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BAYESIAN BULLSHIT

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ABSTRACT

A bullshitter neither knows nor cares about the truth, and therefore, it has been asserted, is more pernicious than a liar. We examine this assertion within the standard model of cheap talk communication where a bullshitter is modeled as an uninformed Sender. We show that in some circumstances, uncertainty about whether the Sender is informed or not can increase the welfare of the Receiver.

Keywords: Bayesian persuasion, information design, cheap talk.

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1. INTRODUCTION

> Political language is designed to make lies sound truthful and murder respectable, and to give an appearance of solidity to pure wind.

George Orwell

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2. INTRODUCTION

A LIAR knows the truth and consciously seeks to conceal it. Liars are absent in the canonical economic model of communication: cheap talk. As Sobel (2020) points out, lying requires a common understanding of what words mean, which is not a feature of cheap talk models. To account for lying, Sobel (2020) adds the concept of a common language to a standard model of cheap talk. This allows one to interpret every equilibrium type-action distribution as involving lies of omission. The central finding is that fully anticipated lies don't interfere with the exchange of information.

To lie is not the same as to bullshit because a bullshitter, unlike a liar, neither knows nor cares about the truth. A bullshitter does care about concealing both their ignorance of and interest in the truth.

The Oxford English Dictionary defines bullshit as 'to talk nonsense or rubbish' and 'to bluff one's way through something by talking nonsense'. According to Partridge et al. (2006), the word entered into general use during the first world war to describe a focus on the inessential (spit & polish) over the essential (combat readiness). Frankfurt (2009) argues that bullshit is now widespread because the opportunities to engage in it have expanded:

> "....the production of bullshit is stimulated whenever a person's obligations or opportunities to speak about some topic are more excessive than his knowledge of the facts that are relevant to that topic."

He also asserts that it is more dangerous than lying.

> "When an honest man speaks, he says only what he believes to be true; and for the liar, it is correspondingly indispensable that he considers his statements to be false. For the bullshitter, however, all bets are off . . . He does not reject the authority of the truth, as the liar does, and oppose himself to it. He pays no attention to it at all. By virtue of this, bullshit is a greater enemy of truth than lies are."

Bullshitting does not fit into the cheap talk model because the Sender is no more informed than the Receiver. This lacunae is filled by presuming the existence of an uninformed Sender. We use it to examine Frankfurt's assertion that bullshitting is the 'greater enemy'. His assertion relies on a skepticism about the Receiver's ability to identify bullshit (Frankfurt 2009):

> "… most people are rather confident of their ability to recognise bullshit and to avoid being taken in by it"

Just as the Receiver in cheap talk accounts for the fact that an informed Sender will obscure the truth, why can't they do the same with a bullshitter? If so, is it obvious that they are worse off compared to the case where the Sender is informed?

We answer this question in a standard cheap talk environment. There is an underlying one dimensional state of nature unknown to the Receiver who wishes to choose an action that matches the state. The Receiver faces a Sender who may or may not be informed about the underlying state. The probability that the Sender is informed is p. An informed Sender's preferences differ from that of the Receiver and depend upon the state. They wish the Receiver to choose an action equal to the state plus a commonly known *positive bias*. As the Sender's preferences differ from the Receiver's, they have an interest in obscuring the truth.

As remarked earlier, to account for lying one must have a common language. In the standard cheap talk model, messages correspond to subsets of the state space and these subsets will be the common language. Thus, if the informed sender says S, where S is some subset of the state space, this is understood to mean that the Sender is saying that the underlying state is in S. If the Sender says S and the state is outside S, this is an outright lie. If the state is in S but S is not a singleton, this is a lie of omission. The non-babbling equilibria of the standard model of cheap talk correspond to lies of omission and we focus on these. This makes the informed Sender the 'liar' in our model.

With probability $1-p$, the Sender is uninformed and has preferences that are state-independent. The uninformed Sender is the bullshitter in our model because they are ignorant of the underlying state (i.e. the truth) and, as will be seen, have no incentive to acquire information about it. The uninformed Sender wishes to be perceived as informed, and this is hard wired into the model by assuming $p > 0$. We consider two possibilities. The uninformed Sender prefers higher to lower actions or the converse. One can interpret the uninformed Sender as adding noise to the message sent by the informed Sender. Blume et al. (2007) considers the impact of noise in the standard cheap talk model by assuming every message has a chance of being uninformative. In our case, *some* but not all messages have a chance of being uninformative. This is because there are some messages an uninformed Sender never sends, making some of the informed Sender's messages noise free.

The possibility of an uninformed Sender ($p < 1$) changes equilibrium outcomes (modulo selection) in the cheap talk game by *changing* the behavior of the informed Sender rather than through direct manipulation of the Receiver's beliefs. Thus, our model highlights the *indirect* effect an uninformed Sender has on dialog by influencing the communication strategies of the informed Sender. Whether the Receiver is the worse for it, compared to the case $p = 1$, *depends* on the commonly known preference of the uninformed Sender.

If the uninformed Sender prefers higher to lower actions, the Receiver is worse off compared to the case when $p = 1$. An informed Sender is also ex-ante worse off. In this case bullshit dilutes informative communication, consistent with Frankfurt's lament about the prevalence of bullshit.[1]

If, however, the uninformed Sender prefers lower to higher actions, the Receiver's expected payoff can be *higher* compared to the case when $p = 1$ provided p is not too small. The informed Sender with positive bias is ex-ante always better off provided the bias is not too small. This contrasts with Frankfurter's unqualified assertion that bullshitting is worse than lying. It highlights that whether bullshitting has a negative effect or not depends upon the uninformed Sender's preferences relative to those of the informed Sender.

One might conjecture that an uninformed Sender is no different than an informed Sender with a very large bias. If true, it would mean that the uninformed Sender's ignorance of the underlying state plays no role. We show this not to be case. The equilibrium messaging strategy of an informed but highly biased Sender *differs* from that of an uninformed Sender.

We also consider the possibility that the preferences of the uninformed Sender may be unknown. Specifically, the uninformed Sender may prefer higher actions or lower actions. We find that if both possibilities are equally likely, the Receiver can be better off, provided that the likelihood of facing an uninformed Sender is sufficiently small.

Our model also highlights another way in which a bullshitter can affect communication. The standard Crawford & Sobel (1982) cheap talk model supports equilibria where the Sender can partition the message space into finer intervals. This increases the precision of information transmitted and leads to higher payoffs for both Receiver and informed Sender. This is no longer true in the presence of a bullshitter. In particular, an equilibrium with three messages as opposed to two, only leads to higher payoffs if p is sufficiently

[1] Many find it hard to follow Wittgenstein's injunction: "Whereof one cannot speak, thereof one must be silent."

large. Thus, the presence of a bullshitter reduces the incentives to use a larger message space.

The remainder of this paper is organized as follows. The next section reviews the basic cheap talk model of Crawford & Sobel (1982). The subsequent section investigates what happens in this model when the possibility of an uninformed Sender being present is allowed. The next section discusses some variations on the base model to highlight the robustness of our conclusions. The final section draws comparisons between bullshit in a cheap talk environment and in a Bayesian persuasion environment.

3. CHEAP TALK MODEL

We recall the textbook cheap talk communication game between an informed Sender with known positive bias and an uninformed Receiver who must choose an action (Crawford & Sobel 1982).

1. The Sender's type is the underlying state, $\theta \in [0,1]$.
2. The Receiver is ignorant of the state but believes it to be drawn uniformly from $[0,1]$. This belief is known to the Sender.
3. Receiver's action space is $A = [0,1]$.
4. Receiver's payoff from action $a \in [0,1]$ is $-(a-\theta)^2$.
5. Sender's payoff from action $a \in [0,1]$ is $-(a-(\theta+b))^2$ where $b > 0$ is the bias.
6. Sender's message space is $[0,1]$.

The cheap talk game admits multiple equilibria. One reason is that the meaning of messages is arbitrary. Given any equilibrium, one can generate another equilibrium by changing the interpretation of messages.

Rather than assuming the Sender's strategy is a signaling rule (a function from type into message), we follow Gordon (2010) in assuming the informed Sender's strategy is an increasing set of thresholds. Adjacent thresholds identify an interval and distinct types of the informed Sender send a common message if and only if their types are in the same interval. This formulation eliminates message indeterminacy. It also means that the informed Sender never lies, but they don't disclose the whole truth either.

It is straightforward to see that an equilibrium of the cheap talk game generates a type-action distribution if and only if it is an equilibrium of the threshold game. Crawford & Sobel (1982) show that there is a finite upper bound,

n^*, on the number of thresholds. Furthermore, for each integer k between 1 and n^* there is a choice of b such that it is a unique equilibrium outcome with k thresholds. For economy of exposition, we focus primarily on equilibria with a single threshold. Section 5.3, discusses what happens when this assumption is relaxed. In the appendices we explore other variations of this model.[2]

Suppose there exists a threshold $t \in [0,1]$ and the Sender sends the message B (for bottom) if state $\theta \in [0,t]$ and the message T (for top) otherwise. In the first case the Receiver chooses action $0.5t$ to maximize their expected payoff. In the other case, the Receiver's optimal action choice is $0.5(1+t)$. At an equilibrium threshold t the following indifference condition for the informed Sender must hold:

$$t+b-0.5t = 0.5(1+t)-(t+b) \;\Rightarrow\; t = 0.5-2b.$$

We need $b \leq 0.25$ to ensure this is well defined and this will be assumed throughout.

When the possibility of an uninformed Sender is introduced, the equilibrium threshold t changes. We are interested in how this change affects the Receiver and Sender's expected payoffs. This is summarized in the next two Lemmas which are standard.

Lemma 3.1. *The Receiver's expected payoff as a function of t, denoted $\rho(t)$, is increasing for $t \leq 0.5$, and decreasing otherwise.*

Proof.

$$\begin{aligned}
\rho(t) &= -t \times t^{-1}\int_0^t (0.5t-x)^2 dx - (1-t)\times(1-t)^{-1}\int_t^1 (0.5(1+t)-x)^2 dx \\
&= -\int_0^t (0.5t-x)^2 dx - \int_t^1 (0.5(1+t)-x)^2 dx \\
&= \frac{(t-1)^3 - t^3}{12}.
\end{aligned}$$

Now, $\rho(t)$ achieves a maximum in $[0,1]$ at $t = 0.5$ (where the derivative is zero). $\square$

The informed Sender's payoff will depend upon the underlying state. Hence, we focus on the informed Sender's ex-ante expected payoff.

[2] For more on equilibrium selection in this context see Gordon et al. (2021).

Lemma 3.2. *The Sender's ex-ante expected payoff as a function of t, denoted $\sigma(t)$ is increasing for $t \leq 0.5$ and decreasing otherwise.*

Proof.

$$\begin{aligned}\sigma(t) &= -t \times t^{-1} \int_0^t (0.5t - b - x)^2 dx - (1-t) \\ &\quad \times (1-t)^{-1} \int_t^1 (0.5(1+t) - b - x)^2 dx \\ &= -\int_0^t (0.5t - b - x)^2 dx - \int_t^1 (0.5(1+t) - b - x)^2 dx \\ &= \frac{(t-1)^3 - t^3}{12} - b^2.\end{aligned}$$

The derivative of $\sigma(t)$ with respect to t is $-0.5t + 0.25$. The second derivative is -0.5 which is negative. Therefore $\sigma(t)$ achieves a maximum at $t = 0.5$. □

Under cheap talk $t = 0.5 - 2b$. In this case, the Sender's ex-ante expected payoff is

$$\begin{aligned}&\frac{(0.5 - 2b - 1)^3 - (0.5 - 2b)^3}{12} - b^2 \\ &= \frac{16.5b^3 + 10.25 - 19.25b + 10.25b^2 - 12b^2 - 9}{12}.\end{aligned}$$

As observed already in Crawford & Sobel (1982), for any $t \in [0,1]$, that $\rho(t) - \sigma(t) = b^2$. Therefore, the informed Sender's and the Receiver's payoffs vary with t in the same way.

