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THE EXISTENCE OF EQUILIBRIUM FLOWS

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ABSTRACT

In this paper, we establish conditions for the existence of an equilibrium in the equilibrium flow problem studied by Galichon et al. (2024). The problem nests several classical economic models such as bipartite matching models, hedonic pricing models, shortest-path and minimum-cost flow problems, and time-dependent routing problems.

Keywords: Equilibrium flow, unified gross substitutes, matching.

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1. INTRODUCTION

Galichon, Samuelson and Vernet introduced a class of problems, *equilibrium flow problems*, that nests several classical economic models such as bipartite matching models, hedonic pricing models, shortest-path and minimum-cost flow problems, and time-dependent routing problems (Galichon et al. 2024). This paper first provides a more detailed presentation and

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motivation for the equilibrium flow problem. Our main result then provides conditions sufficient to ensure the existence of an equilibrium flow.

The framework for the equilibrium flow problem is a network, consisting of a set of nodes and a collection of directed arcs. A subset of the nodes are source nodes, each characterized by a negative quantity that can be interpreted as a quantity that must be conveyed away from the source node. Another subset of nodes are destination nodes, each characterized by a positive quantity that can be interpreted as a quantity that must be conveyed to the destination node.

A price vector attaches a price to each node, and a connection function assigns a payoff to each arc as a function of the prices at the arc's origin and destination. The structure of the network is exogenous, and the quantities assigned to source and destination nodes are also typically exogenous, while prices are determined endogenously as part of an equilibrium specification.

An equilibrium flow is a specification of the prices assigned to the nodes and a specification of flows along the arcs with the properties that (i) all of the quantity assigned to each source node is transported away from the source node and each destination node receives its specified quantity, (ii) the prices are such that no arc exhibits a positive payoff, and (iii) no flow occurs on an arc with a negative payoff.

The equilibrium flow framework captures a variety of applications. If all nodes are either source nodes or destination nodes, and an arc runs from every source node to every destination node, then the equilibrium flow problem can be used to model one-to-one matching, either with transfers (as in Crawford & Knoer 1981 or Demange & Gale 1985) or without transfers (as in Gale & Shapley 1962), with prices taking the role of utilities. Interpreting source nodes as firms, destination nodes as consumers, and intermediate nodes as products produced by the firms, the equilibrium flow problem can model hedonic pricing (Rosen 1974; Chiappori et al. 2010; Ekeland et al. 2004). If there is a single source node and single destination node (but perhaps many intermediate nodes), the equilibrium flow problem generalizes the shortest path problem of Bellman (1958). Expanding to multiple sources and destinations gives the minimum-cost flow problem of optimal transport (Galichon 2018). Time dependent routing problems (Gendreau et al. 2015) are similarly a special case. We expect there to be other applications.

The equilibrium flow problem has two useful features. First, the framework requires no assumptions on the payoff functions other than the appropriate monotonicity conditions, obviating the need to work with transferable utility or analogous linearity assumptions. Second, Galichon et al. (2024) show that the equilibrium correspondence, associating with each price vector the set of quantities consistent with an equilibrium flow, satisfies their condition of unified gross substitutes. The unified gross substitutes condition is a natural extension to correspondences of the familiar weak gross substitutes assumption for functions and is a natural strengthening of Kelso & Crawford's (1982) gross substitutes condition for correspondences. This characterization of the equilibrium correspondence in turn provides the foundation for comparative static results. Section 4 illustrates.

Section 2 presents the equilibrium flow problem, making precise this section's intuitive description.

Section 3 presents the existence argument that constitutes the contribution of this paper. The argument rests on two building blocks, each a variation on familiar ideas. First, there must exist a set of feasible flows capable of carrying the quantities from the source nodes to the destination nodes. We rely on a generalization of Hall's (1935) theorem to obtain a sufficient condition for feasibility. Second, we must be able to find prices such that flows occur only along zero-payoff arcs, with no arcs exhibiting positive payoffs. Here, we rely on a condition reminiscent of Rochet's (1987) cyclical monotonicity.

Section 4 presents an example of the applications of the equilibrium flow problem.

2. EQUILIBRIUM FLOW PROBLEM

This section expands on the presentation of the equilibrium flow problem in Galichon et al. (2024).

