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GOLDEN RULE IN COOPERATIVE COMMONS

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ABSTRACT

This paper considers common use of natural, renewable resources. It identifies good prospects for efficiency and welfare. To be precise, a core outcome –hence cooperation – can be secured *over time* by principal planning of total quotas, and *in time* by agents who share these in short-term markets. Information flows in two directions: to the principal as market *prices* and from him as total *quantities*. Of particular interest is eventual convergence to a golden-rule, steady state.

Keywords: Renewable resources, commons, golden rule.

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1. INTRODUCTION

ECONOMIC THEORY, like *political science*, has long regarded use of common resources as largely determined by self-concerned individuals. Then, if such parties see no proper rights or frames, outcomes have often been grossly inefficient.

It's fortunate therefore that the two fields increasingly note, and often criticise, lack of mitigating or welfare-enhancing institutions (March & Olsen, 1989). Both stress that dedicated agencies ought balance competition against coordination and conflict against cooperation. Clearly, many instances are overwhelmingly complex. Yet *two* major features stand out. *First*, since nobody can attend to everything, competencies must be divided and responsibilities delineated. *Second*, for efficiency, short-term concerns must comply with those of the long run.

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Addressing these features, this paper argues for a two-level splitting of roles and tasks. At the upper level, a directorate or *principal* sets aggregate quotas in *macro and over time*. At the lower level, legitimate *agents* recurrently use short-term allowances in *micro and time.* Between the two levels, and between the agents, market-like mechanisms channel information, determine prices, and allocate quantities.

Thus the principal cares, in the aggregate, for the natural resources, their abundance, dynamics and sustainable yields. Lacking knowledge as to minor details, mainly of an economic or operative nature, that agency must leave short-term calculations and small-scale decisions to *resource users*. Each among the latter parties, applying their own quota and technology, appropriates some resource rent or profit, period after period. Set up this way, the organizational design divorces central governance from decentralized discretion. Yet, as argued below, it can coordinate choices and elicit necessary information.

The renewable resources, considered here, are confined to a "region" or local community. By assumption, property and user rights therein are of intermediary nature, between *sole ownership* and *open access*, but regulated or well defined.[1] Multi-agent fishery and forestry are cases in point.[2] Suppose that legitimate parties are so few and foresighted as to honour short and long term commitments. On these premises, the questions addressed here are: *Can the users share resources efficiently and fairly in and across time? Can they operate throughout as would a well informed, highly competent syndicate?*

This paper answers these questions in the positive. Broadly, it decouples long-term control from short-term choices.[3] Further, it aligns the principal's foresight with the agents' myopia. And finally, it marries centralized planning to decentralized sharing. In other terms, a directorate governs an open-ended cooperative game *in the large*. Play unfolds, however, *in the small*, among resource users, like a sequence of competitive equilibria. At each stage, money makes for transferable utility, and it facilitates exchange of quotas or information.

The motivation comes from the prospects for improving efficiency. To that end, the paper goes far by way of decomposition across governance, parties,

[1] Establishing (or enforcing) rights is important, but not an issue discussed here.

[2] See "*Rainforests need laws, not saws*" in The Economist, 4th-10th March, 2023.

[3] Single-agent control dominates parts of the received literature (Gale, 1967; Majumdar & Roy, 2009; Peleg & Ryder, 1974; Rockafellar, 1976).

resources, species and times. The framing fits a resource-based society in which rights are respected, supervised and well defined. There is centralized control of aggregate take-outs. The economy moves iteratively, by efficient sharing of single-period quotas, and it heads towards a long-run steady state. Quota allocations are determined *as if* supported by market-clearing, price-taking behaviour (Flåm, 2020).[4] Consequently, all along the system trajectory, competitive equilibria prevail in quota markets, hence so do Pareto optimality and core solutions as well.[5]

These novelties appear importnt. They indicate constitutional and institutional reforms. However, the optic doesn't directly or easily fit public cost-benefit analysis (Millner & Heal, 2018).

This paper addresses economists concerned with management and use of renewable resources.[6] The main thrusts and novelties are *threefold*: *First*, invoking convex preferences, convoluted criteria and money, it indicates, by way of decomposition, good prospects for overall efficiency.[7] *Second*, it emphasises the chief roles of agencies: decentralized trade of quotas unfolds besides centralized programming of aggregate take-outs. *Third*, the paper adds to the single-agent, single-stock, institution-free setting which dominates much received literature.

The rest of the paper goes as follows. Section 2 presents preliminaries. Section 3 considers single-period sharing of total take-out. It is organized around efficient, *short-term allocation,* implemented and held up by shadow pricing of available resources. Such allocation is realized period after period. Most important, the outcomes form a sequence of contingent *competitive equilibria*.
Section 4 introduces the ecosystem, its dynamics and principal manager. Being science-based, the latter casts planning as a syndicated problem of dynamic programming. Section 5 considers a long-run steady state: a *golden rule in cooperative commons*. Section 6 concludes.

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[4] Transactions can, but need not, be mediated by markets. Efficiency is described here *as though* facilitated and upheld by clearing prices. In terms of strategic behavior, single-period equilibrium might emerge as a Coase or "folk" theorem (Dutta, 1995).

[5] In principle, there is *open access* to quota markets but *not* directly to the natural resources at hand. Resource rent won't be dissipated but rather restored and shared.