If the informed Sender can commit to a choice of t before the state is realized, i.e., engage in Bayesian persuasion as in Kamenica & Gentzkow (2011), then, $\sigma(t)$ would represent the informed Sender's payoff at threshold t. From the proof of Lemma 3.2 we see that the threshold that maximizes the Sender's ex-ante expected payoff does not depend on her bias.

4. THE UNINFORMED SENDER

We introduce an uninformed Sender who prefers higher to lower actions *independent* of the state (in Section 4.1, we consider the reverse). In a sense the uninformed and informed Sender are 'pulling' in the same direction, as each would choose a higher action than the Receiver would.

The Receiver does not know if the Sender they are facing is informed or not. They believe that the Sender is informed with probability p. When $p = 1$, we recover the standard cheap talk model, and when $p = 0$, there is no possibility of valuable information being transmitted to the Receiver and, thus, we only have a "babbling" equilibrium in which the Receiver's best response to any message is to choose the action that corresponds to the mean of the distribution from which the state is drawn; in our model with $\theta \sim U[0,1]$, this entails setting $a = 0.5$. Therefore, in what follows we assume that $p \in (0,1)$.

The uninformed Sender considers the messages that an informed Sender will send with positive probability and picks from among these. It is easy to see that the uninformed Sender always sends the message T (see Section 5.1). Even if the uninformed Sender were to learn the state but the Receiver is unaware of this, the previously uninformed Sender will always send message T. Hence, the uninformed Sender gains no benefit from knowing the state.

One might imagine that an uninformed Sender is no different than an informed Sender with a very large positive bias (exceeding 0.25). Section 5.1 shows this is not the case because in some instances the high bias informed Sender will send message B.

The Receiver accounts for the possibility that the Sender is uninformed when interpreting message T. This causes the informed Sender to adjust her equilibrium threshold t. Our first result shows that the Receiver and the informed Sender are ex-ante worse off compared to the case $p = 1$.

The intuition for this is straightforward. Given the focus on two message equilibria, to determine the effect of a bullshitter, it is enough to look at the indifferent informed Sender in equilibrium. The equilibrium without bullshitters is a partition in which all Senders with types in $[0,t^*)$ send the message B and all Senders with types in $[t^*,1]$ send message T. Given $b > 0$, the indifferent type t^* satisfies $t^* < 0.5$. The introduction of a bullshitter will change the indifferent type to $\bar{t}$, say. If $\bar{t} < t^*$, the equilibrium provides less information (the partition is more asymmetric), while if it is higher than t^* then the equilibrium is more informative, as long as $|\bar{t} - 0.5| < |t^* - 0.5|$. This is because partitions that lead to lower expected variance lead to a higher payoff. Equal sized partitions always have lower variance than unequal ones (see Goltsman et al. 2009).

Henceforth, we denote by IS the event that the Sender is informed and denote by US the event that Sender is uninformed. We use T to denote message T as well as the event that message T was sent. Similarly with B.

Theorem 4.1. *Suppose the uninformed Sender prefers higher to lower actions. Let t^* be the equilibrium threshold in a non-babbling equilibrium. Then, the following are true.*

1. $t^* \leq 0.5 - 2b$.
2. *The ex-ante payoff of an informed Sender with bias $b \leq 0.25$ and the Receiver is lower than their ex-ante payoff when $p = 1$.*

4.1. Preference for Lower Actions

Now, suppose the uninformed Sender prefers **lower** to higher actions independent of the state. Now, the uninformed Sender and informed Sender are "pulling" in different directions. In Appendix A, we show that the Receiver's expected action is increasing in the probability that the uninformed Sender sends message T. Therefore, the uninformed Sender with preference for *lower* actions will always send message B. The Receiver takes this into account when interpreting message B and in turn this will cause the informed Sender to adjust their equilibrium threshold t.

Theorem 4.2. *Suppose the uninformed Sender prefers lower to higher actions. Then, there is a non-babbling equilibrium where the equilibrium threshold exceeds* $0.5 - 2b$.

Whether the informed Sender or Receiver is better off will depend on whether this threshold is smaller or larger than 0.5. From the proof of Theorem 4.2 in Appendix A, the equilibrium threshold t^* satisfies:

$$t^* = \frac{-(3+4(b-1)p) + \sqrt{16b^2p^2 - 8(b+1)p + 9}}{4p}.$$

When $b = 0$, it is straightforward to verify that $t^* > 0.5$ for all $p \in (0,1)$. Therefore, by Lemma 3.2, the informed Sender will be ex-ante worse off compared to $p = 1$. By continuity it follows that for some small $b > 0$, the possibility of a bullshitter lowers the informed Sender's ex-ante expected payoff.

5. ROBUSTNESS

In this section we consider several variations of our model to verify the robustness of our main qualitative insights. Section 5.1 replaces the uninformed Sender with a highly biased informed Sender. Section 5.2 introduces an additional uninformed Sender with opposing preferences. Section 5.3 examines equilibria when the informed Sender is permitted to partition the message space into more intervals.

5.1. High bias vs Bullshit

Suppose now both types of Senders are informed but differ in their bias and we exclude the possibility that they can send cheap talk messages about their biases. With probability p, the informed Sender has low bias and this event is denoted IS^L. With probability $1-p$, the informed Sender has high bias and this event is denoted IS^H. We compare the value of the information sent to the Receiver in this case with the case in which the Receiver faces either a low-biased informed Sender or an uninformed Sender with preference for higher actions.

Denote by $t_l \in [0,1]$ the equilibrium threshold used by IS^L to determine her message. The corresponding threshold for IS^H is denoted $t_h \in [0,1]$. Without loss of generality, we assume $t_h > t_l$. The Receiver's best response upon seeing message T is given by:

$$a_T = 0.5\left(\frac{p(t_l^2 - t_h^2) + t_h^2 - 1}{p(t_l - t_h) + t_h - 1}\right).$$

The Receiver's best response upon seeing message B is:

$$a_B = 0.5\left(\frac{p(t_l^2 - t_h^2) + t_h^2}{p(t_l - t_h) + t_h}\right).$$

Now, t_l must satisfy the following indifference condition:

$$\left(0.5\left(\frac{p(t_l{}^2 - t_h{}^2) + t_h{}^2 - 1}{p(t_l - t_h) + t_h - 1}\right) - (t_l + b_l)\right)^2 = \left((t_l + b_l) - 0.5\left(\frac{p(t_l{}^2 - t_h{}^2) + t_h{}^2}{p(t_l - t_h) + t_h}\right)\right)^2$$

Similarly, t_h must satisfy the following indifference condition:

$$\begin{aligned}&\left(0.5\left(\frac{p(t_l{}^2-t_h{}^2)+t_h{}^2-1}{p(t_l-t_h)+t_h-1}\right)-(t_h+b_h)\right)^2\\&=\left((t_h+b_h)-0.5\left(\frac{p(t_l{}^2-t_h{}^2)+t_h{}^2}{p(t_l-t_h)+t_h}\right)\right)^2\end{aligned}$$

Interior solutions to the above are unavailable in closed form. However, we show below that these equilibrium conditions imply that a highly biased informed Sender may behave in a way distinct from an uninformed Sender.

When p is close to 0.5, there exist many (b_l, b_h) pairs that support an interior equilibrium. However, b_h and b_l must be close to zero. Thus, for some p there may be no pure strategy two message cheap talk equilibrium with a highly biased informed Sender. If there is a mixed strategy equilibrium, it would mean that the informed Sender with high bias must send message B with positive probability. Therefore, a highly biased informed Sender may behave differently from an uninformed Sender with a preference for higher actions. In particular, a highly biased informed Sender may inhibit communication altogether.

Lemma 5.1. *In a binary message space, there is a choice of $b_l < 0.25$ and b_h such that an informed Sender with high bias is distinguishable from an uninformed Sender.*

Proof. Suppose there is an equilibrium where $a_T \neq a_B$, i.e., some information is transmitted. The equilibrium threshold t of an informed Sender with bias b must satisfy the following indifference condition:

$$(a_T-(t+b))^2=((t+b)-a_B)^2$$

$$\Rightarrow [(a_T-(t+b))+(t+b-a_B)][(a_T-(t+b))-(t+b-a_B)]=0$$

$$\Rightarrow [a_T-a_B][a_T+a_B-2b-2t]=0.$$

Given $a_T \neq a_B$ (to exclude babbling), it follows that $a_T+a_B-2b-2t=0$.

At an equilibrium we need the indifference condition for both the high *and* low bias informed Sender to be satisfied, i.e.:

$$a_T+a_B-2b_h-2t_h=0,\ a_T+a_B-2b_l-2t_l=0.$$

Here t_h and t_l are the equilibrium thresholds for the high and low bias informed Sender respectively. Combining these equations, we obtain:

$$-2b_h + 2b_l - 2t_h + 2t_l \;\Rightarrow\; t_h - t_l = b_h - b_l.$$

Suppose b_h and b_l take on values such that $t_h, t_l \in (0,1)$. Then, the high-bias informed Sender will send message B with positive probability. Recall that the uninformed Sender always sends message T. $\square$

5.2. Two Uninformed Senders

Here we consider the possibility that the preferences of the uninformed Sender are unknown. We model this by assuming the possibility of two uninformed Senders in addition to an informed Sender. There is an uninformed Sender who prefers higher actions, denoted by US^H, and an uninformed Sender who prefers lower actions, denoted by US^L. Nature selects one from among US^H, US^L or the informed Sender with probabilities p_1, p_2, and $(1 - p_1 - p_2)$, respectively. We use IS to denote the event that nature selects an informed Sender.

The informed Sender sends a state-dependent message, while US^H and US^L send their respective messages without regard to the state. We determine whether the informed Sender and the Receiver, denoted R, are better or worse off than in the case of no uninformed Senders (i.e. $p_1 = p_2 = 0$).

As before, $t \in (0,1)$ denotes the threshold which determines which message IS sends. IS sends message B when $\theta \in [0,t)$, and message T when $\theta \in [t,1]$. The strategy of the uninformed senders is, possibly, mixed. US^H sends message T with probability α and B with probability $(1-\alpha)$ for all $\theta \in [0,1]$, and US^L sends message T with probability β and B with probability $(1-\beta)$ for all $\theta \in \Theta$. Once the Sender is chosen and the message sent, R observes it, but not the identity of the Sender.