2.1. Formulation

Network. Consider a network $(\mathcal{Z}, \mathcal{A})$ where $\mathcal{Z}$ is a finite set of nodes and $\mathcal{A} \subseteq \mathcal{Z} \times \mathcal{Z}$ is the set of directed arcs. If $xy \in \mathcal{A}$, we say that xy is the arc from $x \in \mathcal{Z}$ to $y \in \mathcal{Z}$. We say that x is the starting point of the arc, while y is its end point. We assume that there is no arc in $\mathcal{A}$ whose starting point coincides with the end point, but make no other assumptions on the set of arcs at this point. For example, we allow the possibility that $\mathcal{A}$ contains the arc xy and also the arc yx.

We describe the network with an $|\mathcal{A}|\times|\mathcal{Z}|$ *arc-node incidence matrix* ∇, defined by letting, for $xy \in \mathcal{A}$ and $z \in \mathcal{Z}$

$$\nabla_{xy,z} = \mathbf{1}_{\{z=y\}} - \mathbf{1}_{\{z=x\}}.$$

Each arc thus contributes a pair of nonzero values to the matrix ∇, with $\nabla_{xy,z} = 1$ if xy is an arc ending at z (i.e., $y = z$), and $\nabla_{xy,z} = -1$ if xy is an arc beginning at z (i.e., $x = z$). Otherwise $\nabla_{xy,z} = 0$.

For example, when using the equilibrium flow problem to describe a bipartite matching problem in which all agents are matched, one would have a set of nodes representing sellers (or workers, or men, and so on). Each seller node would be connected by an arc to each element of a set of nodes representing buyers (or firms, or women, and so on), and there would be no other arcs. To expand the model to allow agents to go unmatched, one would add a "dummy" node, with appropriate arcs, interpreting an agent who matches with the dummy node as remaining unmatched. In a routing problem, one would have a collection of nodes representing traffic origins, another set of nodes representing destinations, and a collection of arcs and intermediate nodes describing the possible routes between origins and destinations. **Quantities and Flows.** Let $q \in \mathbb{R}^{\mathcal{Z}}$ with $\sum_{z\in\mathcal{Z}} q_z = 0$ attach a quantity to each node $z \in \mathcal{Z}$. If $q_z < 0$, then the net quantity $|q_z|$ must flow away from node z, while $q_z > 0$ indicates that the net quantity $|q_z|$ must flow into node z. We call q the vector of *quantities*. In many applications, we will have $q_z = 0$ for many nodes.

We let $\mu \in \mathbb{R}_+^{\mathcal{A}}$ be the vector of *internal flows* along arcs, so that μ_{xy} is the flow through arc xy. The feasibility condition connecting these notions is that, for any node $z \in \mathcal{Z}$, the total internal flow that arrives at z minus the total internal flow that leaves z equals the quantity at z, that is

$$\sum_{x:xz\in\mathcal{A}} \mu_{xz} - \sum_{y:zy\in\mathcal{A}} \mu_{zy} = q_z,$$

which we call the *mass balance equation*, and which can be rewritten as

$$\nabla^\top \mu = q. \tag{1}$$

In a matching application, without a dummy node, a quantity $q_z < 0$ would identify for each seller node z the quantity of good the node has for sale. A quantity $q_z > 0$ would identify the quantity to be purchased at buyer node z. The restriction $\sum_{z\in\mathcal{Z}} q_z = 0$ would capture the market balance condition that the total quantities supplied and demanded are equal. The mass balance equation stipulates that each seller sells all of her good, and each buyer purchases all she desires. With the addition of a dummy node, one could capture an imbalance in the total quantities supplied and demanded, as well as the possibilities that some sellers do not sell all of their good, or some buyer demands go unfilled. In a routing problem, a quantity $q_z < 0$ would identify the number of departures from source node z, while $q_z > 0$ would identify the number of arrivals at destination node z. The mass balance equation indicates that every traveler leaves her source and arrives at a destination.

Prices and connection functions. To capture incentives, we let $p \in \mathbb{R}^{\mathcal{Z}}$ be a price vector, where we interpret p_z as the price at node z. If modeling a competitive economy, we could think of a trader as being able to purchase a unit of a commodity at node x (or in market x) at price p_x and then sell it at a node y (for which $xy \in \mathcal{A}$) at price p_y. If thinking about a bipartite matching problem, we could interpret p_x as the utility of the agent at node x and $-p_y$ as the utility of the agent at node y (with the negative sign explained in footnote 1).

