ becomes a permutation matrix.

HZ introduced a Pseudo market with a rent system $(h_1, h_2, \cdots, h_m)$ over the m houses in C. In their market each agent $i \in A$ is assigned an equal budget $B_i > 0$ of an artificial money and chooses a probability distribution or lottery over all houses $(p_{i1}, p_{i2}, \cdots, p_{im})$ to maximize his expected utility $\sum_{j=1}^{m} p_{ij} u_{ij}$ subject to his budget constraint $\sum_{j=1}^{m} p_{ij} h_j \leq B_i$. They showed that there always exists a rent system $(h_1^*, h_2^*, \cdots, h_m^*)$ and a lottery allocation $[p_{ij}^*]$ that form a competitive equilibrium. That is, the lottery $(p_{i1}^*, p_{i2}^*, \cdots, p_{im}^*)$ maximizes the expected utility for each agent i among all lotteries $(p_{i1}, p_{i2}, \cdots, p_{im})$ that satisfy agent i's budget constraint and the market clearing condition (by the property of probability matrix $[p_{ij}^*]$). Note that entries in $[p_{ij}^*]$ are shares, not necessarily zero or 1. The random allocation or assignment $[p_{ij}^*]$ is an ex-ante EF and PO. Most importantly, the competitive equilibrium random assignment $[p_{ij}^*]$ can be implemented by a centralized draw of a lottery over all pure assignments. To see how this may be done, we may assume that $q_j = 1$ for all $j \in C$ so that $n = m$. A competitive equilibrium random assignment $[p_{ij}^*]$ is doubly stochastic, i.e., a matrix each of whose rows and columns sums to 1. The noted Birkhoff-von Neumann decomposition theorem shows that a doubly stochastic matrix can be represented by a convex combination (i.e., a centralized lottery) of permutation matrices. This result can be extended to the more general case where each house has multiple identical copies because each identical copy may be considered as a house with a unit capacity. Note that identical houses must have identical prices at a competitive equilibrium.

HZ's model does not apply to affordable housing directly. There are three obstacles for applying it to affordable housing. First, the capacity in the economy will be much smaller than the number of agents in the economy for affordable housing. Second, the random assignment at an equilibrium in HZ may not satisfy the rule–no duplicate lottery entries, which means that not all IS allocations in our model can be realized by a competitive equilibrium in a Pseudo market in HZ and a competitive equilibrium in HZ may not be feasible for our model. Third, more importantly, the Pseudo market in HZ is quite complicated and may not be applicable to affordable housing. This is because the competitive rents $(h_1^*, h_2^*, \cdots, h_m^*)$ are unknown, an auctioneer must find an algorithm to integrate the demand information of an individual random assignment from each agent so that an adjustment of the rents, which are out of equilibrium, can be made to approach $(h_1^*, h_2^*, \cdots, h_m^*)$. It is unclear if there is an effective algorithm to do the job. If there is one, it will not be a simple task for an economy with a large scale of, say, 48,000 agents. In fact, the same question regarding applicability of the HZ Pseudo market approach has been raised and discussed before in BCKM.

Under one specific case, we are able to relate our IS allocation to the result in Hylland & Zeckhauser (1979), using a much simpler way to compute the competitive equilibrium of their Pseudo market. Let us add a "null" community $m+1$ with $n-m$ number of the same copies into our model and that in HZ. For all agents i, we set $u_{i(m+1)} = 0$ for both models. A lottery allocation $[p_{ij}]$ in HZ for the null community satisfies $\sum_{i=1}^{n} p_{i(m+1)} = n-m$. A lottery allocation in our model has a specific restriction such that for all agent $i = 1, 2, \cdots, n$ and communities $j = 1, 2, ..., m$, there is one and only one p_{ij} that is greater than zero. That is, each agent can only be assigned with a positive probability to one house that is not null in order to satisfy the rule. By the equal access principle, each eligible agent has applied for one unit in a community $j \in C$.

Example 5. Let $C = \{1, 2\}$ and $q = (1, 1)$. There are three agents whose vNM utilities are given by $u_1 = (1, 0)$ and $u_2 = u_3 = (3, 2)$.

Now we set $B_i = 1$ for all $i \in A$. Then under the Pseudo market approach, we get the market-clearing rent vector $h^* = (\frac{9}{5}, \frac{6}{5}, 0)$ and the market-clearing probability matrix

$$P_{HZ} = [p_{ij}^*] = \begin{pmatrix} \frac{5}{9} & 0 & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{2} & \frac{5}{18} \\ \frac{2}{9} & \frac{1}{2} & \frac{5}{18} \end{pmatrix},$$

which yields total expected utility of $3\frac{8}{9}$. The expected utilities for the three agents are given by $(\frac{5}{9}, 1\frac{2}{3}, 1\frac{2}{3})$. Under our Tickets algorithm, with the agent order $\sigma = (1, 2, 3)$, we get

$$x^n(\sigma) = \begin{pmatrix} x_1^n \\ x_2^n \\ x_3^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad and \quad P(x^n(\sigma)) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

with total expected utility of 4. The expected utilities for the three agents are given by $(\frac{1}{2}, 2, 1\frac{1}{2})$. Clearly, $P(x^n(\sigma)) \neq P_{HZ}$. There are five more orders under the Tickets algorithm, each of which gives rise to an IS allocation with the same total expected utility of 4. So, the Tickets algorithm achieves a better

PO allocation than the competitive equilibrium in HZ. An interesting aspect of this observation is the fact that these IS allocations are all at pure Nash equilibria (see Section VI), which outperform the Pseudo market at competitive equilibrium. Nonetheless, this is not a general result because there are examples in which the Pseudo market at competitive equilibrium outperforms the Tickets algorithm or the NE.

Theorem 7. *Let x^* be the SEF allocation and $P(x^*)$ be the induced probability distribution over communities C and the null community. Then $P(x^*) = P_{HZ}$, where P_{HZ} is the competitive equilibrium random assignment under the Pseudo market in HZ with each agent a budge of 1.*

An important observation in the proof of Theorem 7 is that the equilibrium rent of a house h that is not null equals the reciprocal of the lottery drawing probability in $P(x^*)$ for the house h under the SEF allocation. That is, the lottery drawing probability matrix uniquely determines the equilibrium prices of the houses. A house's price at equilibrium is higher because it is more overly demanded. Thus, for an economy with an SEF allocation, the Tickets algorithm provides a simple way to implement the unique desired EF and PO competitive equilibrium in HZ. This overcomes the obstacle using a centralized auctioneer to integrate individual demand information into the prices in the Pseudo market.

7. LOTTERY CONGESTION GAMES

Rosenthal (1973) originated the study of a class of symmetric congestion games. Milchtaich (1996) studied a class of unweighted (singleton) congestion games with player-specific payoff functions. A congestion game in the class of Rosenthal has a potential function and the Finite Improvement property (FIP) (Monderer & Shapley, 1996). Milchtaich showed that a game in his class may not have the FIP or a potential function. He found out that a best-reply improvement path or dynamics may also last infinitely by forming a cycle. Nevertheless, he found a finite best-reply improvement path that ends up with a pure Nash equilibrium (NE). Konishi et al. (1997) provided a general class of congestion games that are defined differently from that in Milchtaich. But their assumptions on the payoff functions, namely, the *independence of irrelevant choices*, *anonymity*, and *partial rivalry*, make their game equivalent to that in Milchtaich (Voorneveld et al., 1999). Konishi et al. showed that a pure strategy

SNE (Aumann, 1959) always exists. The existence theorem about SNE for the class of congestion games in Rosenthal was proved in Holzman & Law-Yone (1997). Note that not all potential games have an SNE. In this paper, we go one step further to provide a condition such that there is a unique SNE, by means of the SEF notion. If the condition is not satisfied, then the Tickets-TTC algorithm always achieves an SNE, not necessarily unique.

Our Tickets algorithm is initially designed by following the two-sided matching literature. We recognize later that our algorithm is in fact an explicit way to identify a finite best-reply improvement path implicitly stated in Milchtaich once we transfer our affordable housing lottery into a lottery congestion game along his framework.[16] Our recognition that an unweighted congestion game in Milchtaich may be transferred into an affordable housing lottery, and vice versa, provides a new link between the housing allocation model in HZ and the congestion models in Rosenthal, Konishi et al., and Milchtaich (see Tables 3 and 4) or the potential games in Monderer and Shapley. In particular, the unique SNE is equivalent to the unique Pseudo market equilibrium outcome in HZ, a result that justifies the artificial Pseudo market economy in HZ. Moreover, the Tickets algorithm always reaches the unique SNE, if any SEF exists, for any order of applicants, a novel contribution to the literature of congestion or potential games. The reason the Tickets algorithm can always reach the unique SEF or SNE is that it achieves an NE that is not Pareto dominated by others (via Konishi et al.).

Beyond the unique SNE, our main contribution in this section is to show how to transfer an affordable housing lottery into a congestion game in Milchtaich. Moreover, an unweighted congestion game in Milchtaich that may be transferred into an affordable housing lottery along the framework of HZ provides a new link between the housing allocation model in HZ and the congestion models. Furthermore, the unique SNE is equivalent to the unique Pseudo market equilibrium outcome in HZ, a result that contributes to a new view of the Pseudo market of HZ.