R observes α, β, p_1, and p_2, using these to inform her beliefs on the Sender's type and, thus, θ. Using these beliefs, R chooses an action to maximize her expected payoff. R's best response to message T is given by:

$$a_T = 0.5\left(\frac{p_1(\alpha + t^2 - 1) + p_2(\beta + t^2 - 1) - t^2 + 1}{p_1(\alpha + t - 1) + p_2(\beta + t - 1) - t + 1}\right).$$

R's best response to message B is then given by:

$$a_B = 0.5\left(\frac{p_1(\alpha + t^2 - 1) + p_2(\beta + t^2 - 1) - t^2}{p_1(\alpha + t - 1) + p_2(\beta + t - 1) - t}\right).$$

US^H will choose α to solve $\max_{\alpha\in[0,1]}[a_T\alpha + a_B(1-\alpha)]$. In Appendix B, we show that the optimal solution is $\alpha = 1$. Thus, US^H will only send message T. Similarly, US^L will only send message B.

The informed Sender's threshold t must satisfy the following indifference condition:

$$-\left((t+b)-a_B\right)^2 = -\left(a_T-(t+b)\right)^2.$$

Focusing on the non-babbling case:

$$\begin{aligned} &0.5\left(\frac{p_1t^2+p_2(t^2-1)-t^2+1}{p_1t+p_2(t-1)-t+1}\right)-t-b \\ &= t+b-0.5\left(\frac{p_1t^2+p_2(t^2-1)-t^2}{p_1t+p_2(t-1)-t}\right). \end{aligned} \tag{1}$$

The solution to equation (1) is not monotone in either p_1 or p_2 but it does decrease with b.

Lemma 5.2. *Suppose $p_1 = p_2 = p_u \in (0,0.5)$. Then, in a non-babbling equilibrium, ex-ante, the informed Sender is always better off.*

In the presence of two uninformed Senders ($p_1, p_2 > 0$), one can support non-babbling equilibria in which $b \in (\frac{1}{4}, \frac{1}{2})$. Recall that in the standard cheap talk setting, such equilibria exist only when $b < \frac{1}{4}$.

Unsurprisingly, the informed Sender's payoff decreases monotonically in b. When the informed Sender is *highly biased* (i.e., $b > 0.25$), the presence of uninformed Senders makes the informed Sender strictly better off for any value of $p_u \in (0, \frac{1}{2})$.

Turning to the other extreme, when $b = 0$, the presence of the uninformed Senders does not reduce the ex-ante expected payoff of the informed Sender for any value of $p_u > 0$. Hence, there will be a threshold bias level, denoted b^* depending on p_u, where for all $b < b^*$, the informed Sender is no worse off under $p_u > 0$ compared to $p_u = 0$. As p_u increases, the informed Sender's payoff peaks and then decreases, eventually falling below the standard cheap talk payoff. Essentially, an informed Sender with sufficiently large bias achieves higher utility levels when there is a positive but small prevalence of bullshit. Note that in this setting bullshit is, on average, neutral; the expected bias of the uninformed message is 0.

In the presence of a single uninformed Sender, the informed Sender's message is less informative. However, two uninformed Sender's with opposing preferences– which is essentially ”noise” in the model – can benefit the informed Sender and the Receiver! This may appear unsurprising given Blume et al. (2007) which allows for noisy communication in the standard cheap talk model. In that paper the Receiver with some probability $\lambda \in [0,1]$ receives a state independent random message from a continuous distribution. They show for λ sufficiently small, there exists an equilibrium that is Pareto superior to all noiseless cheap talk equilibria. This result does not imply the claim we make here because Blume et al. (2007) relies on flexibility in the choice of the number of messages. In their example 1, for instance, the Pareto optimal equilibrium in the noiseless cheap talk model has two messages, while their Pareto superior equilibrium with noise has three messages. In our case we hold the number of messages *fixed* at two.

5.3. The Enlarged Message Space

Crawford & Sobel (1982) and Gordon (2010) show that, for sufficiently small bias, there exist cheap talk equilibria that support more than two messages. In the standard cheap talk model (where $p = 1$), both the informed Sender and the Receiver are better off with a larger message space. Here we examine whether the same is true in the presence of an uninformed Sender. We restrict ourselves to a three message equilibrium where the uninformed sender does not mix.

The informed Sender uses thresholds $t_1 \in [0,1]$ and $t_2 \in [0,1]$ where $t_2 > t_1$ to determine which message to send. Let the informed Sender send message B when $\theta \in [0,t_1]$, message M when $\theta \in [t_1,t_2]$, and message T when $\theta \in [t_2,1]$. In Appendix C, we show that there is an equilibrium where the uninformed Sender sends only message T.

In the absence of an uninformed Sender, a three message equilibrium exists provided the following condition on the informed Sender's bias is satisfied (Crawford & Sobel 1982):

$$2 \leq \frac{1}{2}\left(\sqrt{1+\frac{2}{b}}-1\right) \Rightarrow b \leq \frac{1}{12}.$$

When the uninformed Sender is present with probability $1-p$, a two message equilibrium is supported when

$$b \leq \frac{2-\sqrt{4-3p}}{12} \text{ or } b \geq \frac{2+\sqrt{4-3p}}{12}.$$

Hence, the value of b under which a three message equilibrium exists under cheap talk *and* in the presence of an uninformed Sender must satisfy $b \leq \left(2-\sqrt{4-3p}\right)/12$. For such a b, the possible values of p consistent with a three message equilibrium in the presence of an uninformed Sender must be at least as large as $16b(1-3b)$.

We have determined numerically that for $b = 1/20$ there is a threshold such that for p below it, the informed Sender's non-babbling equilibrium payoff with three messages is *below* their payoff in the two message non-babbling equilibrium. This is illustrated in Figure 1. A closed form expression for this threshold is unavailable. The possible values of p are plotted on the horizontal while the vertical axes corresponds to the expected payoff of the informed Sender.

The orange curve shows how the expected payoff of the informed Sender in a three message equilibrium changes with p. The blue curve shows how the expected payoff of the informed Sender in a two message equilibrium changes with p. Both curves are plotted only for the range of p under which a three message non-babbling equilibrium exists, i.e. $p \geq 16b(1-3b)$. The red region corresponds to the values of p for which both non-babbling two and three message equilibria exist but where a two message equilibrium is preferable to the informed Sender.

To understand why this is happening, we examined numerically the case $b = 1/20$. The resulting partition induced by the three message equilibrium is very uneven, increasing expected variance. This effect offsets the benefits of being able to more closely tailor the message to the realized state.

In pure cheap talk, a larger message space allows the informed Sender to transmit more information to the Receiver. However, in the presence of an uninformed Sender, enlarging the message space need not increase the informed Sender's or Receiver's expected payoff. For $p \geq 16b(1-3b)$ the informed sender's expected payoff under both a two and three message equilibrium increases in p. However, there exists exactly one $p_s \in (16b(1-3b), 1)$ below which the informed Sender's expected payoff in a three message equilibrium is strictly below that in a two message equilibrium. For $p > p_s$ the converse is true. Now, the Receiver's expected payoffs in both the two and three message equilibria also increase in p. There exists a $p_r \in (16b(1-3b), 1)$ in which the Receiver's expected payoff in the three message equilibria is below that of the two message equilibrium for $p < p_r$. It appears that $p_s > p_r$ for all $b \in [0, 1/12]$. Additionally, it seems that p_s and p_r each increase in b, while

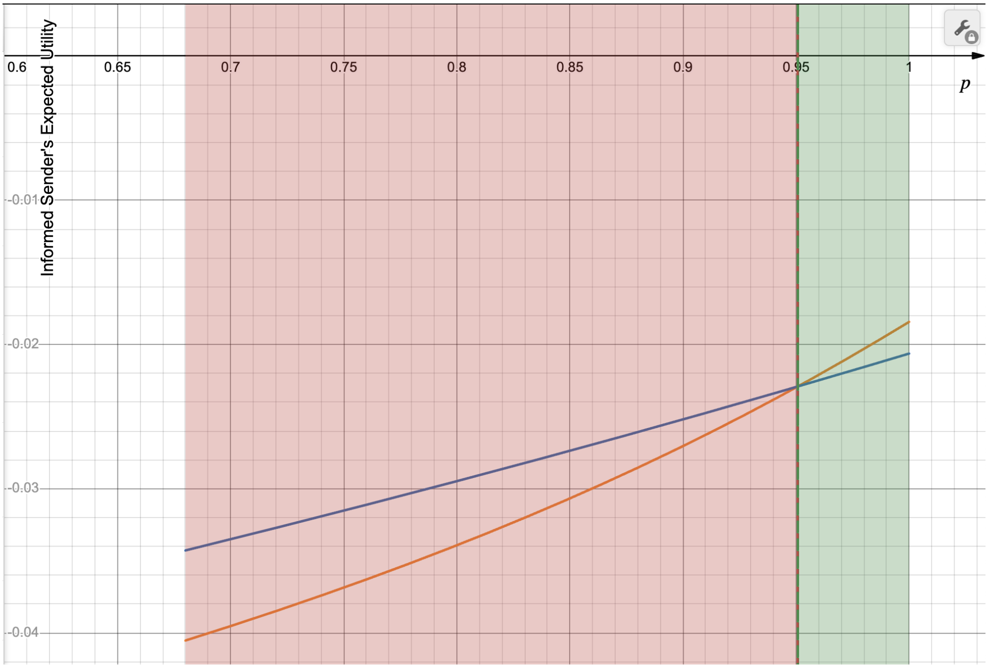

Figure 1: IS's ex ante expected utility as function of p

$p_s - p_r$ decreases in b.

Thus, the presence of an uninformed Sender harms communication in two ways. First, it may cause the informed Sender and Receiver to prefer a smaller message space. This limits the *amount* of information the informed Sender is able to convey to the Receiver. Second, the informed Sender's equilibrium thresholds change, which changes the *type* of information the informed Sender is able to convey to the Receiver.

6. CHEAP TALK VS. BAYESIAN PERSUASION

It is natural to ask what would happen if the informed Sender could commit, before learning the state, to a disclosure policy. This is the Bayesian persuasion setting of Kamenica & Gentzkow (2011). Persuasion differs from cheap talk in that the informed Sender's bias influences the disclosure policy but not the inferences that the Receiver draws.

In this section, we allow the informed Sender with bias $b > 0$ to influence the Receiver's action by committing to a single threshold prior to learning the state and promising the Receiver to faithfully send one message when the state is above the threshold and a different message when the state is below the threshold. As before, the uninformed Sender prefers higher actions to lower ones independent of the state.