Given the price at node y, there is a threshold value of the price at node x such that a trader in a competitive economy is indifferent between buying at price p_x to sell at price p_y, and not doing so. Equivalently, given the utility $v_y = -p_y$ in a matching market, there is a utility $u_x = p_x$ such that the agents at nodes x and y are just able to match and achieve the utilities u_x and v_y. To capture this, we assume there exists a function $G_{xy} : \mathbb{R} \to \mathbb{R}$, for each $xy \in \mathcal{A}$, with the following interpretation. If $p_x > G_{xy}(p_y)$, the purchase price at node x is excessive, and the trader in our competitive economy interpretation will not engage in the trade. On the contrary, if $p_x < G_{xy}(p_y)$, the purchase price is strictly below indifference level and positive payoff can be made from the trade on the arc xy. The value $G_{xy}(p_y)$ thus identifies the highest price at which a trader able to sell at price p_y at node y is willing to buy at node x. Alternatively, value $G_{xy}(p_y)$ identifies the highest utility agent x in a matching market can achieve when matching with agent y in a way that allows y to achieve utility $-p_y$. In keeping with these interpretations, we maintain the following throughout:

Assumption 1. *For each $xy \in \mathcal{A}$, the function $G_{xy} : \mathbb{R} \to \mathbb{R}$ is increasing and continuous, with limits $\lim_{p_y \to \infty} G_{xy}(p_y) = \infty$ and $\lim_{p_y \to -\infty} G_{xy}(p_y) = -\infty$.*

The function G_{xy} is called the *connection function*, as it connects the price at the endpoint to the price at the starting point.[1]

We refer to a triple $(\mathcal{Z}, \mathcal{A}, G)$ as an equilibrium flow problem.

2.2. Equilibrium

The triple $(q, \mu, p) \in \mathbb{R}^{\mathcal{Z}} \times \mathbb{R}_+^{\mathcal{A}} \times \mathbb{R}^{\mathcal{Z}}$ is an equilibrium outcome if it satisfies three conditions. The first condition is the conservation of the flow given by the mass balance equation (1). In the case of a competitive economy, this is the statement that markets balance. In the matching interpretation, this is the statement that agents are matched into pairs.

The second condition is that there is no positive payoff on any arc, that is:

$$p_x \geq G_{xy}\left(p_y\right) \quad \forall xy \in \mathcal{A}. \tag{2}$$

In the interpretation as a competitive economy, this is the statement that excess profits will be dissipated by entry into the market. In the matching interpretation, a positive payoff ($p_x < G_{xy}\left(p_y\right)$) indicates that the utilities for the agents at nodes x and y lie inside the utility frontier characterizing their match.

The third condition is that arcs with negative payoffs carry no flow. Hence

$$\mu_{xy} > 0 \implies p_x \leq G_{xy}\left(p_y\right), \tag{3}$$

which combines with the no-positive-payoff requirement to yield

$$\mu_{xy} > 0 \implies p_x = G_{xy}\left(p_y\right).$$

This is a complementary slackness condition, which can be written

$$\sum_{xy \in \mathcal{A}} \mu_{xy}\left(p_x - G_{xy}\left(p_y\right)\right) = 0.$$

In the case of a competitive economy, a negative payoff would induce exit from the market. In case of a matching market, a negative payoff signifies that the equilibrium utilities of the agents at nodes x and y lie outside their utility frontier, with the complementary slackness condition in turn indicating that these agents must not be matched with each other.

In summary, we define:

[1] This is related to the idea of Galois connections, which explains the choice of the letter G; see Nöldeke & Samuelson (2018). In the matching market, we let the utility of the agent at node y be $-p_y$ so that it makes sense for G_{xy} to be increasing, since an increase in p_y is then a decrease in agent y's utility, allowing an increase agent x's utility p_x.

Definition 1 (Equilibrium Outcome). *The triple* $(q, \mu, p) \in \mathbb{R}^{\mathcal{Z}} \times \mathbb{R}_+^{\mathcal{A}} \times \mathbb{R}^{\mathcal{Z}}$ *is an equilibrium outcome when the following conditions are met:*

(i) $\nabla^\top \mu = q$
(ii) $p_x \geq G_{xy}\left(p_y\right) \quad \forall xy \in \mathcal{A}$
(iii) $\sum_{xy \in \mathcal{A}} \mu_{xy}\left(p_x - G_{xy}\left(p_y\right)\right) = 0.$

The first condition implies our structural requirement that $\sum_{z \in \mathcal{Z}} q_z = 0$.