Next we construct a lottery congestion game that belongs to the class studied by Milchtaich from an affordable housing lottery. Our first result of this section is that every pure Nash equilibrium outcome of the lottery congestion game

[16] We thank Abraham Neyman for communications and comments that brought our attention to the congestion game in Milchtaich during the 2015 International Conferences in honor of Abraham Neyman and Sergiu Hart in which our main results about the Tickets algorithm were presented.

induces an IS allocation, and vice versa. First, we introduce the congestion game in Milchtaich:

Definition 7. (Milchtaich, 1996). An unweighted congestion game with player-specific payoff functions $\Gamma_{(A,C,U)}$ is defined as follows: Let A be the set of players and C be the common set of strategies with $q_j = 1$ for all $j \in C$. Let $s = (s_1, s_2, \cdots, s_n) \in C^n$ be a profile of strategies. The payoff player i receives for playing strategy j, i.e., $s_i = j$, is a monotonically nonincreasing function U_{ij} of the total number $n_j(s)$ of players playing the jth strategy, i.e., $n_j(s) = |\{i \in A | s_i = j\}|$ for all $j \in C$.

An unweighted congestion game with player-specific payoff functions becomes a symmetric (singleton) congestion game studied by Rosenthal if there is a payoff function U_j such that $U_{ij} = U_j$ for all $i \in A$ and $j \in C$. It is easy to see how a game $\Gamma_{(A,C,U)}$ may be transferred into an affordable housing lottery problem. Each $j \in C$ may be seen as a house and the vNM utility for a house is derived by limiting the strategy profiles to those s such that $n_j(s) = 1$ for all $j \in C$. If the utility does not satisfy the vNM condition, then we can derive the ordinal preferences from U_{ij} by varying all $s \in C^n$.

Definition 8. A strategy profile s is a Nash equilibrium (NE) of an unweighted congestion game with player-specific payoff functions if and only if each s_i is a best-reply strategy:

$$U_{is_i}(n_{s_i}(s)) \geq U_{ij}(n_j(s) + 1), \quad for\ all\ i \in A\ and\ j \in C. \tag{7.1}$$

Monderer and Shapley showed that an ordinal potential game is equivalent to a game that has FIP. They defined a path by a sequence $\gamma = (s(0), s(1), s(2), \cdots)$ such that for every $k \geq 1$, there exists a unique deviator i, say, such that $s(k) = (s_{-i}(k-1), s_i(k))$ with $s_i(k) \neq s_i(k-1)$. If the path is finite, $s(0)$ is the initial point of γ and the last element is called the terminal point. It is a finite improvement path if for all $k \geq 1$, $U_{is_i(k)}(n_{s_i(k)}(s(k))) > U_{is_i(k-1)}(n_{s_i(k-1)}(s(k-1)))$ for the unique deviator i at step k. The game has FIP if every improvement path is finite. It is a best-reply improvement path (BRIP) if for all $k \geq 1$, $U_{is_i(k)}(n_{s_i(k)}(s(k))) > U_{is_i(k-1)}(n_{s_i(k-1)}(s(k-1)))$ for the unique deviator i at step k and $s_i(k)$ is a best reply strategy against $s_{-i}(k-1)$. A game that has FIP must have a finite BRIP, but the converse is not true. Even worse, a best-reply improvement path may last infinite for a game in Milchtaich.

Example 6. (Milchtaich, 1996). Let $A = \{1,2,3\}$ and $C = \{C_1, C_2, C_3\}$. The ordinal preferences of the three players over (C_j, n_{C_j}) of communities $C_j \in C$ and its congestion n_{C_j} are given in Table 3:

Table 3: Players' preferences

Agents' Orders	*Agents in Community C_1*	*Agents in Community C_2*
$(1,2,3,4)$; $(1,2,4,3)$; $(1,4,2,3)$; $(3,2,1,4)$ $(3,2,4,1)$; $(3,4,2,1)$; $(4,2,3,1)$; $(4,2,1,3)$	1, 3, 4	2
$(1,3,2,4)$; $(1,3,4,2)$; $(1,4,3,2)$; $(2,3,1,4)$ $(2,4,3,1)$; $(3,2,4,1)$; $(4,3,2,1)$; $(4,3,1,1)$	1, 2, 4	3
$(2,1,3,4)$; $(2,1,4,3)$; $(2,4,1,3)$; $(3,1,2,4)$ $(3,1,4,2)$; $(3,4,1,2)$; $(4,1,3,2)$; $(4,1,2,3)$	2, 3, 4	1

The numerical game is given in Table 4, in which we assign numerical numbers 5, 4, 3, 2, and 1, respectively, for the first, the second, $\cdots$, and the fifth ranked choice in a player's preferences and zero for the rest of the choices.

Let $\gamma = (s(0), s(1), s(2), \cdots, s(5), \cdots)$ be an improvement path given by $s(0) = (C_2, C_1, C_1)$, $s(1) = (C_3, C_1, C_1)$, $s(2) = (C_3, C_3, C_1)$, $s(3) = (C_3, C_3, C_2)$, $s(4) = (C_2, C_3, C_2)$, $s(5) = (C_2, C_1, C_2)$, with a cycle

$$s(0) \to s(1) \to s(2) \to s(3) \to s(4) \to s(5) \to s(0)$$

with corresponding deviator sequences of $(1 : C_2 \to C_3)$, $(2 : C_1 \to C_3)$, $(3 : C_1 \to C_2)$, $(1 : C_3 \to C_2)$, $(2 : C_3 \to C_1)$, $(3 : C_2 \to C_1)$, where $(i : C_j \to C_{j'})$ denotes player i deviates from C_j to $C_{j'}$. One can check that each deviator uses a best-reply strategy at each $k \geq 1$. Thus, γ is a best-reply improvement path that lasts infinitely. Nonetheless, the game has two pure NE: $s^* = (C_3, C_1, C_2)$ and $s^0 = (C_2, C_3, C_1)$; it is the former that Pareto dominates the later.

Note that a best-reply improvement path that starts with an NE must stop there. Thus, a best-reply improvement path approach can reach both NE, including the less efficient one with equilibrium payoffs $(4,4,4)$. Under the Tickets algorithm, the more efficient NE (C_3, C_1, C_2) is the only one that is obtained.

Theorem 8. (Milchtaich, 1996). *Every unweighted congestion game with player-specific payoff functions has a Nash equilibrium in pure strategies.*

Table 4: A Lottery Game with Infinite BRIP

		A_2		
		C_1	C_2	C_3
	C_1	(0,0,0)	(0,1,3)	(0,4,3)
A_1	C_2	(4,2,3)	(2,0,4)	**(4,4,4)**
	C_3	(5,2,3)	(3,0,4)	(2,3,4)

A_3: C_1

		A_2		
		C_1	C_2	C_3
	C_1	(0,2,5)	(1,0,2)	(1,4,5)
A_1	C_2	(3,5,2)	(0,0,0)	(3,4,2)
	C_3	**(5,5,5)**	(1,3,1)	(2,3,5)

A_3: C_2

		A_2		
		C_1	C_2	C_3
	C_1	(0,2,1)	(1,1,1)	(1,3,0)
A_1	C_2	(4,5,1)	(3,0,1)	(4,3,0)
	C_3	(2,5,0)	(2,1,0)	(0,0,0)

A_3: C_3

The strategy in the proof of Theorem 8 is to find a best-reply improvement path that is finite. Milchtaich started with a one player game $\Gamma_{(A,C,U)}$ with $\mid A \mid= 1$, which has a pure NE. Then let $\Gamma_{(A,C,U)}$ be a game with $\mid A \mid= i$ and assume that it has a pure NE s^i. Starting with the equilibrium profile s^i, Milchtaich found a finite best-reply improvement path that ends up with a pure NE s^{i+1} in the game $\Gamma_{(A,C,U)}$, where $\mid A \mid= i + 1$. His induction approach is motivated by the definitions of FIP and the best-reply improvement path, each of which allows a single deviator at each step k. The Tickets algorithm follows the random order rematching process in Ma (1996) for the marriage problem that allows a single player to enter the market to make a proposal. The induction approach consists of a sequence of games such as the game defined in Table 4. The Tickets algorithm consists of a sequence of applicants working directly on preferences given in Table 3, with each applicant knowing her private preference but without knowing the private preferences of all others.

In fact we can transfer a lottery allocation problem (A,C,q,u) into a lottery congestion game $\mathcal{G}$ as follows. A is the set of players and C is the common set of strategies. Each strategy $j \in C$ has an integer $q_j \geq 1$. A profile of strategies $s = (s_1, s_2, \cdots, s_n) \in \Pi_{i=1}^{n} C$ induces a lottery allocation $x(s)$ such that $x_{ij}(s) = 1$ if $s_i = j$ and $x_{ij}(s) = 0$ otherwise. Note that $x(s) \in X$. Then we define a vector of congestion by $n(s) = (n_1(x(s)), n_2(x(s)), \cdots, n_m(x(s)))$, where $n_j(x(s))$ is defined by (4.1) by replacing x with $x(s)$. Given a strategy profile s, the payoff vector $U(s) = (U_1(s), U_2(s), \cdots, U_n(s))$ is defined by $U_i(s) = EU_i(x(s))$ for all $i \in A$ via (4.3). Equivalently, using notation in Milchtaich, we have $U_{is_i}(n_{s_i}(s)) = EU_i(x(s))$.