It is easy to see that the informed Sender will do better when they can commit to a message strategy. It is less clear that the Receiver will benefit in the presence of an uninformed Sender. As before, we restrict all Senders to one of two messages. The uninformed Sender sends message T with probability 1. The Receiver's best response to each message remains as before:

$$a_T = 0.5\left(\frac{pt^2-1}{pt-1}\right),\ a_B = 0.5t.$$

The informed Sender chooses her threshold t *before* the state is realized so as to maximize her expected ex ante utility:

$$\begin{aligned}\max_t E[u_{IS}] &= \max_{t\in[0,1]} E[u_{IS}|T]P(T) + E[u_{IS}|B]P(B)\\ &= \max_{t\in[0,1]} -\frac{pt(3t^2-6t+4)-1}{12(pt-1)} - b^2.\end{aligned}$$

The first-order optimality condition for an interior optimal is:

$$2pt^3 - 2pt^2 - 3t^2 + 4t - 1 = 0.$$

The optimal choice of t, denoted t^* satisfies

$$t^* = \frac{3-\sqrt{9-8p}}{4p}.$$

The expected payoff of the Receiver is

$$\frac{(4p+3\sqrt{9-8p}-9)(9-8p)}{96p^2}.$$

The expected payoff of the informed Sender is

$$\frac{(4p+3\sqrt{9-8p}-9)(9-8p)}{96p^2} - b^2.$$

Lemma 6.1. *If the uninformed Sender has a preference for higher actions, the informed Sender and Receiver are strictly better off under persuasion than cheap talk.*

Proof. If not, the equilibrium payoff of the Receiver under cheap talk exceeds that of its payoff under persuasion. In other words,

$$\frac{27(\sqrt{8p(2b^2p+b-1)+9}-3)}{96p^2} - \frac{4p(12b^2p+(3b+6)\sqrt{8p(2b^2p+b-1)+9}}{96p^2} + \frac{8p-27)}{96p^2} > \frac{(4p+3\sqrt{9-8p}-9)(9-8p)}{96p^2}.$$

This implies that

$$27\left(\sqrt{8p(2b^2p+b-1)+9}-3\right) - 4p\left(12b^2p+(3b+6)\sqrt{8p(2b^2p+b-1)+9}+8p-27\right) > \left(4p+3\sqrt{9-8p}-9\right)(9-8p).$$

Therefore $p < 0$ which cannot be. $\square$

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Appendix A

Suppose two types of Senders: an informed one and an uninformed one who prefers higher actions. In this section we derive the equilibrium actions of each Sender and the Receiver.

Nature draws the state θ uniformly at random from $[0,1]$ and selects the type of Sender (informed or uninformed), with probabilities p and $(1-p)$, respectively with $p \in (0,1)$. If the uninformed is selected, the uninformed Sender sends message T with probability α and message B with probability $(1-\alpha)$ for all $\theta \in \Theta$. If the informed Sender is selected, the informed Sender chooses a threshold $t \in [0,1]$ and sends message B when $\theta \in [0,t)$, and T when $\theta \in [t,1]$.

Lemma 6.2. *Let a_T denote the Receiver's best response upon seeing message T. Then,*

$$a_T = 0.5\left(\frac{p(\alpha+t^2-1)-\alpha}{p(\alpha+t-1)-\alpha}\right).$$

Proof. The probability that the Receiver sees message T is $p(1-t)+(1-p)\alpha$. Given message T, the probability that the Sender is informed is

$$\frac{p(1-t)}{p(1-t)+(1-p)\alpha}.$$

The probability that the Sender is uninformed is

$$\frac{(1-p)\alpha}{p(1-t)+(1-p)\alpha}.$$

The Receiver's expected payoff from choosing action a is given by:

$$E[u_R|T] = E[u_R|IS,T]P(IS|T) + E[u_R|US,T]P(US|T) =$$

$$\frac{p(1-t)}{p(1-t)+(1-p)\alpha}\int_t^1 -\frac{(a-\theta)^2}{1-t}d\theta + \frac{(1-p)\alpha}{p(1-t)+(1-p)\alpha)}\int_0^1 -(a-\theta)^2 d\theta =$$

$$-\frac{p\int_t^1 (a^2-2a\theta+\theta^2)\,d\theta}{p(1-t)+(1-p)\alpha} - \frac{(1-p)\alpha\int_0^1 (a^2-2a\theta+\theta^2)\,d\theta}{p(1-t)+(1-p)\alpha}$$

To determine the Receiver's optimal choice of a we determine when the derivative of (A) is zero:

$$\partial\left[-\frac{p\int_t^1(a^2-2a\theta+\theta^2)\,d\theta}{p(1-t)+(1-p)\alpha}-\frac{(1-p)\alpha\int_0^1(a^2-2a\theta+\theta^2)\,d\theta}{p(1-t)+(1-p)\alpha}\right]\Big/\partial a=0$$

$$\Rightarrow -2\left(\frac{p\int_t^1(a-\theta)\,d\theta}{p(1-t)+(1-p)\alpha}+\frac{(1-p)\alpha\int_0^1(a-\theta)\,d\theta}{p(1-t)+(1-p)\alpha}\right)=0$$

$$\Rightarrow \frac{p\Big[a\theta-0.5\theta^2\Big]_t^1}{p(1-t)+(1-p)\alpha}+\frac{(1-p)\alpha\Big[a\theta-0.5\theta^2\Big]_0^1}{p(1-t)+(1-p)\alpha}=0$$

$$\Rightarrow a_T=0.5\left(\frac{p(\alpha+t^2-1)-\alpha}{p(\alpha+t-1)-\alpha}\right).$$

□

Lemma 6.3. *Let a_B denote the Receiver's best response upon seeing message B. Then,*

$$a_B=0.5\left(\frac{p(\alpha+t^2-1)-\alpha+1}{p(\alpha+t-1)-\alpha+1}\right).$$

Proof. The probability the Receiver sees message B is $pt+(1-p)(1-\alpha)$. Given that the message is B, the probability that the Sender is informed

$$\frac{pt}{pt+(1-p)(1-\alpha)}.$$

Given that the message is B, the probability that the Sender is uninformed is

$$\frac{(1-p)(1-\alpha)}{pt+(1-p)(1-\alpha)}.$$

The Receiver's expected payoff from choosing action a is given by:

$$E[u_R|B]=E[u_R|IS,B]P(IS|B)+E[u_R|US,B]P(US|B)$$

$$=\frac{pt}{pt+(1-p)(1-\alpha)}\int_0^t-(a-\theta)^2\frac{1}{t}d\theta+\frac{(1-p)(1-\alpha)}{pt+(1-p)(1-\alpha)}\int_0^1-(a-\theta)^2d\theta$$

$$=-\frac{p\int_0^t(a^2-2a\theta+\theta^2)\,d\theta}{pt+(1-p)(1-\alpha)}-\frac{(1-p)(1-\alpha)\int_0^1(a^2-2a\theta+\theta^2)\,d\theta}{pt+(1-p)(1-\alpha)}. \qquad (2)$$

To determine the Receiver's optimal choice of a we set the derivative of (2) with respect to a equal to zero:

$$\partial\left[-\frac{p\int_0^t (a^2-2a\theta+\theta^2)\,d\theta}{pt+(1-p)(1-\alpha)}-\frac{(1-p)(1-\alpha)\int_0^1 (a^2-2a\theta+\theta^2)\,d\theta}{pt+(1-p)(1-\alpha)}\right]\Big/\partial a=0$$

$$\Rightarrow -2\left(\frac{p\int_0^t (a-\theta)\,d\theta}{pt+(1-p)(1-\alpha)}+\frac{(1-p)(1-\alpha)\int_0^1 (a-\theta)\,d\theta}{pt+(1-p)(1-\alpha)}\right)=0$$

$$\Rightarrow \frac{p\Big[a\theta-0.5\theta^2\Big]_0^t}{pt+(1-p)(1-\alpha)}+\frac{(1-p)(1-\alpha)\Big[a\theta-0.5\theta^2\Big]_0^1}{pt+(1-p)(1-\alpha)}=0$$

$$\Rightarrow a_B=0.5\left(\frac{p(\alpha+t^2-1)-\alpha+1}{p(\alpha+t-1)-\alpha+1}\right).$$

□

Lemma 6.4. *The uninformed Sender's optimal choice of α is $\alpha=1$.*

Proof. The uninformed Sender's problem is:

$$\begin{aligned}\max_{\alpha} E[a] &= \max_{\alpha}[\alpha a_T+(1-\alpha)a_B]\\ &= \max_{\alpha\in[0,1]}\Bigg(0.5\left(\frac{p(\alpha+t^2-1)-\alpha}{p(\alpha+t-1)-\alpha}\right)\times\alpha\\ &\qquad +0.5\left(\frac{p(\alpha+t^2-1)-\alpha+1}{p(\alpha+t-1)-\alpha+1}\right)\times(1-\alpha)\Bigg)\end{aligned}$$

The first-order condition for an interior optimal yields the following:

$$\alpha=-\frac{p^2(t-1)-2p(t-1)+\sqrt{(p-1)^2(t-1)t}+t-1}{(p-1)^2}. \tag{3}$$

Given $t\in(0,1)$ and $p\in(0,1)$, equation (3) has no solution with $\alpha\in(0,1)$. Hence, the optimal value of α is either 0 or 1. Direct computation shows that it must be $\alpha=1$. □

Theorem 4.1

Proof. It is clear that in equilibrium the uninformed Sender will always send message T (a formal proof is in Appendix A). If t is the threshold used by the informed Sender, the Receiver's best response to messages T and B as a function of t, are given by (see Appendix A):

$$a_T = 0.5\left(\frac{pt^2-1}{pt-1}\right),\ a_B = 0.5t.$$

The informed Sender's indifference condition is:

$$-\left((t+b)-a_B\right)^2 = -\left(a_T-(t+b)\right)^2.$$

This leads to two possibilities. The first is that

$$(t+b)-a_B+a_T-(t+b) = 0 \ \Rightarrow a_B = a_T.$$

Hence,

$$pt^2-t = pt^2-1 \ \Rightarrow t = 1,$$

which corresponds to a babbling equilibrium which we ignore. The second possibility is that

$$\begin{aligned}
(t+b)-a_B &= a_T-(t+b)\\
\Rightarrow t+b-0.5t &= 0.5\frac{pt^2-1}{pt-1}-t-b\\
\Rightarrow 0.5t+b &= 0.5\frac{pt^2-1}{pt-1}-t-b\\
\Rightarrow 1.5t+2b &= 0.5\frac{pt^2-1}{pt-1}\\
\Rightarrow (pt-1)(1.5t+2b) &= 0.5(pt^2-1)\\
\Rightarrow 1.5pt^2+2bpt-1.5t-2b &= 0.5(pt^2-1)\\
\Rightarrow pt^2+2bpt-1.5t-2b+0.5 &= 0\\
\Rightarrow 2pt^2+4bpt-3t-4b+1 &= 0\\
\Rightarrow 2pt^2+(4bp-3)t-4b+1 &= 0\\
\Rightarrow t &= \frac{-(4bp-3)\pm\sqrt{16b^2p^2+8(b-1)p+9}}{4p}.
\end{aligned}$$

For t to be well defined we need the term in the square root to be non-negative. The expression $16b^2p^2+8(b-1)p+9$ has a derivative with respect to p of $32b^2p-8(1-b)$ which is negative given $b^2 \leq 1/8$. Therefore, the expression achieves a minimum at $p=0$, so the term in the square root is non-negative and the roots are real.

We eliminate the root

$$\frac{-(4bp-3)+\sqrt{16b^2p^2+8(b-1)p+9}}{4p}$$

from consideration because it exceeds 1. To see why, suppose otherwise:

$$\frac{-(4bp-3)+\sqrt{16b^2p^2+8(b-1)p+9}}{4p}<1$$

$$\Rightarrow -(4bp-3)+\sqrt{16b^2p^2+8(b-1)p+9}<4p$$

$$\Rightarrow 16b^2p^2+8(b-1)p+9<16p^2(1+b)^2+9-24p(1+b)$$

$$\Rightarrow 8(b-1)p<16p^2+32p^2b-24p-24bp$$

$$\Rightarrow 1<p,$$

which cannot be.