[6] It connects to game theory, optimization and operation research, but no links are spelled out.

[7] For brevity and readability - and for political science - technical prerequisites are minimal.

2. PRELIMINARIES

Each vector space is real, finite-dimensional - with standard partial order $\leq$, "dot $\cdot$" inner product, and associated norm $\|\cdot\|$. Multi-dimensional *quotas* form such a space $\mathbb{Q}$.[8] Any vector *"quantity"* $q \in \mathbb{Q}$ is seen as a bundle of consumption (or harvest), taken out, during a single season, from one and the same local habitat of natural, renewable resources.

$q^* \in \mathbb{Q}^*$ is shorthand for a linear *price* system on $\mathbb{Q}$. Thus, under price-taking behavior, take-out $q \in \mathbb{Q}$ is worth $q^*q := q^*(q) := q^* \cdot q \in \mathbb{R}$.

Each economic agent, considered below, operates with a *pecuniary payoff function* $\pi : \mathbb{Q} \to \mathbb{R} \cup \{-\infty\}$ of his own. The value $\pi(q) = -\infty$ serves as a conceptual but convenient device to account for implicit constraints. It precludes infeasible or nonsensical choice q. Declare a payoff function $\pi(\cdot)$ *proper* iff finite-valued somewhere, meaning $dom(\pi) := \{q \mid \pi(q) \in \mathbb{R}\}$ is non-empty.

As usual, optimality conditions invoke differentiability. That notion is generalized here - and global in nature:

Definition 1. (Generalized derivatives) *A payoff criterion $\pi : \mathbb{Q} \to \mathbb{R} \cup \{-\infty\}$ has a **supgradient** $q^* \in \mathbb{Q}^*$ at $q \in \mathbb{Q}$, written $q^* \in \partial\pi(q)$, iff q maximizes the function $\hat{q} \mapsto \pi(\hat{q}) - q^*\hat{q}$ with finite value. The set $\partial\pi(q)$ of all such supgradients is called the **supdifferential**.*

$\pi(q) = -\infty$ makes $\partial\pi(q)$ empty. Otherwise, provided $\pi(q)$ be finite,

$$q^* \in \partial\pi(q) \Longleftrightarrow \pi(\hat{q}) \leq \pi(q) + q^*(\hat{q} - q) \text{ for all } \hat{q} \in \mathbb{Q}. \tag{1}$$

By (1), the set $\partial\pi(q)$ is closed convex. When a concave $\pi(\cdot)$ is classically differentiable, $\partial\pi(q)$ reduces to the customary gradient $\pi'(q)$. For both practical computation and realism, it is important though, to accommodate non-smooth instances:

Example 1. (Generating payoff by programs) *Suppose that a resource user, while holding bundle $q \in \mathbb{Q}$ as input, faces a linear primal problem:*

$$\pi(q) := \max_y \{y^*y \mid Ay \leq q \ \&\ y \in \mathbb{Y}_+\},$$

presumed solvable with finite optimal value. His output y belongs to a Euclidean space $\mathbb{Y}$, and $y^* \in \mathbb{Y}^*$ prices output linearly. The matrix A maps $\mathbb{Y}$

[8] In geometrical terms, $\mathbb{Q}$ is Euclidean.

into $\mathbb{Q}$. If A^* denotes the transposed matrix, a price $q^* \in \mathbb{Q}^*$, not necessarily unique, belongs to $\partial\pi(q)$ iff it solves the dual problem:

$$\pi(q) = \min_{q^*} \left\{ q^* q \;\middle|\; A^* q^* \geq y^* \;\&\; q^* \in \mathbb{Q}^*_+ \right\}. \tag{2}$$

For any subset $\mathscr{R}$ of the real numbers $\mathbb{R}$, its *supremum*, denoted $\sup \mathscr{R}$ is the smallest $\bar{r} \in \mathbb{R} \cup \{\pm\infty\}$ which is $\geq$ each $r \in \mathscr{R}$. [By convention here, $\sup \varnothing = -\infty$.] *Infimum*, denoted $\inf \mathscr{R}$, equals $-\sup(-\mathscr{R})$. The following derives from elementary convex analysis:

Proposition 1. (On price-taking profit) *Suppose that a resource user, who has a proper and pecuniary payoff function $\pi : \mathbb{Q} \to \mathbb{R} \cup \{-\infty\}$, faces cost regime $q^* \in \mathbb{Q}^*$ when taking out any $q \in \mathbb{Q}$.*
• *Then, his (price-taking)* ***competitive profit function*** *is given by*

$$q^* \in \mathbb{Q}^* \mapsto \pi^*(q^*) := \sup \left\{ \pi(q) - q^* q \mid q \in \mathbb{Q} \right\}, \tag{3}$$

is closed[9] *and convex;*
• *The supremum $\pi^*(q^*)$ in (3) is attained as maximum at q iff $q^* \in \partial\pi(q)$. Then, $q^* \in \partial\pi(q) \Longleftrightarrow \pi(q) = \pi^*(q^*) + q^* q$, meaning that payoff $\pi(q)$ splits into profit $\pi^*(q^*)$ plus competitive factor cost $q^* q$;*
• *If the payoff function $\pi(\cdot)$ is concave and bounded below near $q \in \mathbb{Q}$, the supdifferential $\partial\pi(q)$ is non-empty;*
• *Reasonably, $\pi(0) \geq 0$, and in that case, $\pi^*(q^*) \geq 0$ for any $q^* \in \mathbb{Q}^*$;*
• *If the agent faces user cost q^*, and holds private "property" $\underline{q} \in \mathbb{Q}$, he may aim at taking home $\pi^*(q^*) + q^* \underline{q} = \pi(q) + q^*(\underline{q} - q)$, composed of pure profit $\pi^*(q^*)$ plus "resource rent" $q^* \underline{q}$.*