In $\mathcal{G}$, $n_j(x(s))$ is a measure for congestion of strategy j under strategy profile s. It follows from (4.2) that $z_j(x(s))$ is a monotonically nonincreasing function of $n_j(x(s))$. Thus the payoff function $EU_i(s)$ is a monotonically nonincreasing function of $n_j(x(s))$. So our lottery congestion game $\mathcal{G}$ belongs to the class of unweighted congestion games with player-specific payoff functions in Milchtaich. As in the construction of our lottery game, a pure strategy profile s induces a feasible lottery allocation $x(s)$ in X. Moreover, a lottery allocation x naturally induces a strategy profile $s(x)$ for the game $\mathcal{G}$ by setting $s_i(x) = j$ for $x_{ij} = 1$ for all $i \in A$. It is easy to see that a pure Nash equilibrium s induces a lottery allocation that is IS. An IS allocation x induces a Nash equilibrium in pure strategies $s(x)$ for the game $\mathcal{G}$. Thus, Theorem 1 proves Theorem 8, and vice versa.

Definition 9. (Envy-freeness). A strategy profile s is envy-free (EF) in an unweighted congestion game with player-specific payoff functions if and only if no player i wants to change his strategy with any other's, including those strategies that may not be played:

$$U_{is_i}(n_{s_i}(s)) \geq U_{ij}(n_j(s)) \;\; for\ all\ i \in A\ and\ j \in C. \tag{7.2}$$

Thus, a strategy profile s is strictly envy-free (SEF) if and only if the inequality above strictly holds for all $j \neq s_i$. For example, the NE s^* in Example 6 is SEF but the NE s^0 is not. Clearly, an EF or SEF strategy profile s must be an NE; the converse is not true. Next we discuss how an SNE is related to the SEF lottery allocation.

Definition 10. A profile s^* is a strong (pure) Nash equilibrium (SNE) of the game $\mathcal{G}$ if there is no coalition $T \subset A$ and strategies s_T for the coalition T such that

$$EU_i(s^*_{-T}, s_T) > EU_i(s^*) \quad \forall i \in T.$$

An SNE is a Nash equilibrium whose outcome is weakly PO. Because a symmetric congestion game in Rosenthal is an exact potential game (Monderer and Shapley), SNE, if any exists, coincides with NE, which is in the Argmax set of the potential function, as shown by Holzman and Law-Yone. This is no longer true for the game $\mathcal{G}$. In Example 6, s^0 is Pareto dominated by s^* (Konishi et al.).

Theorem 9. *(Konishi et al., 1997) Every unweighted congestion game with player-specific payoff functions has a pure strategy strong Nash equilibrium.*

Proof. A CS allocation induces an SNE. The existence follows from Theorem 2. □

With the SEF notion for the game $\mathcal{G}$, we prove a result that is much stronger than Theorem 9. The following directly follows from Theorem 6.

Theorem 10. *An unweighted congestion game with player-specific payoff functions $\mathcal{G}$ has an NE that is SEF if and only if the outcome of the Tickets algorithm is SEF in the related affordable housing lottery.*

Theorem 11. *Let s^* be an NE in the game $\mathcal{G}$ that is also SEF. Then the NE s^* is the unique SNE in the game $\mathcal{G}$. Moreover, such an NE s^* induces a unique probability distribution matrix at a competitive equilibrium under the Pseudo market in HZ with each agent being assigned an equal budget of* 1.

Proof. An SEF allocation is the unique CS allocation which induces a unique SNE; Moreover, it can be achieved by the Tickets algorithm for any given order (Theorems 5 and 6). The connection between SNE and the competitive equilibrium in HZ follows from Theorem 7. □

8. CONCLUSIONS

There are so many affordable housing lotteries using a simple rule–preventing duplicate lottery entries on ex ante lottery allocations, a rule that is used in the design that guarantees an equal access and opportunity for eligible applicants to affordable housing. The same type rule is used as well in many school choice lotteries. This rule is incompatible with, in the general domain, the competitive equilibrium of a Pseudo market in HZ and the EF and PO allocations generated by the algorithms in BM and BCKM. We study an affordable housing lottery with a model in HZ with a more restrictive domain, which fits well those affordable housing or school choice lotteries implemented in practice. These lotteries have a flaw in their designs because they generate outcomes that are not EF and PO, even if they exist. We provide a fix by a Tickets algorithm that incorporates an optimization process with the rejection-acceptance idea in the noted deferred acceptance algorithm. We have proposed a new fairness notion of individual stability along the lines of a stable matching in Gale and Shapley. Such a notion is justified with the Nash equilibrium in a lottery congestion game in Milchtaich. In fact, we provide a way to transfer an affordable housing lottery that is modeled by the housing allocation problem in HZ into a congestion game in Milchtaich, and vice versa, to establish several equivalence results between IS and NE, between SEF and the competitive equilibrium in HZ, and between the CS allocations and the SNE outcomes. In particular, we provide conditions for the existence of a unique SNE for the lottery congestion game and show that the Tickets algorithm always reaches the unique SNE for any order of applicants, a result contributing to two different strands of the literature of the housing allocation problem in HZ and the congestion or potential games. This link paves a new way to address other important issues in affordable housing lotteries by means of congestion or potential games, which have been more extensively studied.

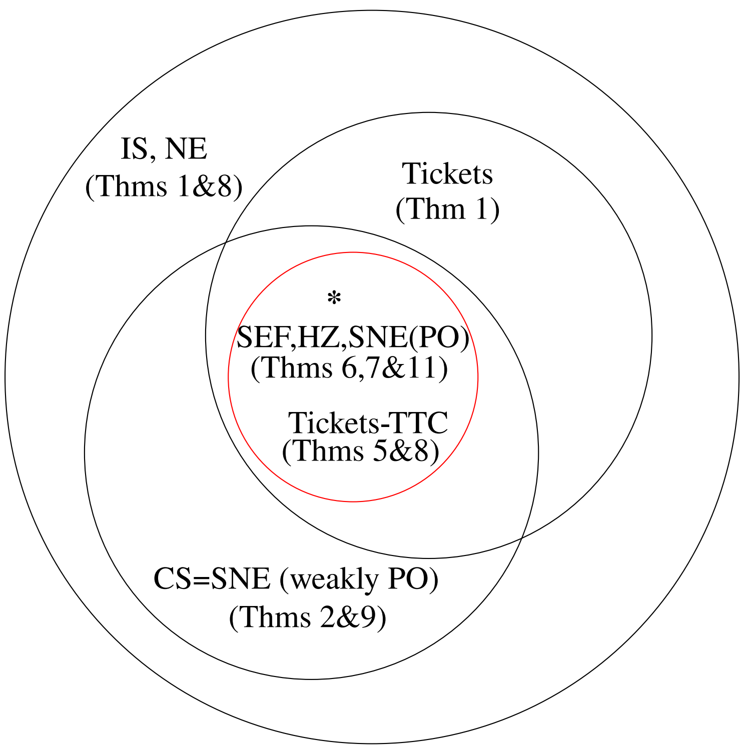
No issue with affordable housing lotteries is small. Households in the millions have been living in affordable housing in the U.S. alone and there are more to come as time goes by. There are also many different lottery protocols for affordable housing in the U.S. and across the world, each of which may be considered fair. But fairness should not be taken for granted as with the NYC lottery. Any slight change in a lottery protocol can affect the lottery allocations ex ante or ex post. A change in the rule also changes the outcome completely because agents play very different games, as noted in the celebrated papers by Crawford (1979, 1980). It remains open if our main results in this paper hold for other lottery mechanisms with different rules.

The central goal of this research is to explore the HZ economy and provide a model for achieving a fair lottery allocation to practical problems like affordable housing or school choice lotteries. The HZ model and our Tickets algorithm may be useful for other applications involving lotteries. Subsequent works should target other rules over allocation–like rental control–so that the unique SEF lottery allocation always exists. The rental control literature in Talman & Yang (2008); Andersson & Svensson (2014, 2016); Andersson et al. (2015) may be an excellent start. Since ex post allocations can affect the ways agents behave ex ante, another interesting extension of this paper would be a combination of this paper's model with the ex post allocation proposed in Andersson et al. (2016).

9. APPENDICES: SUMMARY OF MAIN RESULTS AND PROOFS

9.1. Summary of Main Results

Figure 1. The Relations of the Main Results



Note: Figure 1 summarizes our main results in this paper. Note that Figure 1 is drawn for an economy with and without an SEF lottery allocation in the same graph. For an economy with an SEF allocation, the three inner circles in Figure 1 will all shrink to a single point $*$: the SEF allocation is a singleton that is achieved by the Tickets algorithm without using TTC; it is also the unique SNE outcome and the unique Pseudo market competitive equilibrium in HZ. For an economy that does not have an SEF allocation, the allocations that can be achieved by the Tickets-TTC algorithm are a subset of all CS lottery allocations or SNE outcomes. The Tickets-TTC algorithm is selective because it only achieves those NE outcomes that are not Pareto dominated. The Tickets algorithm achieves an IS allocation or an NE outcome that may not be CS or SNE. The IS allocations or NE outcomes achieved by the Tickets algorithm are a subset of all IS allocations or NE outcomes.