Notice that if p satisfies condition (ii), then setting $q = 0$ and $\mu = 0$ ensures that the remaining conditions are satisfied. Indeed, if (q, μ, p) is a equilibrium outcome, then so is $(\lambda q, \lambda \mu, p)$ for any nonnegative scalar λ. Hence, there may be no equilibrium outcome (for example, there may be no p satisfying condition (ii)), but there will often be multiple equilibrium outcomes.

3. EXISTENCE OF EQUILIBRIUM

Our task is now to fix an equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$ and a vector of quantities q (with $\sum_{z \in \mathcal{Z}} q_z = 0$), and then ask whether there exists a price vector p and flow μ such that (q, μ, p) is an equilibrium outcome of the equilibrium flow problem.

3.1. The Sufficient Conditions

We will unsurprisingly require some sufficient conditions for the existence of an equilibrium outcome, which are developed in this section.

3.1.1. Feasible Flows

Suppose we have a particularly simple equilibrium flow problem with $\mathcal{Z} = \{x, y\}$ and $\mathcal{A} = \emptyset$, with quantities $q = (-1, 1)$. It is then immediately obvious that there is no equilibrium flow, as there are no arcs along which to carry mass from node x to node y. Our first task is thus to ensure that the collection of arcs is rich enough to ensure there exist flows satisfying the mass balance condition. To formulate this requirement, we introduce the idea of a retaining set.

For a subset B of $\mathcal{Z}$, an arc xy is said to be outward if $x \in B$ and $y \notin B$. A subset B of $\mathcal{Z}$ is called retaining if there is no arc outward from B. Hence, subset B is retaining if and only if

$$\nabla \mathbf{1}_B \geq 0,$$

where $\mathbf{1}_B$ is a column vector of dimension $|\mathcal{Z}|$ identifying the nodes contained in the set B, and recalling that ∇ is the $|\mathcal{A}|\times|\mathcal{Z}|$ arc-node incidence matrix.

We state the feasibility condition in terms of retaining sets:

Assumption 2. *If B is a retaining set, then $q(B) = \sum_{z \in B} q_z \geq 0$.*

It is immediately clear that Assumption 2 is necessary for the existence of a flow satisfying the mass balance condition. If Assumption 2 fails, then there is a set (such as the set $\{x\}$ in the motivating example for this subsection) with the property that some quantity must flow away from this set, but there are no arcs to carry such flow. It is perhaps more surprising that this assumption suffices to ensure the existence of a slow satisfying mass balance:[2]

Lemma 1. *Let $q \in \mathbb{R}^{\mathcal{Z}}$ be such that $\sum_{z \in \mathcal{Z}} q_z = 0$. Then the following statements are equivalent.*

[1.1] There exists $\mu \in \mathbb{R}_+^{\mathcal{A}}$ such that $q = \nabla^\top \mu$.

[1.2] For each retaining set $B \subseteq \mathcal{Z}$, one has $q\,(B) \geq 0$.

Proof. That [1.1] implies [1.2] is straightforward: If there is no arc from a node in B to a node in $\mathcal{Z} \backslash B$, then mass can only enter into B. If a flow exists satisfying mass balance, it must then be that $q\,(B) \geq 0$.

To show the converse, we exploit Hoffman's (1960) circulation theorem. For any subset $B \subset \mathcal{Z}$, let $\mathcal{O}\,(B) = \{xy \in \mathcal{A} : x \in B, y \notin B\}$, and $\mathcal{I}\,(B) = \{xy \in \mathcal{A} : x \notin B, y \in B\}$ be respectively the set of outward and inward arcs of B. Hoffman's theorem states that, given $q \in \mathbb{R}^{\mathcal{Z}}$ with $\sum_{z \in \mathcal{Z}} q_z = 0$ and given the vectors $\underline{\mu}$ and $\overline{\mu}$ in $\mathbb{R}_+^{\mathcal{A}}$, there exists $\mu \in \mathbb{R}_+^{\mathcal{A}}$ such that

$$\nabla^\top \mu = q \qquad \text{and} \qquad \underline{\mu} \leq \mu \leq \overline{\mu}$$

if and only if for all subsets B of $\mathcal{Z}$, one has

$$\sum_{z \in B} q_z \leq \sum_{a \in \mathcal{I}(B)} \overline{\mu}_a - \sum_{a \in \mathcal{O}(B)} \underline{\mu}_a.$$

[2] For example, one might have speculated that sufficient conditions would also require a separate stipulation that if we have a set B with no inbound arcs, such as the set $\{y\}$ in our example, then $\sum_{z \in B} q_z \leq 0$.