3. EFFICIENCY WITHIN SEASON

The paper attempts to decompose a *long-term* problem of resource management into *short-term* parts. This section begins by considering allocation of the seasonal total quota(s) across legitimate parties. Such quotas are presumed perfectly divisible, marketable and transferable - with no externalities, fees or frictions.

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[9] It's *lower semicontinuous*, meaning that $\{q^* \in \mathbb{Q}^* \mid \pi^*(q^*) \leq r\}$ comes closed for each $r \in \mathbb{R}$.

Emphasis is here on efficient sharing during any fixed season (Flåm, 2020). An impatient reader, chiefly concerned with intertemporal allocation and long run, may first skip this section and later return to it.

Accommodated henceforth is a fixed finite ensemble $\mathscr{I}$ of economic agents, $\#\mathscr{I} \geq 2$, construed as resource users. In each period, member $i \in \mathscr{I}$ applies the same proper, pecuniary *payoff function* $\pi_i : \mathbb{Q} \to \mathbb{R} \cup \{-\infty\}$, and he targets maximal profit.

Absent externalities *in season*, arguments as to short-term *allocative efficiency* will revolve around *convolution* (4) of individual payoffs:

Definition 2. (Supremal convolution) *If each $i \in \mathscr{I}$ has a proper payoff function $q_i \in \mathbb{Q} \mapsto \pi_i(q_i) \in \mathbb{R} \cup \{-\infty\}$, their sup-convoluted payoff function is defined by*

$$q_{\mathscr{I}} \in \mathbb{Q} \mapsto \pi_{\mathscr{I}}(q_{\mathscr{I}}) := \sup_{(q_i)} \left\{ \sum_{i \in \mathscr{I}} \pi_i(q_i) \;\middle|\; \sum_{i \in \mathscr{I}} q_i = q_{\mathscr{I}} \right\}. \tag{4}$$

If some $\pi_i(\cdot)$ increases on $dom\pi_i$, then

$$\pi_{\mathscr{I}}(q_{\mathscr{I}}) = \sup_{(q_i)} \left\{ \sum_{i \in \mathscr{I}} \pi_i(q_i) \;\middle|\; \sum_{i \in \mathscr{I}} q_i \leq q_{\mathscr{I}} \right\}.$$

With each $\pi_i(\cdot)$ (quasi-)concave, $\pi_{\mathscr{I}}(\cdot)$ also becomes (quasi-)concave.

Proposition 2. (On efficient allocation by equal margins (Flåm, 2020, 2021b)) *For any best choice (q_i) in (4) it holds*

$$\partial \pi_{\mathscr{I}}(q_{\mathscr{I}}) \subseteq \cap_{i \in \mathscr{I}} \partial \pi_i(q_i).$$

Conversely, provided $q_{\mathscr{I}} = \sum_{i \in \mathscr{I}} q_i$, it also holds the turned-around inclusion:

$$\partial \pi_{\mathscr{I}}(q_{\mathscr{I}}) \supseteq \cap_{i \in \mathscr{I}} \partial \pi_i(q_i).$$

If moreover, $\cap_{i \in \mathscr{I}} \partial \pi_i(q_i)$ is non-empty, then (q_i) solves (4).

In this section, but this only, function $\pi_{\mathscr{I}}(\cdot)$ (4) is mainly a conceptual construct, serving analysis. No member $i \in \mathscr{I}$ states problem (4). Indeed, maybe not knowing more than his own criterion $\pi_i(\cdot)$ - or, seeing just a local approximation to it - he can neither identify $\pi_{\mathscr{I}}(\cdot)$ nor solve any particular instance $\pi_{\mathscr{I}}(q_{\mathscr{I}})$ (4). It suffices though, that the agents together solve problem (4) themselves; see Remark 2 below.

It is worth mentioning that Proposition 2 presumes no concavity of objectives. It just needed that $\partial \pi_{\mathscr{I}}(q_{\mathscr{I}})$ be non-empty.[10]

Added here are some results on *Pareto optimality*, *core outcome* and *competitive equilibrium*. As is well known, these concepts connect to theory on games and economic welfare (Luenberger, 1995; Osborne & Rubinstein, 1994). Their relevance for resource economics appears less noticed:

Proposition 3. (On single-season Pareto optimality, core solution and competitive equilibrium (Flåm, 2021b, 2023))

• *Any solution (q_i) to (4) is a **Pareto optimal** allocation of $q_{\mathscr{I}}$.*

• *If agent $i \in \mathscr{I}$ already "owns" $\underline{q}_i$, "coalition" $I \subseteq \mathscr{I}$ can - by going alone, in autarky - aim at no less joint payoff than*

$$\underline{\pi}_I(\underline{q}_I) := \sup_{(q_i)} \left\{ \sum_{i \in I} \pi_i(q_i) \;\middle|\; \sum_{i \in I} q_i = \sum_{i \in I} \underline{q}_i =: \underline{q}_I \right\}.$$

*So, for any **shadow price** $q^* \in \partial \pi_{\mathscr{I}}(q_{\mathscr{I}})$ on actual resource use, the cash payment profile $i \in \mathscr{I} \mapsto \kappa_i(q^*) := \pi_i^*(q^*) + q^* \underline{q}_i$ constitutes a **core solution** in that*

$$\sum_{i \in I} \kappa_i(q^*) \geq \underline{\pi}_I(\underline{q}_I)$$

for each $I \subset \mathscr{I}$ with equality for $I = \mathscr{I}$ and $\underline{q}_{\mathscr{I}} = q_{\mathscr{I}}$.