Proof of Theorem 1. The sequence $\{x^i\}, i = 1, 2, 3, ..., n$, consists of lottery allocations such that each x^i is an IS allocation with respect to agents $A^i = \{1, 2, \cdots, i\}$ in the room at step i and the communities C, with the same q. We have to show that the algorithm ends in a finite number of steps and the final allocation x^n is an IS allocation for the problem (A, C, q, u). We can compare the two lottery allocations x^{i-1} and x^i. After i enters the room, the expected utilities for agents from 1 to $(i-1)$ under x^i are non-increasing and there exists at least one agent from 1 to $(i-1)$ whose expected utility has been decreased if there is anyone from 1 to $(i-1)$ who returns his ticket, due to agent i's entry. This is because only one agent enters the door each time during the process. The number of agents who hold a lottery ticket from a machine j, for all j, under x^i is also nondecreasing under x^{i-1} after agent i enters the room. Moreover, there exists one and only one machine such that the number of agents who hold a lottery ticket from that machine increases by one from x^{i-1} to x^i. Note that u_{aj} is positive for at least one $j \in C$ and for any $a \in A$.

It is clear that x^1 is IS with respect to agent 1, when he is the only agent in the room and he chooses the lottery ticket that he likes the most. Assume x^{i-1} is IS with respect to agents $\{1, 2, \cdots, i-1\}$ and C. After agent i enters the room, if there is no agent h, $h = 1, 2, \cdots, i-1$, who would like to return his lottery ticket and goes back to the door, then we obtain x^i, which is IS with respect to $\{1, 2, \cdots, i\}$ and C, by the algorithm and the fact that x^{i-1} is IS with respect to $\{1, 2, \cdots, i-1\}$ and C. If there is an agent h who would like to return his lottery ticket and goes out of the door, then agent h is seen as i. Note that the lottery allocation in the room before h reenters the room is IS with respect to $\{1, 2, \cdots, i\} \setminus \{h\}$ and C. After h reenters, he chooses the lottery ticket that he likes the most. Such a lottery ticket cannot be the one he just returned.

Since h has chosen the lottery ticket he likes the most when he reenters, he will not return his ticket again and go back to the door within the process from x^{i-1} to x^i. Note that as long as there is an agent by the door during the process, the number of lottery tickets issued by each machine remains the same. This implies that the process from x^{i-1} to x^i will not form a cycle because the number of agents in the room is finite at $(i-1)$ and no agent will visit the door twice in the process. Therefore, the process will end with a lottery allocation x^i in a finite number of steps. During the process, starting with i, x^i is obtained such that no agent from $\{1, \cdots, i\}$ would like to return his lottery ticket anymore. Thus, x^i must be IS with respect to $\{1, 2, \cdots, i\}$ and C. This shows that x^n is an IS allocation for (A, C, q, u). Note that the total number of steps to achieve x^n is bounded by $(n(n+1))/2$. Thus, this completes the proof of Theorem 1. □

Proof of Theorem 4. Let x^* be an SEF allocation. We know from Theorem 3 that x^* is PO. Assume, on the contrary, that there is another allocation $x' \in X$ which is also individually stable and Pareto optimal. As in the proof of Theorem 3, we consider two cases: (a) $n_j(x^*) = n_j(x')$ for all $j \in C$; and (b) there is a community j_0 such that $n_{j_0}(x^*) < n_{j_0}(x')$.

We prove Case (a) first. Because $x^* \neq x'$, there exists at least an agent i' such that $x^*_{i'} \neq x'_{i'}$. By the assumption x^* is SEF, it follows that $EU_{i'}(x^*) > EU_{i'}(x')$. Using the fact that x^* is SEF and $n_j(x^*) = n_j(x')$, we also have that $EU_i(x^*) \geq EU_i(x')$ for all agents i. But, this implies that $x^* P x'$, which contradicts the assumption that x' is Pareto optimal. This completes the proof of Case (a).

Now we prove Case (b): $\exists j_0$ such that $n_{j_0}(x') > n_{j_0}(x^*)$. In this case, we show that we can find an infinite chain of communities, $j_0, j_1, j_2, \cdots$, with distinct vertexes, such that

$$n_{j_0}(x^*) < n_{j_0}(x'),$$

$$n_{j_1}(x^*) + n_{j_0}(x^*) < n_{j_1}(x') + n_{j_0}(x'),$$

$$n_{j_2}(x^*) + n_{j_1}(x^*) + n_{j_0}(x^*) < n_{j_2}(x') + n_{j_1}(x') + n_{j_0}(x'),$$

$$\cdots\cdots$$

But this is impossible because the number of communities is finite.

To complete the proof, we start with a chain, $j_0, j_1, \cdots, j_k$, with distinct vertexes, such that

$$n_{j_k}(x^*) + \cdots + n_{j_1}(x^*) + n_{j_0}(x^*) < n_{j_k}(x') + \cdots + n_{j_1}(x') + n_{j_0}(x'). \quad (9.1)$$

Then we claim that there exists j_{k+1} such that $j_{k+1} \notin \{j_0, j_1, \cdots, j_k\}$ and

$$n_{j_{k+1}}(x^*)+n_{j_k}(x^*)+\cdots+n_{j_1}(x^*)+n_{j_0}(x^*) < n_{j_{k+1}}(x')+n_{j_k}(x^*)+\cdots+n_{j_1}(x')+n_{j_0}(x'). \quad (9.2)$$

By assumption, (9.1) holds for $k = 0$, i.e., $n_{j_0}(x^*) < n_{j_0}(x')$, there exists at least one agent i_0 such that $x'_{i_0} = j_0$ and $x^*_{i_0} \neq j_0$. For notational convenience, let $j_1 = x^*_{i_0}$. Clearly, $j_0 \neq j_1$.

Because x' is IS, we have, by the IS definition,

$$EU_{i_0}(x') \geq EU_{i_0}(x'_{-i_0}, j_1),$$

which leads to

$$u_{i_0 j_0} z_{j_0}(x') \geq u_{i_0 j_1} z_{j_1}(x'_{-i_0}, j_1). \quad (9.3)$$

By the assumption $n_{j_0}(x^*) < n_{j_0}(x')$, we have $z_{j_0}(x^*) \geq z_{j_0}(x')$. Then

$$u_{i_0 j_0} z_{j_0}(x') \leq u_{i_0 j_0} z_{j_0}(x^*) \quad (9.4)$$

must hold. Because x^* is SEF, it follows from the SEF definition that

$$u_{i_0 j_0} z_{j_0}(x^*) < u_{i_0 j_1} z_{j_1}(x^*). \quad (9.5)$$

Now it follows from inequalities (9.3)-(9.5) that

$$u_{i_0 j_1} z_{j_1}(x^*) > u_{i_0 j_1} z_{j_1}(x'_{-i_0}, j_1),$$

which gives us $z_{j_1}(x^*) > z_{j_1}(x'_{-i_0}, j_1)$ and then

$$n_{j_1}(x^*) \leq n_{j_1}(x'). \quad (9.6)$$

It follows from (9.6) and $n_{j_0}(x^*) < n_{j_0}(x')$ that

$$n_{j_1}(x^*) + n_{j_0}(x^*) < n_{j_1}(x') + n_{j_0}(x'). \quad (9.7)$$

Now, we can follow the procedure to find j_2, etc, up to k for the chain $j_0, j_1, \cdots, j_k$. We want to show that we can find the next j_{k+1}. Once again, it follows from (9.1) that there must exist at least one agent i_k such that $x'_{i_k} \in \{j_0, j_1, \cdots, j_k\}$ and $x^*_{i_k} \notin \{j_0, j_1, \cdots, j_k\}$. Let $x'_{i_k} = j_\alpha \in \{j_0, j_1, \cdots, j_k\}$. For convenience, let $j_{k+1} = x^*_{i_k}$. Note that $j_{k+1} \notin \{j_0, j_1, \cdots, j_k\}$.