Therefore, the informed Sender's equilibrium threshold, denoted t^*, satisfies:

$$t^*=\frac{3-4bp-\sqrt{16b^2p^2+8(b-1)p+9}}{4p}. \tag{4}$$

We verify that $t^*\leq 0.5-2b$:

$$\frac{3-4bp-\sqrt{16b^2p^2+8(b-1)p+9}}{4p}\leq 0.5-2b$$

$$\Rightarrow 3-4bp-\sqrt{16b^2p^2+8(b-1)p+9}\leq 2p-8bp$$

$$\Rightarrow 3+4bp-2p\leq\sqrt{16b^2p^2+8(b-1)p+}$$

$$\Rightarrow (3+4bp-2p)^2\leq 16b^2p^2+8(b-1)p+9$$

$$\Rightarrow 9+(4bp-2p)^2+6(4bp-2p)\leq 16b^2p^2+8(b-1)p+9$$

$$\Rightarrow 9+16b^2p^2+4p^2-16bp^2+24bp-12p\leq 16b^2p^2+8(b-1)p+9$$

$$\Rightarrow 4p^2-16bp^2+16bp-4p\leq 0$$

$$\Rightarrow 4p^2(1-4b)+4p(4b-1)\leq 0$$

$$\Rightarrow 4p(p-1)(1-4b)\leq 0,$$

which is clearly true.

We also verify that $t^* \geq 0$:

$$\frac{3-4bp-\sqrt{16b^2p^2+8(b-1)p+9}}{4p} \geq 0$$

$$\Rightarrow 3-4bp \geq \sqrt{16b^2p^2+8(b-1)p+9}$$

$$\Rightarrow 9+16b^2p^2-24bp \geq 16b^2p^2+8(b-1)p+9$$

$$\Rightarrow -24bp \geq +8(b-1)p$$

$$\Rightarrow 8p \geq 32bp$$

$$\Rightarrow 0.25 \geq b,$$

which is true by assumption.

Now, we compare the informed Sender's ex-ante payoff which is

$$-\int_0^{t^*}(\theta+b-0.5t^*)^2 d\theta - \int_{t^*}^1 (0.5\frac{p(t^*)^2-1}{pt^*-1}-\theta-b)^2 d\theta,$$

with their ex-ante payoff when $p=1$ which is $\sigma(0.5-2b)$.

Consider the function $\phi(x) = \int_{t^*}^1 (x-\theta-b)^2 d\theta$. A straightforward calculation shows that it is convex and achieves a minimum at $x = 0.5(1+t^*)+b$. Therefore, for $x \leq 0.5(1+t^*)+b$, $\phi(x)$ is decreasing. Now, $\frac{p(t^*)^2-1}{pt^*-1} \leq 1+t^*$. Hence,

$$-\int_0^{t^*}(\theta+b-0.5t)^2 d\theta - \int_{t^*}^1 (0.5\frac{p(t^*)^2-1}{pt-1}-\theta-b)^2 d\theta$$

$$\leq -\int_0^{t^*}(\theta+b-0.5t)^2 d\theta - \int_{t^*}^1 (0.5(1+t^*)-\theta-b)^2 d\theta$$

$$= \sigma(t^*) \leq \sigma(0.5-2b).$$

The last inequality follows from Lemma 3.2 and the fact that $t^* \leq 0.5-2b$.

The Receiver's expected payoff is

$$-\int_0^{t^*}(\theta-0.5t)^2 d\theta - \int_{t^*}^1 (0.5\frac{p(t^*)^2-1}{pt-1}-\theta)^2 d\theta.$$

We will use Lemma 3.1 to compare it with their expected payoff when $p=1$ which is $\rho(0.5-2b)$. Consider the function $\mu(x) = \int_{t^*}^1 (x-\theta)^2 d\theta$. A straightforward calculation shows that it is convex and achieves a minimum at $x = 0.5(1+t^*)$. Therefore, for

$x \leq 0.5(1+t^*)$, $\mu(x)$ is decreasing. As $\frac{p(t^*)^2-1}{pt^*-1} \leq 1+t^*$ it follows that the Receiver's expected payoff is bounded above by

$$-\int_0^{t^*}(\theta-0.5t)^2 d\theta - \int_{t^*}^1 (0.5(t^*+1)-\theta)^2 d\theta = \rho(t^*) \leq \rho(0.5-2b).$$

□

Theorem 4.2

Proof. Given that the uninformed Sender strictly prefers to send message B, the Receiver's best response – as functions of the informed Sender's threshold t – to messages T and B, respectively, are given by (see Appendix A):

$$a_T = 0.5(t+1),\ a_B = 0.5\left(\frac{p(t^2-1)+1}{p(t-1)+1}\right).$$

The informed Sender's indifference condition is:

$$-\left((t+b)-a_B\right)^2 = -\left(a_T-(t+b)\right)^2.$$

There are two possibilities. The first is that

$$a_T = a_B \Rightarrow p(t+1)(t-1)+(t+1) = p(t^2-1)+1 \Rightarrow t=0.$$

This corresponds to a babbling equilibrium which we ignore. The second possibility is that

$$\begin{aligned}
t+b-a_B &= a_T-t-b \\
\Rightarrow t+b-0.5\left(\frac{p(t^2-1)+1}{p(t-1)+1}\right) &= 0.5(t+1)-t-b \\
\Rightarrow \left(\frac{p(t^2-1)+1}{p(t-1)+1}\right)+t+1 &= 4(t+b) \\
\Rightarrow p(t^2-1)+1+p(t-1)(t+1)+t+1 &= 4(t+b)[p(t-1)+1] \\
\Rightarrow 2p(t^2-1)+t+2 &= 4(t+b)[p(t-1)+1] \\
\Rightarrow 2p(t^2-1)+t+2 &= 4pt^2-4pt+4bpt-4bp+4(t+b) \\
\Rightarrow 2pt^2+(4(b-1)p+3)t-4bp+2p+4b-2 &= 0 \\
\Rightarrow 2pt^2+(4(b-1)p+3)t+2p(1-2b)+2(2b-1) &= 0 \\
\Rightarrow 2pt^2+(4(b-1)p+3)t+2p(1-2b)-2(1-2b) &= 0 \\
\Rightarrow 2pt^2+(4(b-1)p+3)t+2(p-1)(1-2b) &= 0
\end{aligned}$$

$$\Rightarrow 2pt^2 + (4(b-1)p+3)t - 2(1-p)(1-2b) = 0.$$

There are two roots:

$$t^* = \frac{-(3+4(b-1)p) \pm \sqrt{16b^2p^2 - 8(b+1)p+9}}{4p}.$$

Consider first

$$t^* = \frac{-(3+4(b-1)p) + \sqrt{16b^2p^2 - 8(b+1)p+9}}{4p}. \tag{5}$$

We verify that it is in $[0.5-2b, 1]$:

$$\frac{-(3+4(b-1)p) + \sqrt{16b^2p^2 - 8(b+1)p+9}}{4p} \leq 1$$

$$\Rightarrow -(3+4(b-1)p) + \sqrt{16b^2p^2 - 8(b+1)p+9} \leq 4p$$

$$\Rightarrow \sqrt{16b^2p^2 - 8(b+1)p+9} \leq 4p + (3+4(b-1)p)$$

$$\Rightarrow \sqrt{16b^2p^2 - 8(b+1)p+9} \leq 4p + 3 + 4bp - 4p$$

$$\Rightarrow \sqrt{16b^2p^2 - 8(b+1)p+9} \leq 3 + 4bp$$

$$\Rightarrow 16b^2p^2 - 8(b+1)p + 9 \leq 9 + 16b^2p^2 + 24bp$$

$$\Rightarrow -8(b+1)p \leq 24bp,$$

which is certainly true.

Next, we show that this root exceeds $0.5-2b$. Suppose not, then

$$\frac{-(3+4(b-1)p) + \sqrt{16b^2p^2 - 8(b+1)p+9}}{4p} \leq 0.5 - 2b$$

$$\Rightarrow -(3+4(b-1)p) + \sqrt{16b^2p^2 - 8(b+1)p+9} \leq 2p - 8pb$$

$$\Rightarrow \sqrt{16b^2p^2 - 8(b+1)p+9} \leq 2p - 8pb + 3 + 4(b-1)p$$

$$\Rightarrow \sqrt{16b^2p^2 - 8(b+1)p+9} \leq 2p - 8pb + 3 + 4bp - 4p$$

$$\Rightarrow \sqrt{16b^2p^2 - 8(b+1)p+9} \leq -2p - 4pb + 3$$

$$\Rightarrow 16b^2p^2 - 8(b+1)p + 9 \leq 9 + (2p+4pb)^2 - 6(2p+4pb)$$

$$\Rightarrow 16b^2p^2 - 8bp - 8p \leq (2p+4pb)^2 - 12p - 24bp$$

$$\Rightarrow 16b^2p^2 \leq (2p+4pb)^2 - 4p - 16bp$$

$$\Rightarrow 16b^2p^2 \leq 4p^2 + 16p^2b^2 + 16p^2b - 4p - 16bp$$

$$\Rightarrow 0 \leq 4p^2 + 16p^2b - 4p - 16bp$$

$$\Rightarrow 0 \leq 4p^2(1+4b) - 4p(1+4b) = 4p(p-1)(1+4b),$$

which is impossible given $p \leq 1$.

Now, we consider the other root

$$t^* = \frac{-(3+4(b-1)p) - \sqrt{16b^2p^2 - 8(b+1)p + 9}}{4p}. \tag{6}$$

This root can be eliminated because it is negative. To see why, suppose otherwise:

$$\frac{-(3+4(b-1)p) - \sqrt{16b^2p^2 - 8(b+1)p + 9}}{4p} \geq 0$$

$$\Rightarrow -(3+4(b-1)p) - \sqrt{16b^2p^2 - 8(b+1)p + 9} \geq 0$$

$$\Rightarrow -3 - 4(b-1)p \geq \sqrt{16b^2p^2 - 8(b+1)p + 9}$$

$$\Rightarrow -3 + 4(1-b)p \geq \sqrt{16b^2p^2 - 8(b+1)p + 9}.$$

Squaring both sides yields:

$$9 + 16(1-b)^2p^2 - 24(1-b)p \geq 16b^2p^2 - 8(b+1)p + 9$$

$$\Rightarrow 16(1-2b+b^2)p^2 - 24(1-b)p \geq 16b^2p^2 - 8(b+1)p$$

$$\Rightarrow 16p^2 - 32bp^2 + 16b^2p^2 - 24(1-b)p \geq 16b^2p^2 - 8(b+1)p$$

$$\Rightarrow 16p^2 - 32bp^2 - 24(1-b)p \geq -8(b+1)p$$

$$\Rightarrow 16p^2 - 32bp^2 \geq 24(1-b)p - 8(b+1)p$$

$$\Rightarrow 16p^2(1-2b) \geq 16p(1-2b)$$

which is a contradiction given $p \leq 1$. □

Appendix B

Suppose there are three types of Senders: an informed Sender, an uninformed Sender who prefers higher actions (US^H), and an uninformed Sender who prefers lower actions (US^L). The Receiver believes that the Sender is US^H with probability p_1, US^L with probability p_2, and informed with probability $(1-p_1-p_2)$.

The informed Sender sets threshold $t \in [0,1]$ and sends message B when $\theta \in [0,t)$, and T when $\theta \in [t,1]$. Suppose US^H sends message T with probability α and B with probability $(1-\alpha)$ for all $\theta \in \Theta$, and US^L sends message T with probability β and B with probability $(1-\beta)$ for all $\theta \in \Theta$.