• *Still suppose agent $i \in \mathscr{I}$ "owns" $\underline{q}_i$ with $\sum_{i \in \mathscr{I}} \underline{q}_i = \underline{q}_{\mathscr{I}}$. Then, any shadow price $q^* \in \partial \pi_{\mathscr{I}}(\underline{q}_{\mathscr{I}})$, alongside any optimal allocation (q_i) of $\underline{q}_{\mathscr{I}} = q_{\mathscr{I}}$ to (4) constitutes a **competitive equilibrium** in that the quota market clears, and agent i takes home maximal total profit $\pi_i^*(q^*) + q^* \underline{q}_i$.*[11]

Clearly, activities, quotas q_i or rents $q^* \underline{q}_i$ can be distributed rather unevenly.

Concluding this section is a summary on how resource users enter a season and proceed therein:

[10] For that, presence of many and minor agents may help. While (4) preserves concavity, it may also contribute to "create" it.

[11] These items could - or ought - be taxed.

Assumption 1. (On sharing of seasonal quotas)
(I) At the very beginning of a season, ***prior to any trade of quotas****, the principal sets a total quota $q_{\mathscr{I}}$. Immediately thereafter $q_{\mathscr{I}}$ is split among legitimate users by some time-invariant rule*

$$i \in \mathscr{I},\ q_{\mathscr{I}} \in \mathbb{Q} \mapsto \underline{q}_i = Q_i(q_{\mathscr{I}}), \quad \sum_{i \in \mathscr{I}} Q_i(q_{\mathscr{I}}) = q_{\mathscr{I}}. \tag{5}$$

(II) Subsequently, but ***prior to any use of quotas****, the said users settle on a profile $i \in \mathscr{I} \mapsto q_i \in \mathbb{Q}$ which solves (4).*

Remark 1. (On allotted shares) $\underline{q}_i$ in (5) can reflect grand-fathered or traditional rights; see Flåm (2020). If $\underline{q}_i \neq 0$ and $q_i = 0$, agent i just owns rights, but uses none. He only collects rent. Conversely, if $\underline{q}_i = 0$ and $q_i \neq 0$, being "propertyless," he fully rents his user rights. When $\underline{q}_i \neq 0$ and $q_i \neq 0$, agent i acts in twin capacities: as owner and user. Part (I) of Assumption 1 is *constitutional*. It relates to established law, presumed here, but not discussed.

Remark 2. (On allocation of allotted shares) Part (II) of Assumption 1 is *institutional*. It points to auctions, bargaining, barter *quid pro quo*, direct deals or markets. These institutions or platforms may help agents to solve (4) by themselves, but no mechanism is singled out or modelled here; see Flåm (2021b, 2023).[12]

4. EFFICIENCY ACROSS SEASONS

Henceforth, suppose quotas will be traded, *in each season*, for money, up to price-supported Pareto efficiency (Proposition 2) - in fact, up to single-period, competitive equilibrium (Proposition 3). On that premise, this section considers how a harvest might be allocated *across seasons*?

The ecosystem comprises a finite set S of renewable resources or species $s \in S$. Let the vector space $\mathbb{X} = \mathbb{R}^S$ comprise all bundles $x = (x_s)$. A *system state* $x = (x_s) \in \mathbb{X}_+$ informs about the actual biomass $x_s \geq 0$ of each species $s \in S$. That state remains in some rectangle set $X := [\underline{x}, \bar{x}] \subset \mathbb{X}$, bounded by sustainable stock levels $0 \leq \underline{x}_s < \bar{x}_s$, $s \in S$.

[12] Circumvented here are issues on control of prices versus quantities (Weitzman, 1974).

Natural growth $x \in \mathbb{X}_+ \mapsto G_s(x) \in \mathbb{R}$ of species $s \in S$ - *after harvest, during the season* - is presumed concave. Let $G(x) := [G_s(x)]_{s\in S} \in \mathbb{X}$ and $g(x) := G(x) - x$.[13]

Most often, *the state affects payoffs.*[14] So, from here onwards, $\pi_i(x, q_i)$ takes the place of the above simplified version $\pi_i(q_i)$. By assumption, $(x, q_i) \mapsto \pi_i(x, q_i)$ is jointly concave. It's separately differentiable and increasing in $x \in X$, but maybe neither in q_i; see Example 1.

Naturally, for the quota space, introduced earlier, let $\mathbb{Q} := \mathbb{X}$, and posit $Q := X$ for the set of short-term total quotas.

Time $t \in T := \{0, 1, ..\}$ is discrete, with open horizon. An initial point $x_{-1} \in X$ is specified.