When $j_\alpha = j_0$, the proof (9.3)-(9.5) yields $n_{j_1}(x^*) \le n_{j_1}(x')$, which gives us

$$z_{j_1}(x^*) \ge z_{j_1}(x'). \tag{9.8}$$

Now we show the case when $j_\alpha = j_1$. Once again, we can follow the proof (9.3)-(9.5) (in which (9.8) has been used) to get $n_{j_2}(x^*) \le n_{j_2}(x')$, which gives us the desired inequality

$$z_{j_2}(x^*) \ge z_{j_2}(x'). \tag{9.9}$$

Thus, by duplicating the proof of (9.3)-(9.5) for $\alpha = 0, 1, 2, \cdots, k$, we get the following sequence

$$\begin{array}{rcl} n_{j_0}(x^*) & < & n_{j_0}(x') \Rightarrow z_{j_0}(x^*) \ge z_{j_0}(x') \\ \Rightarrow n_{j_1}(x^*) & \le & n_{j_1}(x') \Rightarrow z_{j_1}(x^*) \ge z_{j_1}(x') \\ \cdots & & \cdots \\ \Rightarrow n_{j_k}(x^*) & \le & n_{j_k}(x') \Rightarrow z_{j_k}(x^*) \ge z_{j_k}(x') \end{array} \tag{9.10}$$

Note that $j_{k+1} = x^*_{i_k}$ and $j_{k+1} \notin \{j_0, j_1, \cdots, j_k\}$. Once again, due to $z_{j_k}(x^*) \ge z_{j_k}(x')$, we can follow the proof of (9.3)-(9.5) to get $n_{j_{k+1}}(x^*) \le n_{j_{k+1}}(x')$, which, together with (9.1), yields (9.2). This completes the proof. □

Discussion of the proof of Theorem 6

In the proof of Theorem 6, we introduce a special door clock to track agents and their allocated tickets. We set a clock on the door and let it tick once when an agent is entering the room. The clock does not tick when an agent inside goes out of the door. When that agent reenters the room, the clock ticks one more time. For example, if 1 and 2 are the first two agents in an order to enter the room, then, when agent 1 is entering the room, the clock ticks once and records a time $z = 1$. The clock ticks one more time and records a time $z = 2$ when agent 2 is entering the room. Things now become a little bit more complicated because agent 1 may want to return his ticket after agent 2 enters and takes a ticket from the same pool as agent 1's. If that is the case, we allow agent 1 to go out of the room but the clock does not tick when agent 1 goes out. The clock ticks one more time and records a time $z = 3$ when agent 1 is reentering. If agent 1 does not want to return his ticket after agent 2 enters, agent 1 stays in the room and the clock stops at $z = 2$. After that, the next agent

Table 5: Door time for Example 3

agent at the door	door time	(1,2,3,4)
1	1	$(\ , \ , \ , \ )$
2	2	$(C_2, \ , \ , \ )$
1	3	$(\ , C_2, \ , \ )$
3	4	$(C_3, C_2, \ , \ )$
4	5	$(C_3, C_2, C_2, \ )$
null	6	(C_3, C_2, C_2, C_1)

in the order can get in and the clock ticks accordingly. When all agents in A form any fixed or random order to enter the room, the clock records a sequence of door times $\{z\} = 1, 2, \cdots, T$, where T is the door time when the algorithm ends. Note T is at least as large as n, and at most as large as $\frac{n(n+1)}{2}$.

An agent's door time is a dynamic process, depending on the order of agents entering the room and the optimization process taking place in the room. But, we can assign a unique door time $\{t_1, t_2, \cdots, t_n\}$ for agents in A, where t_i is defined as the door time when agent i enters the door the last time. That is, agent i enters the room and gets his ticket at door time t_i, and then keeps the same ticket from the door time t_i to the door time T. Clearly, each t_i is a time in the sequence of door times $1, 2, \cdots, T$.

For each community, there is a unique door time t_j, where t_j is the first door time such that community j has a number of outside tickets that equals $n_j(x^n)$. That is, the total number of outside tickets held by agents remains the same for j from time t_j to time T.

Table 7 presents the door time in Example 2 for the order (1,2,3,4) under the Tickets algorithm. According to our definition, the unique door times of the four agents are given by $t_1 = 3, \ t_2 = 2, \ t_3 = 4, \ t_4 = 5$, while the unique door times of the three communities are given by $t_{C_1} = 6, \ t_{C_2} = 5, \ t_{C_3} = 4$. Agent 1 reenters the room and the clock ticks at $z = 3$ and he chooses C_3 and then holds C_3 to the end. Thus, $t_1 = 3$. At the door time $z = 5$, the number of agents who hold C_2 equals 2, that remains the same from $z = 5$ to $z = 6$, which is the end. Thus, $t_{C_2} = 5$.

We now define a sequence of allocations $(y_1, y_2, \cdots, y_T)$, where each y_z is the allocation in the room at the door time z for those agents who have been inside the room, $1 \leq z \leq T$, and the agent at time z, if any, who is willing to return his lottery, has gone out of the door.

It should be noted that the allocation $(y_1, y_2, \cdots, y_T)$ may be different from the allocation $(x^1, x^2, \cdots, x^n)$ in the algorithm because T may be different from n. But, note that $y_2 = x^1$ and $y_T = x^n$. Let $n_j(y_z)$ denote the number of agents who hold tickets in community j under y_z, $z = 1, 2, \cdots, T$. Let t_1 and t_2 be two door times such that $y_{t_1} = x^i$ and $y_{t_2} = x^{i+1}$. Then, for any door time t such that $t_1 \leq t < t_2$ and any community $j \in C$, we have $n_j(y_t) = n_j(y_{t_2})$. Thus, $n_j(y_z)$ must be monotonically increasing with respect to door time $z = 1, 2, \cdots, T$.

For each community, there is a unique door time t_j, where t_j is the first door time such that community j has a number of outside tickets that equals $n_j(x^n)$. That is, the total number of outside tickets held by agents remains the same for j from time t_j to time T. This definition makes sense because $n_j(y_z) \leq n_j(y_{z+1})$ for $z = 1, 2, \cdots, T-1$. Thus, each community j has a unique t_j.

There are useful properties associated with these definitions of door times.

(A). $t_{i_1} \neq t_{i_2}$ if and only if $i_1 \neq i_2$ for any two agents i_1 and i_2 in A.

(B). $t_{j_1} \neq t_{j_2}$ if and only if $j_1 \neq j_2$ for any two communities j_1 and j_2 in C.

(C). If $t_{i_1} > t_{i_2}$ for two agents i_1 and i_2, then either i_1 is an agent from the order who enters the room at time t_{i_1} and keeps his choice from t_{i_1} to T, or agent i_1 must have gone out of the room and reentered at least once after door time t_{i_2}.

(D). $n_j(y_k)$ is nondecreasing in door time k for all $j \in C$. That is, $n_j(y_1) \leq n_j(y_2) \leq \cdots \leq n_j(y_T)$ for all $j \in C$. That is, no community becomes less congested as more and more agents enter the room.

(E). It follows from the definition of t_j, $n_j(y_t) < n_j(y_T)$ for all $t = 1, 2, \cdots, t_{j-1}$ and $n_j(y_t) = n_j(y_T)$ for all $t = t_j, t_{j+1}, \cdots, T$ for all $j \in C$.

(F). Allocation y_t, $t = 1, 2, \cdots, T$, is always IS with respect to agents in the room. Under our assumption $u_{ij} > 0$ for at least one $j \in C$, each agent in the room under y_t holds one and only one lottery ticket.

(G). The agent who enters the room as the clock ticks and records door time $t_j - 1$ must have chosen community j. For any $t > t_j$, if there exists an agent who chooses j, there must exist the other agent who returns his j ticket and gets out of the door and reenters to choose a ticket that is not j.

Proof of Theorem 6. Assume, on the contrary, there exists an SEF allocation x^* and $x^n \neq x^*$. We consider two cases.

Case 1. There is a community j_0 such that $n_{j_0}(x^*) < n_{j_0}(x^n)$. This leads to

Case (b) in the proof of theorem 4. Note that in the proof of theorem 4 in Case (b), we only use the IS, not PO assumption. From that proof, we can conclude that x^n is not IS, which is a desired contradiction.

Case 2. Assume that $n_j(x^*) = n_j(x^n)$ for all $j \in C$. Because $x^* \neq x^n$, x^* is SEF, x^n is IS, it follows from Theorem 5 that we can get x^* by using the TTC algorithm from x^n. This is due to the fact that x^* is unique and x^n is IS. Let $\Omega_1 = \{i_1, i_2, \cdots, i_\omega\}$ be any TTC cycle from the TTC algorithm such that $\omega \geq 2$. Because Ω_1 is a TTC cycle, without loss of generality, we may suppose the cycle Ω_1 has been formed as follows: Agent i_θ receives agent $i_{(\theta+1)}$'s ticket for all $\theta = 1, 2, \cdots, \omega - 1$ and agent i_ω receives agent i_1's ticket.

For notation convenience, we use θ for i_θ, $\theta = 1, 2, \cdots, \omega$. Let $\mu_\theta = x^*_\theta$ and $\nu_\theta = x^n_\theta$. μ_θ and ν_θ are the tickets held by agent θ under x^* and x^n, respectively. Note that agents θ, $\theta = 1, 2, \cdots, \omega$, trade their tickets under x^n to get their tickets under x^*. We need to show that there exists no Ω_1 cycle such that $|\Omega_1| \geq 2$, by a contradiction.

Because x^* is strictly envy-free, all agents θ become strictly better off by trading their tickets in the cycle Ω_1. The following must hold for all agents θ, $\theta = 1, 2, \cdots, \omega$,

$$EU_\theta(x^n) < EU_\theta(x^*), \tag{9.11}$$

which implies that

$$u_{\theta\mu_\theta} z_{\mu_\theta}(x^*) > u_{\theta\nu_\theta} z_{\nu_\theta}(x^n), \tag{9.12}$$

since $n_j(x^*) = n_j(x_n)$ for all $j \in C$, by assumption.