Lemma 6.5. *Let a_T denote the Receiver's best response upon seeing message T. Then,*

$$a_T = 0.5\left(\frac{p_1(\alpha+t^2-1)+p_2(\beta+t^2-1)-t^2+1}{p_1(\alpha+t-1)+p_2(\beta+t-1)-t+1}\right).$$

Proof. The probability that the Receiver sees message T is $\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)$. Given that the message is T, the probability that the Sender is uninformed and prefers higher actions is

$$\frac{\alpha p_1}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}.$$

Similarly, the probability that the Sender is uninformed and prefers lower actions is

$$\frac{\beta p_2}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}.$$

Finally, given message T, the probability that the Sender is informed is

$$\frac{(1-t)(1-p_1-p_2)}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}.$$

The Receiver's expected payoff from choosing action a is:

$$E[u_R|T] = E[u_R|US^H,T]P(US^H|T)+E[u_R|US^L,T]P(US^L|T)+E[u_R|IS,T]P(IS|T)$$

$$= \frac{\alpha p_1}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}\int_0^1 -(a-\theta)^2 d\theta$$

$$+\frac{\beta p_2}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}\int_0^1 -(a-\theta)^2 d\theta$$

$$+\frac{(1-t)(1-p_1-p_2)}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}\int_t^1 \frac{-(a-\theta)^2}{1-t}\,d\theta$$

$$=-\frac{[\alpha p_1+\beta p_2]\int_0^1 (a-\theta)^2 d\theta}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}-\frac{[1-p_1-p_2]\int_t^1 (a-\theta)^2\,d\theta}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}. \tag{7}$$

To determine the Receiver's optimal choice of a we set the derivative of (7) with respect to a equal to zero:

$$-\frac{[\alpha p_1+\beta p_2]\int_0^1 2(a-\theta)d\theta}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}-\frac{[1-p_1-p_2]\int_t^1 2(a-\theta)\,d\theta}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}=0$$

$$\frac{(\alpha p_1+\beta p_2)\Big[a-0.5\Big]}{\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2)}+\frac{(1-p_1-p_2)\Big[(a-0.5)-(at-0.5t^2)\Big]}{(\alpha p_1+\beta p_2+(1-t)(1-p_1-p_2))(1-t)}=0$$

$$\Rightarrow a_T=0.5\left(\frac{p_1(\alpha+t^2-1)+p_2(\beta+t^2-1)-t^2+1}{p_1(\alpha+t-1)+p_2(\beta+t-1)-t+1}\right). \qquad \square$$

Lemma 6.6. *Let a_T denote the Receiver's best response upon seeing message T. Then,*

$$a_B=0.5\left(\frac{p_1(\alpha+t^2-1)+p_2(\beta+t^2-1)-t^2}{p_1(\alpha+t-1)+p_2(\beta+t-1)-t}\right).$$

Proof. The probability the Receiver sees message B is $(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)$. Given that the message is B, the probability that the Sender is uninformed and prefers higher actions is

$$\frac{(1-\alpha)p_1}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}.$$

Given that the message is B, the probability that the Sender is uninformed and prefers lower actions is

$$\frac{(1-\beta)p_2}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}.$$

Finally, if the message is B, the probability that the Sender is informed is

$$\frac{t(1-p_1-p_2)}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}.$$

The Receiver's expected payoff from choosing action a is $E[u_R|B] =$

$$E[u_R|US^H,B]P(US^H|B) + E[u_R|US^L,B]P(US^L|B) + E[u_R|IS,B]P(IS|B) = \tag{8}$$

$$-\frac{(1-\alpha)p_1\int_0^1 (a-x)^2dx}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} - \frac{(1-\beta)p_2\int_0^1 (a-x)^2dx}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}$$

$$-\frac{t(1-p_1-p_2)}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}\int_0^t \frac{(a-\theta)^2}{t}\,d\theta =$$

$$-\frac{[(1-\alpha)p_1+(1-\beta)p_2]\int_0^1 (a-\theta)^2d\theta}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} - \frac{[1-p_1-p_2]\int_0^t (a-\theta)^2\,d\theta}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}.$$

To determine the Receiver's optimal choice of a, we differentiate (8) with respect to a and set to zero:

$$-\frac{[(1-\alpha)p_1+(1-\beta)p_2]\int_0^1 (2a-2\theta)d\theta}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} - \frac{[1-p_1-p_2]\int_0^t (2a-2\theta)\,d\theta}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} = 0$$

$$-2\left(\frac{[(1-\alpha)p_1+(1-\beta)p_2]\int_0^1 (a-\theta)d\theta}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} + \frac{[1-p_1-p_2]\int_0^t (a-\theta)\,d\theta}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}\right) = 0$$

$$\frac{[(1-\alpha)p_1+(1-\beta)p_2]\int_0^1 (a-\theta)d\theta}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} + \frac{[1-p_1-p_2]\int_0^t (a-\theta)\,d\theta}{((1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} = 0$$

$$\frac{[(1-\alpha)p_1+(1-\beta)p_2]\left[a\theta-0.5\theta^2\right]_0^1}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} + \frac{[1-p_1-p_2]\left[a\theta-0.5\theta^2\right]_0^t}{((1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} = 0$$

$$\frac{[(1-\alpha)p_1+(1-\beta)p_2]\left[a-0.5\right]}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} + \frac{[1-p_1-p_2]\left[at-0.5t^2\right]}{((1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)} = 0$$

$$\frac{[(1-\alpha)p_1+(1-\beta)p_2]\Big[a-0.5\Big]}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}+\frac{[1-p_1-p_2]\Big[a-0.5t\Big]}{(1-\alpha)p_1+(1-\beta)p_2+t(1-p_1-p_2)}=0$$

$$\Rightarrow a_B=0.5\left(\frac{p_1(\alpha+t^2-1)+p_2(\beta+t^2-1)-t^2}{p_1(\alpha+t-1)+p_2(\beta+t-1)-t}\right). \qquad \square$$

The uninformed Sender who prefers higher actions must solve the following problem:

$$\max_{\alpha} E[a]=\max_{\alpha}[a_T P(T)+a_B P(B)]=\max_{\alpha}[a_T\alpha+a_B(1-\alpha)]$$

$$\begin{aligned}=\max_{\alpha\in[0,1]}\Bigg(&0.5\left(\frac{p_1(\alpha+t^2-1)+p_2(\beta+t^2-1)-t^2+1}{p_1(\alpha+t-1)+p_2(\beta+t-1)-t+1}\right)\times\alpha\\ &+0.5\left(\frac{p_1(\alpha+t^2-1)+p_2(\beta+t^2-1)-t^2}{p_1(\alpha+t-1)+p_2(\beta+t-1)-t}\right)\times(1-\alpha)\Bigg)\end{aligned}$$

Lemma 6.7. *The uninformed Sender who prefers higher actions will choose $\alpha = 1$.*

Proof. The first-order condition for an interior optimal is:

$$\begin{aligned}\alpha^* = \frac{1}{(p_1-1){p_1}^2}\Bigg(&-\big({p_1}^2(\beta^2{p_2}^2+2\beta p_1p_2t-\beta p_1p_2+2\beta {p_2}^2t\\ &-2\beta{p_2}^2-2\beta p_2t+\beta p_2+{p_1}^2t^2-{p_1}^2t+2p_1p_2t^2-3p_1p_2t+p_1p_2\\ &-2p_1t^2+2p_1t+{p_2}^2t^2-2{p_2}^2t+{p_2}^2-2p_2t^2+3p_2t-p_2+t^2-t)\big)^{\frac{1}{2}}\\ &-bp^2q-tp^3+p^3-p^2qt+p^2q+p^2t-p^2\Bigg).\end{aligned}$$

Given $t\in(0,1)$, $p_1\in(0,1)$, $p_2\in(0,1)$, and $\beta\in(0,1)$, this expression has no solution in $(0,1)$. Hence, at optimality $\alpha=0$ or $\alpha=1$.

Setting $\alpha=0$ yields the following expected payoff for US^H:

$$0.5\left(\frac{p_1(t^2-1)+p_2(\beta+t^2-1)-t^2}{p_1(t-1)+p_2(\beta+t-1)-t}\right).$$

Setting $\alpha=1$ yields the following expected payoff for US^H:

$$0.5\left(\frac{p_1t^2+p_2(\beta+t^2-1)-t^2+1}{p_1t+p_2(\beta+t-1)-t+1}\right).$$

With $t\in(0,1)$ and $p_1\in(0,1)$, $p_2\in(0,1)$, and $\beta\in(0,1)$, $\alpha=1$ yields a greater payoff for US^H. Thus, the uninformed Sender who prefers higher actions will always send message T. $\square$

By symmetry, the uninformed Sender who prefers lower actions will choose $\alpha = 0$, i.e., will always send message B. The informed Sender will choose threshold t so that

$$((t+b)-a_B)^2 = (a_T-(t+b))^2.$$

Lemma 5.2

Proof. From equation (1), we have:

$$0.5\left(\frac{p_ut^2+p_u(t^2-1)-t^2+1}{p_ut+p_u(t-1)-t+1}\right)+0.5\left(\frac{p_ut^2+p_u(t^2-1)-t^2}{p_ut+p_u(t-1)-t}\right)-2t=2b. \quad (9)$$

Denote the left hand side of equation (9) by $f(t)$. In equilibrium, $f(t^*)=2b$ and the derivative of $f(t)$ with respect to t is:

$$\frac{\partial f}{\partial t}=\frac{0.125p_u^2-0.125p_u}{(p_ut-0.5p_u-0.5t)^2}+\frac{0.125(p_u^3-1.5p_u^2+0.5p_u)}{(p_u-0.5)(p_ut-0.5p_u-0.5t+0.5)^2}-1$$

$$=\frac{p_u(p_u-1)}{8((p_u-0.5)t-0.5p_u)^2}+\frac{p_u(2p_u-1)(p_u-1)}{16(p_u-0.5)((p_u-0.5)t+0.5(1-p_u))^2}-1. \quad (10)$$

In the right hand side of equation (10) let

$$A=\frac{p_u(p_u-1)}{8((p_u-0.5)t-0.5p_u)^2}$$

and

$$C=\frac{p_u(2p_u-1)(p_u-1)}{16(p_u-0.5)((p_u-0.5)t+0.5(1-p_u))^2}.$$

Denote the respective numerators by N_A and N_C and the respective denominators by D_A and D_C. Recall that we must have $p_u \in (0,0.5)$ and $t \in (0,1)$.

Now, $N_A = p_u(p_u-1)$ is negative and $D_A = 8((p_u-0.5)t-0.5p_u)^2$ is positive. Hence, $A<0$. Also, $N_C = p_u(2p_u-1)(p_u-1)$ is positive as it is the product of two negative numbers. In addition, $D_C = 16(p_u-0.5)((p_u-0.5)t+0.5(1-p_u))^2$ is negative as it is the product of a positive and a negative number. Hence, $D_C<0$. Therefore, $C<0$.

As $\frac{\partial f}{\partial t}=A+C-1<0$ it follows that f is decreasing in t over the relevant domain. Denote the standard cheap talk threshold by $\bar{t}=0.5-2b$. Assume, for a contradiction, that $f(\bar{t})<f(t^*)$. This yields the following inequality:

$$f(\bar{t})=\frac{b(2b^2(p_u-0.5)^2+\frac{1}{8}p_u(p_u-1)-\frac{1}{32})}{b^2(p_u-0.5)^2-\frac{1}{64}}<2b=f(t^*). \quad (11)$$

We now show that the denominator on the left hand side of inequality (11) is negative. If not,

$$b^2(p_u-0.5)^2-\frac{1}{64}>0$$
$$\Rightarrow b^2(p_u-0.5)^2>\frac{1}{64}$$
$$\Rightarrow b(p_u-0.5)>\frac{1}{8}$$
$$\Rightarrow p_u>\frac{1}{8b}+\frac{1}{2}.$$

This cannot be as $p_u \in (0,0.5)$ and $b>0$.