Assumption 2. (On the principal's long-term program) *Given initial point* $x_{-1} \in X$*, discount factor* $\delta \in (0,1)$*, and convoluted, single-period payoff function*

$$t \in T \mapsto (x_t, q_t) \in X \times Q \mapsto \pi(x_t, q_t) := \sup \left\{ \sum_{i\in\mathscr{I}} \pi_i(x_t, q_{it}) \;\middle|\; \sum_{i\in\mathscr{I}} q_{it} = q_t \right\}, \quad (6)$$

the principal will

$$\textit{maximize present value} \sum_{t\in T} \delta^t \pi(x_t, q_t) \;\; \textit{s.t.} \;\; x_{t+1} \le G(x_t) - q_t \;\; \forall t \in T. \quad (7)$$

Proposition 4. (Existence of optimal profiles) *Suppose single-period payoff* π *in (6) and growth* g *both be upper semicontinuous. If feasible, problem (7) has a best solution.*

Program (7) is an instance of deterministic, discrete-time optimal control. For discussion, let multiplier vector $\lambda_t \in \mathbb{R}^S_+$ value the time-t *excess* $G(x_t) - x_{t+1} - q_t \ge 0$. Thus emerges a *Lagrangian*

$$L(\mathbf{x}, \mathbf{q}, \lambda) := \sum_{t\in T} \delta^t \{\pi(x_t, q_t) + \lambda_t [G(x_t) - x_{t+1} - q_t]\}. \quad (8)$$

In (8), the *primal planning profiles* $\mathbf{x} := (x_t)$ and $\mathbf{q} = (q_t)$ both belong to the space l^∞ of bounded sequences in $\mathbb{X}$.[15] So, any *dual price profile* λ should

[13] Ecological interdependence is possible and quite likely. Reasonably, $x_s < \underline{x}_s \Longrightarrow g_s(x) \le 0$, whence the system collapses. (Moreover, $g_s(\cdot)$ needs not be concave when $x_s \in [0, \underline{x}_s]$; see Majumdar & Roy (2009) and references therein.) By contrast, $x_s > \bar{x}_s \Longrightarrow g_s(x) < 0$ so that viable biomasses and stocks stay bounded. If left alone, a pristine system would evolve by the law $\Delta x := x_{+1} - x = g(x)$.

[14] For instance, use a density-dependent matrix $A(x)$ in Example 1.

[15] In fact, each $(x_t, q_t) \in X \times Q$.

be a continuous linear mapping from l^{∞} into $\mathbb{R}$. As a matter of natural and reasonable modelling, take $\lambda = (\lambda_t)$ to be a member of the linear space

$$l^1 := \left\{ \lambda = (\lambda_t) \;\middle|\; \sum_{t \in T} \delta^t \|\lambda_t\| < +\infty \right\}. \tag{9}$$

It comprises precisely those price regimes $\lambda = (\lambda_t)$ that associate finite present value $\lambda \mathbf{x} := \sum_{t \in T} \delta^t \lambda_t x_t$ to any bounded sequence $\mathbf{x} = (x_t)$ in $\mathbb{X}$.[16]

For the subsequent argument, introduce the *Hamiltonian*

$$H(x, q, \lambda) := \pi(x, q) + \lambda [g(x) - q] \tag{10}$$

to account for aggregate payoff plus shadow pricing λ of net savings $g(x) - q$.

Theorem 1. (On the principal's long-term control) *Suppose $t \in T \mapsto (x_t, q_t) \in X \times Q$ solves (7). Then there is an adjoint, dual trajectory of shadow prices $t \in T \mapsto \lambda_t \in \mathbb{R}^S_+$ on resources such that for every $t \in T$ and (x_t, q_t) it holds*

$$\begin{cases} \text{the system dynamics:} & x_{t+1} & \leq & G(x_t) - q_t, \\ \text{the adjoint equation:} & \lambda_t - \delta^{-1} \lambda_{t-1} & \in & -\frac{\partial}{\partial x} H(x_t, q_t, \lambda_t), \text{ and} \\ \text{the maximum condition:} & q_t & \in & \arg\max H(x_t, \cdot, \lambda_t), \end{cases} \tag{11}$$

with complementarity $\lambda_t [G(x_t) - x_{t+1} - q_t] = 0$, and specified initial point x_{-1}. Moreover, at each time $t \in T$, the valuation vector $\lambda_t \in \mathbb{R}^S_+$ - on resources saved in situ - equals a market equilibrium price vector $q_t^* \in \mathbb{Q}^*$ on resources actually consumed. Thus, Assumption 1, part II, is satisfied with

$$\lambda_t = q_t^* \in \frac{\partial}{\partial q_t} \pi(x_t, q_t) = \cap_{i \in \mathscr{I}} \frac{\partial}{\partial q_{it}} \pi_i(x_t, q_{it}). \tag{12}$$

Proof. Recall Hamiltonian H (10) to rewrite the Lagrangian (8) as

$$L(\mathbf{x}, \mathbf{q}, \lambda) = \sum_{t \in T} \delta^t \{ H(x_t, q_t, \lambda_t) - \lambda_t (x_{t+1} - x_t) \}.$$

[16] By the *Riesz representation theorem* l^1 has l^{∞} as dual space, but not conversely (Dudley, 1989).