Now, we state three lemmas, which only apply to Case 2.

Lemma 1. *There exists at least one agent i_θ in Ω_1 such that $t_{\nu_\theta} < t_\theta$.*

Recall that ν_θ is agent θ's lottery ticket under x^n, the outcome from the algorithm. From (E), the total number of agents who hold a lottery in ν_θ remains the same for $t \geq t_{\nu_\theta}$. By the definition t_θ, agent θ will keep the same lottery for all $t \geq t_\theta$. Due to (F), Lemma 1 states that there exists one agent θ in the cycle Ω_1 such that agent θ enters the room and chooses the lottery ν_θ and keeps the same lottery to the close of the algorithm. Moreover, the claim $t_{\nu_\theta} < t_\theta$ implies that after agent θ enters at t_θ to choose ν_θ, there exists another agent who must return his lottery ticket in community ν_θ and reenters to choose a lottery ticket that is not ν_θ. We can even make a stronger statement. Define η to be the agent such that $t_\eta = \max\{t_\theta | \theta = 1, 2, \cdots, \omega\}$. That is, all agents in

Ω_1, except η, keep their lottery tickets at least from door time $(t_\eta - 1)$ to the end of the algorithm. η is the last agent in Ω_1 who will keep the same lottery from the door t_η to the end of the algorithm. That is, all agents in Ω_1 will keep the same lottery tickets from door time t_η to the end of the algorithm. The next lemma shows that Lemma 1 also applies to this agent η.

Lemma 2. *The inequality $t_{\nu_\eta} < t_\eta$ also holds.*

We now continue our proof of Theorem 6. By Lemma 2, we have $t_{\nu_\eta} < t_\eta$. It follows from the definition of t_η and Lemma 2 that there must exist at least one agent, denoted γ_1, who holds a ticket in community ν_η under the allocation y_{t_η} at the door time t_η but a ticket in a different community, ν_{γ_1}, say, under the lottery allocation x^n. Because agent γ_1 must go back to the door after t_η at least once, it follows that $t_{\gamma_1} > t_\eta$. The next lemma shows that if that is the case, then we can find another agent γ_2 such that $t_{\gamma_2} > t_{\gamma_1}$.

Let Ω_2 be the cycle where agent γ_1 belongs and $t_{\eta_2} = \max\{t_\delta \mid \delta \in \Omega_2\}$. Note that $\gamma_1 \notin \Omega_1$ since $t_\eta < t_{\gamma_1}$. Thus, Ω_1 and Ω_2 are two different cycles. After we find this γ_1, we show that this γ_2 must be in the third cycle Ω_3.

Lemma 3. *There exists an agent who is in the third cycle Ω_3.*

The remaining proof of Theorem 6 directly follows Lemma 3 because Ω_3 is either a singleton set or consists of at least two agents. In either case, we can find the fourth cycle Ω_4 that is nonempty (see the proof of Lemma 3 in the Appendix). This process leads to an infinite sequence of cycles, $\Omega_1, \Omega_2, \cdots$, a desired contraction because the number of cycles can only be finite. This completes the proof of Theorem 6. □

Proof of Lemma 1. Recall our definitions of the door times t_θ for agent θ and t_{ν_θ} for his lottery in community ν_θ: t_{ν_θ} is the door time such that $n_{\nu_\theta}(t) = n_{\nu_\theta}(x^n)$ for all door time $t = t_{\nu_\theta}, (t_{\nu_\theta} + 1), \cdots, T$. Thus, one agent must give up his lottery in community ν_θ and goes out of the door if there is any other agent who enters and chooses a lottery ticket from ν_θ after time t_{ν_θ}. t_θ is the door time such that agent θ keeps his lottery ticket ν_θ to the end of the algorithm, the door time T. The inequality $t_{\nu_\theta} < t_\theta$ implies that either agent θ returns his ticket at the door time $(t_\theta - 1)$ and reenters to choose ν_θ at the door time t_{ν_θ}; or he is an agent from the original order who just enters at the door time t_θ and chooses ν_θ. By $t_{\nu_\theta} < t_\theta$, Lemma 1 claims that after θ chooses his ticket ν_θ, by property (G), there must exist one other agent in the room who will return his lottery in community ν_θ and gets out of the door. This agent reenters and chooses a ticket that is not in community ν_θ.

Define $t' = \max_{\theta=1,2,\cdots,\omega} t_{\nu_\theta}$. Then, for all t such that $t = (t'+1), (t'+2), \cdots, T$, for all agents θ in Ω_1, the following holds

$$n_{\nu_\theta}(y_t) = n_{\nu_\theta}(y_T).$$

Moreover, for all t such that $t \leq t'$, there is at least one $\theta \in \Omega_1$ such that

$$n_{\nu_\theta}(y_t) < n_{\nu_\theta}(y_T),$$

by property (D). Therefore, there is one and only one community ν^* in $\{\nu | \nu_\theta = \nu, \theta \in \Omega_1\}$ such that

$$n_{\nu^*}(y_{t'}) < n_{\nu^*}(y_T)$$

at the door time t'. Thus, $n_{\nu^*}(y_t) < n_{\nu^*}(y_T)$ for all door time $t < t'$ and $n_{\nu^*}(y_t) + 1 = n_{\nu^*}(y_T)$ for $t = t'$. Moreover, $n_{\nu^*}(y_t) = n_{\nu^*}(y_T)$ for all door time $t > t'$.

Now, assume, on the contrary, that $t_\theta \leq t_{\nu_\theta}$ for all $\theta \in \Omega_1$. This means that all agents $\theta \in \Omega_1$ will keep their lotteries to the door time T. That is, there is no agent $\theta \in \Omega_1$ who will go outside of the door after the door time t'. Let θ^* be the agent in Ω_1 such that $\mu_{\theta^*} = \nu^*$, where μ_{θ^*} is the lottery ticket in community ν^* agent θ^* obtains after forming the cycle Ω_1. Then, at the door time t', the following holds

$$n_{\nu_{\theta^*}}(y_{t'}) = n_{\nu_{\theta^*}}(y_T) \text{ and } n_{\mu_{\theta^*}}(y_{t'}) + 1 = n_{\mu_{\theta^*}}(y_T).$$

Thus, at the door time t', agent θ^* holds the ticket in community ν_{θ^*}, but he strictly prefers a lottery in community ν^*, by (9.11). This implies that the lottery allocation at the door time t' cannot be IS (with respect to the set of agents in the room and communities C) because agent θ^* will return his lottery ticket ν_{θ^*} for the better lottery ticket ν^*. However, this is a contradiction to the property (F). This completes the proof of Lemma 1. □

Proof of Lemma 2. We need to prove that, at the door time t_η when agent η enters the room and chooses a lottery ticket from community ν_η, there exists another agent who will return his lottery ticket in community ν_η and go back to the door. Here we have used property (D).

By Lemma 1, there is an agent $\theta^* \in \Omega_1$ such that $t_{\nu_{\theta^*}} < t_{\theta^*}$. If $\theta^* = \eta$, we are done. If not, then $\theta^* \neq \eta$ and

$$t_\eta > t_{\theta^*} > t_{\nu_{\theta^*}}.$$

Now consider agent $(\theta^*+1) \in \Omega_1$ and his lottery ticket $\nu_{(\theta^*+1)}$. By the definition of Ω_1, it follows from (9.11) that

$$EU_{\theta^*}(x^n) < EU_{\theta^*}(x^*) = u_{\theta^*\mu_{\theta^*}} z_{\mu_{\theta^*}}(x^*). \tag{9.13}$$

The equality "=" above follows from the definition of expected utility and the lottery 'house' μ_{θ^*} is obtained by θ^* in the cycle Ω_1. For all $j \in C$, $z_j(x^n) = z_j(x^*)$ and $\mu_{\theta^*} = \nu_{(\theta^*+1)}$, we have

$$u_{\theta^*\mu_{\theta^*}} z_{\mu_{\theta^*}}(x^*) = u_{\theta^*\nu_{(\theta^*+1)}} z_{\nu_{(\theta^*+1)}}(x^n). \tag{9.14}$$

For notational convenience, let $y_{t_\eta} = O$. Because O is IS by the property (F), we have

$$EU_{\theta^*}(O) \geq EU_{\theta^*}(O_{-\theta^*}, \nu_{(\theta^*+1)}) = u_{\theta^*\nu_{(\theta^*+1)}} z_{\nu_{(\theta^*+1)}}(O_{-\theta^*}, \nu_{(\theta^*+1)}). \tag{9.15}$$

Because $t_{\nu_{\theta^*}} < t_\eta$, we have $n_{\nu_{\theta^*}}(O) = n_{\nu_{\theta^*}}(x^n)$. Thus,

$$EU_{\theta^*}(x^n) = EU_{\theta^*}(O). \tag{9.16}$$

By equations (9.13)-(9.16), we have

$$u_{\theta^*\nu_{(\theta^*+1)}} z_{\nu_{(\theta^*+1)}}(x^n) > u_{\theta^*\nu_{(\theta^*+1)}} z_{\nu_{(\theta^*+1)}}(O_{-\theta^*}, \nu_{(\theta^*+1)}). \tag{9.17}$$