Hence, if we multiply both sides of (11) by $b^2(p_u-0.5)^2-\frac{1}{64}$, the sign of the inequality reverses. Therefore, when $f(\bar{t})<f(t^*)$, the following holds:

$$b(2b^2(p_u-0.5)^2+\frac{1}{8}p_u(p_u-1)-\frac{1}{32})>2b(b^2(p_u-0.5)^2-\frac{1}{64})$$
$$\Rightarrow 2b^3(p_u-0.5)^2+\frac{1}{8}bp_u(p_u-1)-\frac{1}{32}b>2b^3(p_u-0.5)^2-\frac{1}{32}b$$
$$\Rightarrow \frac{1}{8}bp_u(p_u-1)>0.$$

Provided that $b>0$ this is equivalent to $p_u(p_u-1)>0$. Therefore, one of the following must be true:

1. Both p_u and (p_u-1) are negative.
2. Both p_u and (p_u-1) are positive.

Now p_u cannot be negative, nor can it be that (p_u-1) is positive, as $p \in (0,\frac{1}{2})$. Therefore, $f(\bar{t})<f(t^*)$ which is a contradiction. This means that $f(\bar{t})\geq f(t^*)$. Since $f(\cdot)$ is a decreasing function of t, $f(\bar{t})\geq f(t^*)$ implies that $\bar{t}\leq t^*$. Thus, the presence of the uninformed Sender raises the informed Sender's equilibrium threshold when compared to the standard cheap talk model. This implies that, ex ante, the informed Sender who faces two uninformed Senders with opposing preferences, sends message B with greater probability than in the cheap talk case.

Now, we evaluate $f(0.5)$ and show that it is less than $f(t^*)$. This will imply, $t^*<0.5$ and therefore, by Lemma 3.2, the informed Sender is better off. Now,

$$f(0.5)=0.5\left(\frac{0.25p_u-0.75p_u+0.75}{0.5p_u-0.5p_u+0.5}\right)+0.5\left(\frac{0.25p_u-0.75p_u-0.25}{0.5p_u+0.5p_u-0.5}\right)-1$$
$$=0.5\frac{-0.5p_u+0.75}{0.5}+0.5\frac{-0.5p_u-0.25}{-0.5}-1=0<2b=f(t^*). \quad \square$$

Appendix C

We consider the possibility of an equilibrium with multiple messages. Suppose there exists an equilibrium with n messages denoted by M_k for $k = 1,2,...,n$, with M_1 representing the bottom message and M_n representing the top message. The messages partition $[0,1]$ into n intervals, separated by $n-1$ thresholds. Denote by t_k the threshold that separates the intervals M_k and M_{k+1}. Thus, $t_1 \leq t_2 \leq \ldots \leq t_{n-1}$. Suppose the uninformed Sender with a preference for higher actions sends message M_k with probability q_k. Our goal is to identify the choice of q_k's that maximize the uninformed Sender's expected payoff.

Upon observing message M_1, the Receiver chooses action a to maximize her expected payoff:

$$E[u_R|M_1] = E[u_R|IS,M_1]P(IS|M_1) + E[u_R|US,M_1]P(US|M_1)$$

$$= \frac{pt_1}{pt_1 + (1-p)q_1} \int_0^{t_1} -\frac{(a-\theta)^2}{t_1} d\theta + \frac{(1-p)q_1}{pt_1 + (1-p)q_1} \int_0^1 -(a-\theta)^2 d\theta$$

$$\Rightarrow a_{M_1} = 0.5\left(\frac{p({t_1}^2 - q_1) + q_1}{p(t_1 - q_1) + q_1}\right).$$

More generally, upon seeing message M_i (for arbitrary integer $i = 2,3,...,n-1$), the Receiver chooses action a to maximize the following:

$$E[u_R|M_i] = E[u_R|IS,M_t]P(IS|M_t) + E[u_R|US,M_i]P(US|M_i)$$

$$= \frac{p(t_i - t_{i-1})}{p(t_i - t_{i-1}) + (1-p)q_i} \int_{t_{i-1}}^{t_i} -\frac{(a-\theta)^2}{t_i - t_{i-1}} d\theta + \frac{(1-p)q_i}{p(t_i - t_{i-1}) + (1-p)q_i} \int_0^1 -(a-\theta)^2 d\theta$$

$$\Rightarrow a_{M_i} = 0.5\left(\frac{p({t_i}^2 - t_{i-1}^2 - q_i) + q_i}{p(t_i - t_{i-1} - q_i) + q_i}\right).$$

Upon seeing message M_n, the Receiver chooses action a to maximize the following:

$$E[u_R|M_n] = E[u_R|IS,M_n]P(IS|M_n) + E[u_R|US,M_n]P(US|M_n)$$

$$= \frac{p(1 - t_{n-1})}{p(1 - t_{n-1}) + (1-p)q_n} \int_{t_{n-1}}^{1} -\frac{(a-\theta)^2}{1 - t_{n-1}} d\theta + \frac{(1-p)q_n}{p(1 - t_{n-1}) + (1-p)q_n} \int_0^1 -(a-\theta)^2 d\theta$$

$$\Rightarrow a_{M_n} = 0.5\left(\frac{p(1 - t_{n-1}^2 - q_n) + q_n}{p(1 - t_{n-1} - q_n) + q_n}\right).$$

Thus, the expected action of the Receiver is given by:

$$E[a|s=US] = \sum_{t=1}^{n} q_t a_{M_t} = q_1 a_{M_1} + \sum_{t=2}^{n-1} [q_t a_{M_t}] + q_n a_{M_n}$$

$$= q_1 \left(0.5 \left(\frac{p({t_1}^2 - q_1) + q_1}{p(t_1 - q_1) + q_1}\right)\right) + \sum_{i=2}^{n-1} \left[q_i \left(0.5 \left(\frac{p({t_i}^2 - t_{i-1}^2 - q_i) + q_i}{p(t_i - t_{i-1} - q_i) + q_i}\right)\right)\right]$$

$$+q_n \left(0.5 \left(\frac{p(1 - t_{n-1}^2 - q_n) + q_n}{p(1 - t_{n-1} - q_n) + q_n}\right)\right).$$

Let

$$f_1(q_1) = q_1 \left(\frac{p{t_1}^2 + (1-p)q_1}{2(pt_1 + (1-p)q_1)}\right),\ f_i(q_i) = q_i \left(\frac{p({t_i}^2 - t_{i-1}^2) + (1-p)q_i}{2(p(t_i - t_{i-1}) + (1-p)q_1)}\right)$$

for $i = 2, \ldots, n-2$ and

$$f_n(q_n) = f_n = q_n \left(\frac{p(1 - {t_{n-1}}^2) + (1-p)q_n}{2(p(1 - t_{n-1}) + (1-p)q_n)}\right).$$

Then, $E[a|s=US] = \sum_{i=1}^{n-1} f_i(q_i)$. The uninformed Sender's problem is the following:

$$\max \sum_{i=1}^{n-1} f_i(q_i)$$

$$\text{s.t.} \sum_{i=1}^{n-1} q_i = 1;\ q_i \geq 0\ \forall i = 1, \ldots, n-1$$

Let $(q_1^*, \ldots, q_{n-1}^*)$ be an optimal solution to this program. A direct computation shows the following:

- $t_i + t_{i-1} < 1 \Rightarrow f_i(q_i)$ is convex in q_i;
- $t_i + t_{i-1} = 1 \Rightarrow f_i(q_i)$ is linear in q_i;
- $t_i + t_{i-1} > 1 \Rightarrow f_i(q_i)$ is concave in q_i.

Let $r-1$ be the largest index such that $t_{r-1} \leq 0.5$ then $f_1, \ldots, f_{r-1}$ are convex. Consider the following sub-problem:

$$\max \sum_{i=1}^{r-1} f_i(q_i)$$

$$\text{s.t.} \sum_{i=1}^{r-1} q_i = \sum_{i=1}^{r-1} q_i^*;\ q_i \geq 0\ \forall i = 1, \ldots, r-1.$$

As the function being optimized is convex, the maximum of the subproblem must occur at one of the extreme points. A straightforward computation shows that the optimal solution will be $q_{r-1} = \sum_{i=1}^{r-1} q_i^*$. In other words, $q_1 = 0$, i.e., the uninformed Sender never sends the lowest message.

Three Messages

Now, consider the possibility of an equilibrium with three messages, the informed Sender uses thresholds $t_1 \in [0,1]$ and $t_2 \in [0,1]$ where $t_2 > t_1$. The informed Sender sends message B when $\theta \in [0,t_1]$, M when $\theta \in [t_1,t_2]$, and T when $\theta \in [t_2,1]$. The uninformed Sender sends message T with probability α and M with probability $(1-\alpha)$. By the argument above the uninformed Sender *never* sends message B.

The uninformed Sender chooses α to maximize the expected action of the Receiver, since the uninformed Sender strictly prefers higher actions. Thus, the uninformed Sender will choose $\alpha \in [0,1]$ to maximize

$$uR := 0.5\left(\frac{p(\alpha+t_2^2-1)-\alpha}{p(\alpha+t_2-1)-\alpha}\right)\times\alpha + 0.5\left(\frac{p(\alpha+t_2^2-t_1^2-1)-\alpha+1}{p(\alpha+t_2-t_1-1)-\alpha+1}\right)\times(1-\alpha).$$

The first-order condition for optimality is:

$$\begin{aligned}\partial uR/\partial\alpha = 0 \Rightarrow \alpha^* &= 2p^2{t_1}^3 t_2 - 2p^2{t_1}^3 - 2p^2{t_1}^2{t_2}^2 + 2p^2{t_1}^2 \\ &- \sqrt{-2p^2{t_1}^3 t_2 + 2p^2{t_1}^3 + 2p^2{t_1}^2{t_2}^2}.\end{aligned} \tag{12}$$

Note that for some values of p, t_1 and t_2, it may be optimal to choose α strictly between zero and one.[3]

The informed Sender's indifference condition between messages T and M is:

$$0.5\left(\frac{p(\alpha+t_2^2-1)-\alpha}{p(\alpha+t_2-1)-\alpha}\right)-t_2-b = t_2+b-0.5\left(\frac{p(\alpha+t_2^2-t_1^2-1)-\alpha+1}{p(\alpha+t_2-t_1-1)-\alpha+1}\right).$$

The indifference condition between messages M and B is:

$$0.5\left(\frac{p(\alpha+t_2^2-t_1^2-1)-\alpha+1}{p(\alpha+t_2-t_1-1)-\alpha+1}\right)-t_1-b = t_1+b-0.5t_1$$

When $\alpha = 1$,

$$t_2 = \frac{5-36bp-\sqrt{144b^2p^2+24(b-1)p+25}}{12p}$$

[3] See the graph: https://www.desmos.com/calculator/rdv7j9qiwa.

and

$$t_1 = \frac{5-12bp-\sqrt{144b^2p^2+24(b-1)p+25}}{6p}.$$

If we substitute this into equation (12) we see that there is no interior solution. Hence, the uninformed Sender's best response to the choice of thresholds above is $\alpha = 1$.