By $\frac{\partial}{\partial \lambda_t} L(\mathbf{x}, \mathbf{q}, \lambda) \geq 0$, the system dynamics hold for all $t \in T$. Also, the complementarity conditions come up as usual. Note that L features state x_t just *two* times. The corresponding terms are singled out next:

$$L(\mathbf{x}, \mathbf{q}, \lambda) = \cdots +$$

$$\delta^{t-1} \{H(x_{t-1}, q_{t-1}, \lambda_{t-1}) - \lambda_{t-1}(x_t - x_{t-1})\} +$$

$$\delta^t \{H(x_t, q_t, \lambda_t) - \lambda_t(x_{t+1} - x_t)\} + \cdots .$$

From this and $\frac{\partial}{\partial x_t} L(\mathbf{x}, \mathbf{q}, \lambda) = 0$, the adjoint equation follows. Also, because $L(\mathbf{x}, \mathbf{q}, \lambda)$ is additively separable with respect to $q_t, t \in T$, and should be maximized in each q_t, the maximum condition is immediate. Taken together, $\frac{\partial}{\partial q_t} H(x_t, q_t, \lambda_t) = 0$ and Proposition 2 give (12). □

Remark 3. (On qualified constraints) Normally, proper use of the Lagrangian requires some *constraint qualification*. Granted concave functions π and G, as here, the most convenient one - called the *Slater condition* - amounts to strict feasibility. Specifically, suppose the resource system be *productive*, meaning that some state $x \in X$ can be reached at which $G_s(x) > x_s$ for each $s \in S$.

Remark 4. (On value functions, differentiability and duality) Letting

$$\pi(x, x_{+1}) := \sup_q \{\pi(x, q) \mid x_{+1} \leq G(x) - q\},$$

the overall, joint program has optimal value function

$$x \mapsto V(x) := \sup_{\mathbf{x}} \sum_{t \in T} \delta^t \pi(x_t, x_{t+1}), \ \ x_0 = x.$$

Under standard conditions, it holds the Bellman equation

$$V(x) = \max_{x_{+1}} \{\pi(x, x_{+1}) + \delta V(x_{+1}) \mid x_{+1} \in X\},$$

and $\partial V(x) = \partial_x \pi(x, x_{+1})$ with supposedly unique and optimal continuation x_{+1} (Benveniste & Scheinkman, 1979).

5. GOLDEN RULE

Does problem (7) have a globally stable, steady state? For this, it's expedient that both functions π, G be *strongly concave*. For G, this means that $[g'(x)-g'(\tilde{x})]\cdot[x-\tilde{x}] \leq -\mu_g \|x-\tilde{x}\|^2$ with some modulus $\mu_g > 0$. Further, as stressed by Weitzman (1998), δ should be "large enough" - or tuned to - the associated moduli μ_π, μ_g of strong concavity (Rockafellar, 1976).[17] Here, for brevity, stability is simply presumed:

Assumption 3. (On asymptotic stability) *Optimal control leads to a* ***golden steady state*** $x = \lim_{t\to+\infty} x_t$ *- a long-term, fixed point of the system - presumed unique and well defined.*

To characterize the golden state, let

$$\mathscr{H}(x,\lambda) := \max\{H(x,q,\lambda) : q \in \mathbb{Q}\} \tag{13}$$

be the *reduced* Hamiltonian. Define its composite differential by

$$(x^*,\lambda^*) \in \partial\mathscr{H}(x,\lambda) := \partial_x\mathscr{H}(x,\lambda)\times\partial_\lambda\mathscr{H}(x,\lambda) \Longleftrightarrow$$

$$\begin{cases} \mathscr{H}(\hat{x},\lambda) \leq \mathscr{H}(x,\lambda) + x^*(\hat{x}-x) & \text{for all } \hat{x}\in\mathbb{X}, \\ \mathscr{H}(x,\hat{\lambda}) \geq \mathscr{H}(x,\lambda) + \lambda^*(\hat{\lambda}-\lambda) & \text{for all } \hat{\lambda}\in\Lambda. \end{cases}$$

The first inequality captures that $\mathscr{H}(x,\lambda)$ is concave in x. Hence ∂_x denotes a partial supdifferential. The second inequality tells that $\mathscr{H}$ is convex in λ.[18] In these terms, by Theorem 1 and (13), the controlled system moves by

$$x_{t+1} - x_t \in \partial_\lambda\mathscr{H}(x_t,\lambda_t) \quad \& \quad \lambda_t - \delta^{-1}\lambda_{t-1} \in -\partial_x\mathscr{H}(x_t,\lambda_t). \tag{14}$$

Hamiltonian dynamics (14) resemble classical studies of primal-dual programming. As said, stability and limit behavior link to strong concavity (Rockafellar, 1976), notably in state variable x (Hauswith et al., 2020), derived from growth G and payoffs π.

[17] Invoking uncertainty as to the far distant future, (Weitzman, 1998) argues that the largest possible discount factor should be used.

[18] Accordingly, $\partial_\lambda[\cdot] := -\partial_\lambda[-\cdot]$ is the partial *subdifferential*.