Thus, we must have

$$n_{\nu_{y_{(\theta^*+1)}}}(t_\eta) + 1 > n_{\nu_{(\theta^*+1)}}(x^n), \tag{9.18}$$

which yields

$$n_{\nu_{y_{(\theta^*+1)}}}(t_\eta) \geq n_{\nu_{(\theta^*+1)}}(x^n), \tag{9.19}$$

and then,

$$t_{\nu_{(\theta^*+1)}} < t_\eta. \tag{9.20}$$

Now, due to (9.20), we can apply the same idea in the proof with agent θ^* above to agent (θ^*+1) to find agent (θ^*+2) and $\nu_{(\theta^*+2)}$, and then the following inequality also holds

$$t_{\nu_{(\theta^*+2)}} < t_\eta.$$

Continuing with this process, we must eventually reach the agent η since he is also an agent in Ω_1. Thus, the inequality

$$t_{\nu_\eta} < t_\eta$$

must hold. This completes the proof of Lemma 2. □

Proof of Lemma 3. Now we consider two cases for γ_1. Case (α). $|\Omega_2| \geq 2$; Case (β). $\Omega_2 = \{\gamma_1\}$.

First, we prove Case (α). By the assumption, Ω_2 consists of at least two agents. Let $\gamma_1 \in \Omega_2 = \{i'_1, i'_2, \cdots, i'_h\}$. But, for notation convenience, we use δ for i'_δ, $\delta = 1, 2, \cdots, h$. Let $\eta_2 \in \Omega_2$ such that $t_{\eta_2} = \max\{t_\delta | \delta \in \Omega_2\}$. By Lemma 2, we have $t_{\eta_2} > t_{\nu_{\eta_2}}$. Now, property (G) implies that when η_2 enters and chooses the lottery ν_{η_2} in community ν_{η_2}, there exists an agent who returns his lottery in community ν_{η_2} and goes out of the door. This implies that there is an agent γ_2, say, such that $t_{\gamma_2} > t_{\eta_2}$, which implies that γ_2 must be in the third cycle Ω_3. This completes the proof of Case (α).

Next, we prove Case (β). Under this case,

$$EU_{\gamma_1}(x^n) = EU_{\gamma_1}(x^*). \tag{9.21}$$

Recall that $\eta \in \Omega_1$ and γ_1 does not hold a lottery in the community where η is, under x^n. Because x^* is SEF, and η and γ_1 are not in the same communities under x^n, we have

$$EU_{\gamma_1}(x^*) > u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(x^*). \tag{9.22}$$

By our assumption in Case (b), $n_j(x^*) = n_j(x^n)$ for all $j \in C$, which implies that $z_j(x^*) = z_j(x^n)$ for all $j \in C$. Thus, $u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(x^*) = u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(x^n)$. And then (9.22) gives us

$$EU_{\gamma_1}(x^*) > u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(x^n). \tag{9.23}$$

Since $EU_{\gamma_1}(x^n) = u_{\gamma_1 \nu_{\gamma_1}} z_{\nu_{\gamma_1}}(x^n)$, it follows from (9.21) and (9.23) that

$$u_{\gamma_1 \nu_{\gamma_1}} z_{\nu_{\gamma_1}}(x^n) > u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(x^n). \tag{9.24}$$

Because y_{t_η} is IS (with respect to agents in the room at time t_η) and γ_1 holds a lottery ticket in community ν_η, which is not equal to ν_{γ_1}, we obtain, by the IS definition,

$$EU_{\gamma_1}(O) \geq EU_{\gamma_1}((O_{-\gamma_1}, \nu_{\gamma_1})),$$

where $O = y_{t_\eta}$, and then

$$u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(O) \geq u_{\gamma_1 \nu_{\gamma_1}} z_{\nu_{\gamma_1}}((O_{-\gamma_1}, \nu_{\gamma_1})). \tag{9.25}$$

Now, by Lemma 2, at the door time t_η, the inequality $t_{\nu_\eta} < t_\eta$ holds. Thus, by property (E), we get

$$z_{\nu_\eta}(O) = z_{\nu_\eta}(x^n). \tag{9.26}$$

By (9.26), $u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(x^n) = u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(O)$. Then, by (9.24) and (9.25), we get

$$u_{\gamma_1 \nu_{\gamma_1}} z_{\nu_{\gamma_1}}(x^n) > u_{\gamma_1 \nu_\eta} z_{\nu_\eta}(O) \geq u_{\gamma_1 \nu_{\gamma_1}} z_{\nu_{\gamma_1}}((O_{-\gamma_1}, \nu_{\gamma_1})), \tag{9.27}$$

which yields

$$n_{\nu_{\gamma_1}}(x^n) < n_{\nu_{\gamma_1}}(O) + 1,$$

and then

$$n_{\nu_{\gamma_1}}(x^n) \leq n_{\nu_{\gamma_1}}(O). \tag{9.28}$$

But, by property (D), $n_j(t)$ is nondecreasing function in t for all $j \in C$. Thus, it must hold that

$$n_{\nu_{\gamma_1}}(x^n) \geq n_{\nu_{\gamma_1}}(O),$$

which, together with (9.28), gives us

$$n_{\nu_{\gamma_1}}(x^n) = n_{\nu_{\gamma_1}}(O), \tag{9.29}$$

which implies that $t_{\nu_{\gamma_1}} < t_\eta$, and then the following must hold

$$t_{\nu_{\gamma_1}} < t_\eta < t_{\gamma_1}.$$

Now, by property (G), there must exist an agent, denoted γ_2, who holds the ticket of ν_{γ_1} at the door time t_{γ_1}, but not a ticket of a different community, ν_{γ_2}, say, in the lottery allocation x^n. Thus, we have $t_{\gamma_2} > t_{\gamma_1}$, which implies that γ_2 is in the third cycle Ω_3 because Ω_2 is singleton. This completes the proof of Case (β).

□

Proof of Theorem 7. Let $B_i = 1$ be the budgets for all $i \in A$ and $h_j = \frac{1}{z_j(x^*)}$ for $j \in C$ be the prices. For the null commodity, let its price h_{m+1} be zero. Now consider agent i's optimization problem in the choice of random allocations, as if all goods are divisible, under his budget constraint:

$$\max_{(p_{i1},p_{i2},\cdots,p_{im},p_{i(m+1)})} \sum_{j=1}^{m+1} p_{ij}u_{ij}$$

$$\text{subject to} \sum_{j=1}^{m+1} p_{ij}h_j = 1.$$

For all $i \in A$ and $j \in C \cup \{m+1\}$, let

$$p^*_{ij} = \begin{cases} z_j(x^*) & \text{if } x^*_{ij} = 1, j \neq m+1; \\ 1 - z_j(x^*) & \text{if } x^*_{ij} = 1, j = m+1; \\ 0 & \text{if } x^*_{ij} = 0. \end{cases}$$

We claim that $[p^*_{ij}]$ satisfies

a). the budget constraints for all agents i: For all $i \in A$,

$$\sum_{j=1}^{m+1} p^*_{ij} \times h_j = \frac{1}{z_j(x^*)}(z_j(x^*)) + 0 = 1;$$

b). the market clearing condition:

$$\sum_{i=1}^{n} p^*_{ij} = q_j;$$

c). the optimization condition where each agent maximizes his expected utility subject to his budget constraint: Because x^* is SEF, we have, for all i, j such that $x^*_{ij} \neq 1$ and $x^*_{ij^*} = 1$,

$$u_{ij^*}z_{j^*}(x^*) > u_{ij}z_j(x^*). \tag{9.30}$$

Let $P' = [p'_{ij}]$ be a random assignment that satisfies the budget constraints

$$\sum_{j \in C} p'_{ij}h_j = 1, \quad \forall i \in A.$$

Denote $p'_{ij}h_j = o_{ij}$. Then we have

$$\sum_{j\in C} o_{ij} = 1, \quad \forall i \in A. \tag{9.31}$$

Thus, the expected utility of agent i under the random assignment P' is given by

$$EU_i(P') = \sum_{j\in C} u_{ij}p'_{ij} = \sum_{j\in C} u_{ij}\frac{1}{h_j}o_{ij} = \sum_{j\in C} u_{ij}z_j(x^*)o_{ij}.$$

Replacing $u_{ij}z_j(x^*)$ with $u_{ij^*}z_{j^*}(x^*)$ in the above, and then using inequality (9.30) and equality (9.31), we have, $\forall i \in A$,

$$\sum_{j\in C} u_{ij}z_j(x^*)o_{ij} \leq u_{ij^*}z_{j^*}(x^*).$$

Thus, we have $EU_i(P^*) \geq EU_i(P')$ for all $i \in A$. Moreover, the equality holds only if $P' = P^*$. The uniqueness follows. Therefore, $P(x^*) = P_{HZ}$. This completes the proof. □

9.2. Tickets Algorithm versus Lottery Game in Example 2

Example 2 (Continued). In Table 6, we present the Tickets algorithm for the remaining 21 orders in Example 2. In Table 7, in the game of Example 2, we denote agent i by A_i, to avoid confusion. Agent A_1 is the row player, agent A_2 is the column player, agent A_4 is the matrix player, and agent A_3 is the matrix group player. There are two pure Nash equilibria $s = (C_1, C_2, C_2, C_3)$ and $s^* = (C_3, C_2, C_2, C_1)$, with equilibrium payoffs $(2, \frac{1}{2}, \frac{1}{2}, 2)$ and $(3, \frac{1}{2}, \frac{1}{2}, 3)$, respectively. s^* is SNE while s is NE but not SNE (Konishi et al., 1997). We find out that our Tickets algorithm can reach the only SNE s^* no matter which order has been used, as shown in the table.