We now focus on the case $\alpha = 1$ to determine the upper limit on b to support a three message equilibrium. The Receiver's best response to each message is:

$$a_T = 0.5\left(\frac{pt_2{}^2-1}{pt_2-1}\right),\ a_M = 0.5(t_1+t_2),\ a_B = 0.5t_1.$$

The informed Sender's thresholds (ignoring the babbling case) are determined by the following indifference conditions:

$$0.5\left(\frac{pt_2{}^2-1}{pt_2-1}\right)-(t_2+b) = t_2+b-0.5(t_2+t_1). \tag{13}$$

From equation (13) the equilibrium value of t_2 as a function of t_1, denoted $t_2^*(t_1)$, must satisfy:

$$t_2^*(t_1) = \frac{-\sqrt{p^2(t_1-4b)^2+8bp-2p(t_1+4)+9}-4bp+pt_1+3}{4p}.$$

The second indifference condition is

$$0.5(t_2+t_1)-(t_1+b) = t_1+b-0.5t_1. \tag{14}$$

From equation (14) the equilibrium value of t_1 as a function of t_2, denoted $t_1^*(t_2)$, must satisfy:

$$t_1^*(t_2) = 0.5t_2-2b.$$

Solving these equations we find that the equilibrium thresholds satisfy the following:

$$t_1^*(p,b) = \frac{36bp-72b-7p+12-sq}{6(p-3)(2p-3)},$$
$$t_2^*(p,b) = \frac{12bp+7p-12+sq}{6p(2p-3)}$$

for $sq = \sqrt{144b^2p^2-120bp^2+144bp-24p^3+157p^2-276p+144}$. Next, we determine the upper limit of b in order to support such an equilibrium.

In the absence of an uninformed Sender, a three message equilibrium exists provided the following condition on the informed Sender's bias is satisfied (Crawford & Sobel 1982):

$$2 \leq \frac{1}{2}\left(\sqrt{1+\frac{2}{b}}-1\right) \Rightarrow b \leq \frac{1}{12}.$$

When the uninformed Sender is present with probability $1-p$, a two message equilibrium is supported when $t_1 \geq 0$:

$$\frac{36bp-72b-7p+12-sq}{6(p-3)(2p-3)} \geq 0.$$

As $p \in (0,1)$ the denominator in the left hand side of the above is positive. Hence,

$$36bp-72b-7p+12-sq \geq 0$$

$$\Rightarrow 36bp-72b-7p+12 \geq sq$$

$$\Rightarrow (36bp-72b-7p+12)^2 \geq 144b^2p^2-120bp^2+144bp-24p^3+157p^2-276p+144$$

$$\Rightarrow 1152b^2p^2-384bp^2+1728bp-108p^2+108p-5184b^2p+5184b^2+1728b+24p^3 \geq 0$$

$$\Rightarrow 12(p-3)(2p-3)(p+48b^2-16b) \geq 0$$

$$\Rightarrow p+48b^2-16b \geq 0.$$

The roots of the quadratic on the left are $b = (2 \pm \sqrt{4-3p})/12$ and they are real for $p \in [0,1]$. Therefore, $b \leq (2-\sqrt{4-3p})/12$ or $b \geq (2+\sqrt{4-3p})/12$. Hence, the value of b under which a three message equilibrium exists under cheap talk *and* in the presence of an uninformed Sender must satisfy $b \leq (2-\sqrt{4-3p})/12$. For such a b, the possible values of p consistent with a three message equilibrium in the presence of an uninformed Sender must be at least as large as $16b(1-3b)$.

Appendix D

Suppose there exist two types of informed Senders: one with low bias b_l denoted (IS^L) and another with high bias b_h denoted (IS^H). Both Senders send a state-dependent message. We compare the value of the information sent to the Receiver in this case with that in which the Receiver faces either a low-biased informed Sender or an uninformed Sender with preferences for higher actions only.

IS^L sets threshold $t_l \in [0,1]$ to determine which message is sent. IS^L sends message B when $\theta \in [0,t_l)$, and T when $\theta \in [t_l,1]$. Let $t_H \in [0,1]$ be the threshold for IS^H to determine her binary message. Thus, IS^H sends message B when $\theta \in [0,t_h)$, and T when $\theta \in [t_h,1]$. Assume that the informed Sender is IS^L with probability p and IS^H with probability $1-p$.

The Receiver observes the message, but not the type of the informed Sender. The Receiver knows p, b_l, and b_h and uses these to inform her beliefs on the Sender's type and, thus, θ. Then, the Receiver chooses an action to maximize her expected payoff.

Receiver's Action in Response to T

The probability that the Receiver sees message T is:

$$P(T) = P(T|IS^L)P(IS^L) + P(T|IS^H)P(IS^H) = p(1-t_l) + (1-p)(1-t_h).$$

Given that the message is T, the probability that the sender is IS^L is given by:

$$P(IS^L|T) = \frac{P(IS^L)P(T|IS^L)}{P(T)} = \frac{p(1-t_l)}{p(1-t_l)+(1-p)(1-t_h)}.$$

Given that the message is T, the probability that the sender is IS^H is given by:

$$P(IS^H|T) = \frac{P(IS^H)P(T|IS^H)}{P(T)} = \frac{(1-p)(1-t_h)}{p(1-t_l)+(1-p)(1-t_h)}.$$

R's expected payoff from choosing action $a \in [0,1]$ is given by:

$$E[u_R|T] = E[u_R|IS^L,T]P(IS^L|T) + E[u_R|IS^H,T]P(IS^H|T)$$

$$= \frac{p(1-t_l)}{p(1-t_l)+(1-p)(1-t_h)} \int_{t_l}^{1} -\frac{(a-\theta)^2}{1-t_l} d\theta + \frac{(1-p)(1-t_h)}{p(1-t_l)+(1-p)(1-t_h)}$$

$$\int_{t_h}^{1} -\frac{(a-\theta)^2}{1-t_h} d\theta = -\frac{p\int_{t_l}^{1}(a^2-2a\theta+\theta^2)\,d\theta}{p(1-t_l)+(1-p)(1-t_h)} - \frac{(1-p)\int_{t_h}^{1}(a^2-2a\theta+\theta^2)\,d\theta}{p(1-t_l)+(1-p)(1-t_h)}.$$

To determine R's optimal choice of action, we invoke the first-order condition:

$$\partial\left[-\frac{p\int_{t_l}^{1}(a^2-2a\theta+\theta^2)\,d\theta}{p(1-t_l)+(1-p)(1-t_h)} - \frac{(1-p)\int_{t_h}^{1}(a^2-2a\theta+\theta^2)\,d\theta}{p(1-t_l)+(1-p)(1-t_h)}\right] \Big/ \partial a = 0$$

$$-2\left(\frac{p\int_{t_l}^{1}(a-\theta)\,d\theta}{p(1-t_l)+(1-p)(1-t_h)} + \frac{(1-p)\int_{t_h}^{1}(a-\theta)\,d\theta}{p(1-t_l)+(1-p)(1-t_h)}\right) = 0$$

$$\frac{p\left[a\theta-0.5\theta^2\right]_{t_l}^{1}}{p(1-t_l)+(1-p)(1-t_h)}+\frac{(1-p)\left[a\theta-0.5\theta^2\right]_{t_h}^{1}}{p(1-t_l)+(1-p)(1-t_h)}=0$$

$$\frac{p(a-0.5-at_l+0.5{t_l}^2)+(1-p)(a-0.5-at_h+0.5{t_h}^2)}{p(1-t_l)+(1-p)(1-t_h)}=0$$

$$a_T=0.5\left(\frac{p({t_l}^2-{t_h}^2)+{t_h}^2-1}{p(t_l-t_h)+t_h-1}\right).$$

Receiver's Action in Response to B

The probability that the Receiver sees message B is given by:

$$P(B)=P(B|IS^L)P(IS^L)+P(B|IS^H)P(IS^H)=pt_l+(1-p)t_h.$$

Given that the message is B, the probability that the informed Sender has low bias is given by:

$$P(IS^L|B)=\frac{P(IS^L)P(B|IS^L)}{P(B)}=\frac{pt_l}{pt_l+(1-p)t_h},$$

Given that the message is B, the probability that informed Sender has high bias is given by:

$$P(IS^H|B)=\frac{P(IS^H)P(B|IS^H)}{P(B)}=\frac{(1-p)t_h}{pt_l+(1-p)t_h}.$$

The Receiver's expected payoff from choosing action a is given by:

$$E[u_R|B]=E[u_R|IS^L,B]P(IS^L|B)+E[u_R|IS^H,B]P(IS^H|B)$$

$$=\frac{pt_l}{pt_l+(1-p)t_h}\int_0^{t_l}-\frac{(a-\theta)^2}{t_l}d\theta+\frac{(1-p)t_h}{pt_l+(1-p)t_h}\int_0^{t_h}-\frac{(a-\theta)^2}{t_h}d\theta$$

$$=-\frac{p\int_0^{t_l}(a^2-2a\theta+\theta^2)\,d\theta}{pt_l+(1-p)t_h}-\frac{(1-p)\int_0^{t_h}(a^2-2a\theta+\theta^2)\,d\theta}{pt_l+(1-p)t_h}=:uR.$$

To determine the Receiver's optimal choice of a, we invoke the first-order condition for optimality:

$$\partial uR/\partial a=0$$

$$-2\left(\frac{p\int_0^{t_l}(a-\theta)\,d\theta}{pt_l+(1-p)t_h}+\frac{(1-p)\int_0^{t_h}(a-\theta)\,d\theta}{pt_l+(1-p)t_h}\right)=0$$

$$\frac{p}{pt_l+(1-p)t_h}\left[a\theta-0.5\theta^2\right]_0^{t_l}+\frac{(1-p)}{pt_l+(1-p)t_h}\left[a\theta-0.5\theta^2\right]_0^{t_h}=0$$

$$\frac{p(at_l-0.5t_l{}^2)+(1-p)(at_h-0.5t_h{}^2)}{pt_l+(1-p)t_h}=0$$

$$a_B=0.5\left(\frac{p(t_l{}^2-t_h{}^2)+t_h{}^2}{p(t_l-t_h)+t_h}\right). \tag{15}$$

Senders' Strategies

The informed Senders individually set their thresholds such that they are indifferent between sending T and B. In equilibrium, this means that both of the following equations are satisfied by (t_l^*, t_h^*):

$$\begin{aligned}
&-\left(0.5\left(\frac{p(t_l{}^2-t_h{}^2)+t_h{}^2-1}{p(t_l-t_h)+t_h-1}\right)-(t_l+b_l)\right)^2\\
&=-\left((t_l+b_l)-0.5\left(\frac{p(t_l{}^2-t_h{}^2)+t_h{}^2}{p(t_l-t_h)+t_h}\right)\right)^2\\
&\quad-\left(0.5\left(\frac{p(t_l{}^2-t_h{}^2)+t_h{}^2-1}{p(t_l-t_h)+t_h-1}\right)-(t_h+b_h)\right)^2\\
&=-\left((t_h+b_h)-0.5\left(\frac{p(t_l{}^2-t_h{}^2)+t_h{}^2}{p(t_l-t_h)+t_h}\right)\right)^2.
\end{aligned} \tag{16}$$