Proposition 5. (On existence of a steady state) *Under natural assumptions - on π, g being closed concave on compact, convex domains - system (14) has at least one fixed point (x, λ) corresponding to the stationary version*

$$[-(1-\delta^{-1})\lambda, 0] \in \partial \mathscr{H}(x, \lambda)$$

of (14) - with steady harvest $q = g(x)$, and discount factor $\delta = 1/(1+\rho)$ defined by interest rate $\rho > 0$. Further, it holds Hotelling's modified rule:

$$\rho\lambda \in \frac{\partial}{\partial x}\mathscr{H}(x, \lambda). \tag{15}$$

Proof. From (14), the convex-valued point-to-set correspondence $(x, \lambda) \rightrightarrows (x_+, \lambda_+)$ defined by

$$x_+ \in x + \partial_\lambda \mathscr{H}(x, \lambda) \quad \& \quad \lambda_+ \in \delta^{-1}\lambda - \partial_x \mathscr{H}(x, \lambda) \tag{16}$$

is well defined, with closed graph and non-empty values, on some compact convex suitably chosen state space $\Xi \subset \mathbb{X}_+ \times \mathbb{X}_+$. Each "value" - that is, each right hand side of (16) - intersects Ξ. Hence by Kakutani's theorem the said correspondence has a fixed point.[19] □

Proposition 6. (Decomposed golden rule) *The steady state x gives constant user cost $q^* \in \frac{\partial}{\partial q}H(x, q, \lambda)$, shadow price $\lambda = q^* \in \frac{\partial}{\partial x}H_x(x, q, \lambda)/\rho$ on resources saved, and fixed total take-out $g(x) = q \in \arg\max H(x, \cdot, \lambda)$. Any optimal allocation (q_i) of the stable aggregate take-out q solves*

$$\pi(x, q) := \sup_{(q_i)} \left\{ \sum_{i \in \mathscr{I}} \pi_i(x, q_i) \;\middle|\; \sum_{i \in \mathscr{I}} q_i = q \right\}. \tag{17}$$

Whatever price $q^* \in \partial_q \pi(x, q)\big|_{q=g(x)}$ on resource use, agent i gets price-taking maximal profit

$$\pi_i^*(x, q^*) := \sup_{\hat{q}_i} \{\pi_i(x, \hat{q}_i) - q^*\hat{q}_i \mid \hat{q}_i \in \mathbb{Q}\} = \pi_i(x, q_i) - q^* q_i.$$

If he can claim property or user right to $\underline{q}_i$, $\sum_{i \in \mathscr{I}} \underline{q}_i = g(x)$, he takes home overall profit $\pi_i^*(x, q^*) + q^*\underline{q}_i$. In the steady state, an owner of right $\underline{q}_i$ holds capital value $q^*\underline{q}_i/(1-\delta)$.

[19] For existence of a steady state, see also Peleg & Ryder (1974).

Returning to the non-reduced Hamiltonian $H(x,q,\lambda)$ (10), a steady solution (x,q,λ), with $\lambda \in \mathbb{R}^S_{++}$, features

$$\begin{cases} \textit{stationary stock levels:} & g(x) = q = \sum_{i \in \mathscr{I}} q_i, \\ \textit{Hotelling's rule:} & \rho\lambda \in \partial_x H(x,q,\lambda), \ \textit{and} \\ \textit{balanced shadow pricing:} & \lambda \in \partial_q \pi(x,q) \subseteq \cap_{i \in \mathscr{I}} \partial_{q_i} \pi_i(x,q_i). \end{cases} \tag{18}$$

If a function is differentiable, the inclusion sign in (18) should be replaced by equality.[20]

Clearly, with fixed agent ensemble $\mathscr{I}$, payoff functions π_i, state x, and seasonal quota q, convoluted outcome $\pi(x,q)$ (17) *is unaffected by the property rights* in Assumption 1 (I). However, for any specified state x, admission of extra agents, each with $\pi_i(x,0) \geq 0$ - that is, an enlargement of $\mathscr{I}$ - cannot but increase (or at least maintain) π. Thus, the incumbents might see potential entrants.

Ending this section are some second thoughts on the public cost-benefit criterion (7) - and on the principal's access to necessary information.

Remark 5. (On principal discounting) Suppose that agent $i \in \mathscr{I}$ prefers discount factor $\delta_i \in (0,1)$. If $\mathscr{D} := \{\delta_i \mid i \in \mathscr{I}\}$ reduces to a singleton, no issues emerge. Otherwise, many papers consider collective choices among time profiles (Chambers & Echenique, 2018; Harstad, 2020; Jackson & Yariv, 2015; Millner & Heal, 2018). Main objects of study there are social preference orders $\succsim$ on bounded flows $\mathbf{r} = (r_t)$ of some *single and common commodity*. This one-dimensional perspective may fit if the time-t common item $r_t \in \mathbb{R}$ denotes *joint* revenue - or, it represents the amount of some single resource. Here, however, monetary revenue $r_{it} = \pi_i(x_t, q_{it})$ is *private* - or, there are several resources. So, the setting doesn't directly fit *public* cost-benefit analysis.

It's desirable, though, that the principal's order be representative. For that, suppose agent $i \in \mathscr{I}$ enjoys increasing, twice continuously differentiable utility $u_i(r)$ of single-period revenue $r \in \mathbb{R}$. Then, in case all u_i are equal,

[20] System (18) captures directly the steady state. It would be vastly more challenging to find first, the entire value function $x \in X \mapsto V(x) = \max\{\pi(x,q) + \delta V(G(x) - q)\}$ by dynamic programming, invariably using convoluted payoff (17), and finally, to require that $g(x) = q$. Usually, computation requires discretization of spaces, whence clustering techniques.