Table 6: Tickets Algorithm for Example 2

Order steps	(1,3,4,2)	(1,4,2,3)	(1,4,3,2)
1	$(C_2, , ,)$	$(C_2, , ,)$	$(C_2, , ,)$
2	$(C_2, ,C_2,)$	$(C_2, , ,C_1)$	$(C_2, , ,C_1)$
	$(C_3, ,C_2,)$		
3	$(C_3, ,C_2,C_1)$	$(C_2,C_2, ,C_1)$	$(C_2, ,C_2,C_1)$
		$(C_3,C_2, ,C_1)$	$(C_3, ,C_2,C_1)$
4	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)
Order steps	(2,1,3,4)	(2,1,4,3)	(2,3,1,4)
1	$(,C_2, ,)$	$(,C_2, ,)$	$(,C_2, ,)$
2	$(C_3,C_2, ,)$	$(C_3,C_2, ,)$	$(,C_2,C_2,)$
3	$(C_3,C_2,C_2,)$	$(C_3,C_2, ,C_1)$	$(C_3,C_2,C_2,)$
4	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)
Order steps	(2,3,4,1)	(2,4,1,3)	(2,4,3,1)
1	$(,C_2, ,)$	$(,C_2, ,)$	$(,C_2, ,)$
2	$(,C_2,C_2,)$	$(,C_2, ,C_1)$	$(,C_2, ,C_1)$
3	$(,C_2,C_2,C_1)$	$(C_3,C_2, ,C_1)$	$(,C_2,C_2,C_1)$
4	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)
Order steps	(3,1,2,4)	(3,1,4,2)	(3,2,1,4)
1	$(, ,C_2,)$	$(, ,C_2,)$	$(, ,C_2,)$
2	$(C_3, ,C_2,)$	$(C_3, ,C_2,)$	$(,C_2,C_2,)$
3	$(C_3,C_2,C_2,)$	$(C_3, ,C_2,C_1)$	$(C_3,C_2,C_2,)$
4	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)
Order steps	(3,2,4,1)	(3,4,1,2)	(3,4,2,1)
1	$(, ,C_2,)$	$(, ,C_2,)$	$(, ,C_2,)$
2	$(,C_2,C_2,)$	$(, ,C_2,C_1)$	$(, ,C_2,C_1)$
3	$(,C_2,C_2,C_1)$	$(C_3, ,C_2,C_1)$	$(,C_2,C_2,C_1)$
4	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)
Order steps	(4,1,2,3)	(4,1,3,2)	(4,2,1,3)
1	$(, , ,C_1)$	$(, , ,C_1)$	$(, , ,C_1)$
2	$(C_2, , ,C_1)$	$(C_2, , ,C_1)$	$(,C_2, ,C_1)$
3	$(C_2,C_2, ,C_1)$	$(C_2, ,C_2,C_1)$	$(C_3,C_2, ,C_1)$
	$(C_3,C_2, ,C_1)$	$(C_3, ,C_2,C_1)$	
4	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)
Order steps	(4,2,3,1)	(4,3,1,2)	(4,3,2,1)
1	$(, , ,C_1)$	$(, , ,C_1)$	$(, , ,C_1)$
2	$(,C_2, ,C_1)$	$(, ,C_2,C_1)$	$(, ,C_2,C_1)$
3	$(,C_2,C_2,C_1)$	$(C_3, ,C_2,C_1)$	$(,C_2,C_2,C_1)$
4	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)	(C_3,C_2,C_2,C_1)

Table 7: Tickets Algorithm for Example 2

A_3

C_1,

A_1 \ A_2	C_1	C_2	C_3
C_1	$(\frac{1}{2},0,0,\frac{3}{4})$	$(\frac{4}{3},1,0,1)$	$(\frac{4}{3},0,0,1)$
C_2	$(4,0,0,1)$	$(2,\frac{1}{2},0,\frac{3}{2})$	$(4,0,0,\frac{3}{2})$
C_3	$(3,0,0,1)$	$(3,1,0,\frac{3}{2})$	$(\frac{3}{2},0,0,\frac{3}{2})$

A_4: C_1

A_1 \ A_2	C_1	C_2	C_3
C_1	$(\frac{2}{3},0,0,0)$	$(1,\frac{1}{2},0,0)$	$(1,0,0,0)$
C_2	$(2,0,0,0)$	$(\frac{4}{3},\frac{1}{3},0,0)$	$(2,0,0,0)$
C_3	$(3,0,0,0)$	$(3,\frac{1}{2},0,0)$	$(\frac{3}{2},0,0,0)$

A_4: C_2

A_1 \ A_2	C_1	C_2	C_3
C_1	$(\frac{2}{3},0,0,2)$	$(1,1,0,2)$	$(1,0,0,1)$
C_2	$(4,0,0,2)$	$(2,\frac{1}{2},0,2)$	$(4,0,0,1)$
C_3	$(\frac{3}{2},0,0,1)$	$(\frac{3}{2},1,0,2)$	$(1,0,0,\frac{2}{3})$

A_4: C_3

C_2,

A_1 \ A_2	C_1	C_2	C_3
C_1	$(\frac{2}{3},0,1,1)$	$(1,\frac{1}{2},\frac{1}{2},\frac{3}{2})$	$(1,0,1,\frac{3}{2})$
C_2	$(2,0,\frac{1}{2},\frac{3}{2})$	$(\frac{4}{3},\frac{1}{3},\frac{1}{3},3)$	$(2,0,\frac{1}{2},3)$
C_3	$(3,0,1,\frac{3}{2})$	$\boldsymbol{(3,\frac{1}{2},\frac{1}{2},3)}$	$(\frac{3}{2},0,1,3)$

A_4: C_1

A_1 \ A_2	C_1	C_2	C_3
C_1	$(1,0,\frac{1}{2},0)$	$(2,\frac{1}{3},\frac{1}{3},0)$	$(2,0,\frac{1}{2},\frac{1}{2})$
C_2	$(\frac{4}{3},0,\frac{1}{3},0)$	$(1,\frac{1}{4},\frac{1}{4},0)$	$(\frac{4}{3},0,\frac{1}{3},0)$
C_3	$(3,0,\frac{1}{2},0)$	$(3,\frac{1}{3},\frac{1}{3},0)$	$(\frac{3}{2},0,\frac{1}{2},0)$

A_4: C_2

A_1 \ A_2	C_1	C_2	C_3
C_1	$(1,0,1,2)$	$\boldsymbol{(2,\frac{1}{2},\frac{1}{2},2)}$	$(2,0,1,1)$
C_2	$(2,0,\frac{1}{2},2)$	$(\frac{4}{3},\frac{1}{3},\frac{1}{3},2)$	$(2,0,\frac{1}{2},1)$
C_3	$(\frac{3}{2},0,1,1)$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},2)$	$(1,0,1,\frac{2}{3})$

A_4: C_3

C_3,

A_1 \ A_2	C_1	C_2	C_3
C_1	$(\frac{2}{3},0,0,1)$	$(1,1,0,\frac{3}{2})$	$(1,0,0,\frac{3}{2})$
C_2	$(4,0,0,\frac{3}{2})$	$(2,\frac{1}{2},0,3)$	$(4,0,0,3)$
C_3	$(\frac{3}{2},0,0,\frac{3}{2})$	$(\frac{3}{2},1,0,3)$	$(1,0,0,3)$

A_4: C_1

A_1 \ A_2	C_1	C_2	C_3
C_1	$(1,0,0,0)$	$(2,\frac{1}{2},0,0)$	$(2,0,0,0)$
C_2	$(2,0,0,0)$	$(\frac{4}{3},\frac{1}{3},0,0)$	$(2,0,0,0)$
C_3	$(\frac{3}{2},0,0,0)$	$(\frac{3}{2},\frac{1}{2},0,0)$	$(1,0,0,0)$

A_4: C_2

A_1 \ A_2	C_1	C_2	C_3
C_1	$(1,0,0,1)$	$(2,1,0,1)$	$(2,0,0,\frac{2}{3})$
C_2	$(4,0,0,1)$	$(2,\frac{1}{2},0,1)$	$(4,0,0,\frac{2}{3})$
C_3	$(1,0,0,\frac{1}{3})$	$(1,1,0,\frac{1}{3})$	$(\frac{3}{4},0,0,\frac{1}{2})$

A_4: C_3

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