Jackson & Yariv (2015) prove that the principal may use a discount factor $\delta = \sum_{i\in\mathscr{I}} w_i\delta_i$, with weights $(w_i) > 0$. summing to one. Thereby *time-consistency* obtains, and *unanimity* prevails, in that $\mathbf{r} \succsim \hat{\mathbf{r}} \Rightarrow \sum_t \delta_i^t u_i(r_t) \geq \sum_t \delta_i^t u_i(\hat{r}_t)$ for each $i \in \mathscr{I}$.

By contrast, when the single-period utility functions u_i differs, Jackson & Yariv (2015) show that time-consistency requires a dictatorial (or paternalistic) choice $\delta \in \mathscr{D}$.

For these reasons, following Weitzman (1998), patience speaks for itself. That is, a prudent principal could well choose $\delta = \max \mathscr{D}$.[21] This choice also fits concerns with stability, mentioned above.

Remark 6. (On principal information) For his planning the principal needs a firm grip on the function $(x,q) \mapsto \pi(x,q)$. Clearly, if each agent $i \in \mathscr{I}$ honestly hands in his function π_i - say, by a double auction - once and for all,[22] no problems emerge. Otherwise, the principal must somehow estimate, know or synthesize π in (4).

Ignored here, or taken as given, is strategic communication. Even when such mode of play is absent or unimportant, challenges remain. To illustrate, returning to Example 1, suppose agent i faces, time and again, the program form

$$\pi_i(x,q_i) := \sup\left\{y_i^* y_i \mid A_i(x)y_i \leq q_i \;\&\; y_i \in \mathbb{Y}_{i+}\right\}.$$

He knows $y_i^* \in \mathbb{Y}_i^*$ and the state-dependent "technology" $A_i(x) : \mathbb{Y}_i \to \mathbb{R}^S$. By contrast, if the principal knows or sees neither, he can hardly synthesize the corresponding criterion:

$$(x,q) \mapsto \pi(x,q) = \sup\left\{\sum_{i\in\mathscr{I}} y_i^* y_i \;\middle|\; \sum_{i\in\mathscr{I}} A_i(x)y_i \leq q \text{ and } y_i \in \mathbb{Y}_{i+}\right\}.$$

Clearly, his task simplifies when the stock x impacts no technology hence no payoff. In that case, the principal might use time series $t \mapsto [q_t,\ \pi(q_t),\ q_t^* \in \partial\pi(q_t)]$, observed so far - say, up to time τ - to overestimate $\pi(\cdot)$, step by step and "on line," as a one-sided, time-t envelope:

$$\text{For } \tau \in T \text{ and } q \in Q \text{ posit } \pi_\tau(q) := \min_{t\leq\tau}\left\{\pi(q_t) + q_t^*(q - q_t)\right\} \approx \pi(q),$$

[21] It's implicit here that rotation of (finitely lived) principals cause no time inconsistency (Harstad, 2020)

[22] Given any relevant $x \in X$, the function $q_i \mapsto \pi_i(x,q_i)$ could stem from parametric optimization on the part of agent i.

thus $\pi_\tau \geq \pi$.

6. CONCLUDING REMARKS

Considering management of common-property, renewable resources, this paper argues that three agencies can play chief and complementary roles.[23] At the upper level, a competent principal decides total take-outs over time. At the lower level, legitimate users share short-term aggregate quotas in time. Between the two levels, various mechanisms channel information and value resources.

This paper indicates that efficiency and stability may obtain. Indeed, it appears that private and public interests can be aligned, competitive equilibria being constituent components. To this end, the paper has assumed that

- information on payoffs is communicated to the principal by diverse market mechanisms;[24]
- user rights must be clearly defined, perfectly divisible, marketable and transferable (Flåm, 2020);
- qualified users operate, directly or indirectly, with monetary criteria (Flåm, 2021a; Luenberger, 1995);
- there are no single-period externalities and transaction costs;
- aggregate and optimal quotas obtain, via dynamic programming;
- and finally, discounting must be moderate (Rockafellar, 1976; Weitzman, 1998).

Then, modulo oversight and policing, a golden steady state may emerge as limit of short-term, competitive equilibria in quota markets. Clearly, questions remain as to convergence and stability. Others queries include: Who are qualified to which rights - and then, on what grounds (Flåm, 2020)? Who controls compliance or metes out penalties? When viewed as an integrated enterprise, might not the *principal* - or the greater public - tax resource rent?

Among other concerns, important but not considered here, *three issues* merit further study. *First*, do long-lived investments play crucial roles (Clark et al., 1979)?[25] *Second*, how and where should uncertainty be accounted for (Mitra & Roy, 2023; Stokey & Lucas, 1989)? *Third*, can eventual lack

[23] Likewise, *three* fields also enter: *Economic theory* on expediency of markets; *mathematical optimization* on relaxation methods, and *political science* on separation on powers.

[24] Double auctions appear most attractive (Flåm, 2021b).

[25] Ownership to highly specialized capital can be divorced from rental use.

of concavity hence occasional presence of increasing margins in payoffs or growth affect management considerably (Majumdar & Roy, 2009)?

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