This result is usually stated for vectors q that are identically equal to zero, but the adaptation to nonzero vectors q for which $\sum_{z\in\mathcal{Z}} q_z = 0$ is straightforward. With this result in hand, let $\overline{\mu}_a = +\infty$ and $\underline{\mu}_a = 0$ for each $a \in \mathcal{A}$, in which case Hoffman's condition becomes the requirement that for all sets $\tilde{B}$ that have no inbound links,

$$q(\tilde{B}) \leq 0.$$

We then argue that this condition is equivalent to the second condition in Lemma 1. Call a set with no inbound links *repelling*. The result then follows from noting that the set $B \subset \mathcal{Z}$ is retaining if and only if B^c is repelling, and for any set B,

$$q(B) + q(B^c) = 0. \qquad \square$$

Lemma 1 implies Hall's (1935) marriage theorem.[3] We make an additional, technical feasibility assumption.

Assumption 3. *For each $z \in \mathcal{Z}$, either $q_z > 0$ or there exists a path from z to a node $y \in \mathcal{Z}$ with $q_y > 0$.*

To see why we refer to this as a technical assumption, notice that it is automatically satisfied if $q_z > 0$, and is implied by Assumption 2 if $q_z < 0$. However, Assumption 2 is compatible with the existence of a node z with $q_z = 0$ that fails Assumption 3. In this case, it is impossible for an equilibrium to exhibit

[3] To see this, assume we are in the bipartite case to which Hall's marriage theorem applies. Hence, $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ and $\mathcal{A} \subseteq \mathcal{X} \times \mathcal{Y}$. From Lemma 1, we have that there exists a flow satisfying mass balance if and only if for all retaining sets B, one has $q(B) \geq 0$. That is, for all retaining sets B, one has

$$q(B \cap \mathcal{X}) + q(B \cap \mathcal{Y}) \geq 0.$$

Therefore for all subsets $B_{\mathcal{X}}$ of $\mathcal{X}$ and all subsets $B_{\mathcal{Y}}$ of $\mathcal{Y}$, we have

$$[B_{\mathcal{X}} \cup B_{\mathcal{Y}} \text{ retaining }] \Longrightarrow [q(B_{\mathcal{Y}}) \geq -q(B_{\mathcal{X}})]. \tag{4}$$

Now fix $B_{\mathcal{X}} \subset \mathcal{X}$ and let $B_{\mathcal{Y}}$ be the set of nodes y in $\mathcal{Y}$ with the property that there is an arc from a node in $B_{\mathcal{X}}$ to y. Then by construction $B_{\mathcal{X}} \cup B_{\mathcal{Y}}$ is retaining, and so from (4) mass balance is attainable if and only if

$$q(B_{\mathcal{Y}}) \geq -q(B_{\mathcal{X}}),$$

which is Hall's marriage theorem.

flow through node z, and we can without loss of generality eliminate node z from the network. Assumption 3 thus ensures that there are no obviously irrelevant nodes in the network. We could accommodate such nodes, but then would require another step in our equilibrium construction, to attach prices to such nodes.

3.1.2. Absence of Profitable Flows

Assumption 2 ensures that there exists a flow capable of moving quantities from their sources to their destinations. However, in equilibrium the flow must make use of only zero-payoff arcs and there must be no positive-payoff arcs.

To see how these payoff conditions could fail, let $\mathcal{Z} = \{x, y\}$ and let $\mathcal{A} = \{xy, yx\}$. Consider the quantities $q = \{-1, 1\}$. The only retaining set is $\mathcal{Z}$ itself, and so Assumption 2 is satisfied. Let the connection function be:

$$\begin{aligned} G_{xy}(p_y) &= p_y \\ G_{yx}(p_x) &= p_x + 1. \end{aligned}$$

Then there is no set of prices ensuring that no link exhibits a strictly positive payoff. A nonpositive payoff on link xy requires $p_x \geq p_y$, while a nonpositive payoff on link yx requires $p_y \geq p_x + 1$. In effect, the equilibrium flow problem admits a money pump (see Echenique et al. 2011 for a connection to revealed preference theory), with a buyer (in the competitive economy interpretation) able to generate unlimited payoffs by a continuing moving flow through the loop from x to y to x, and so on forever.

We accordingly require a condition on the connection function excluding such profitable loops. One obvious condition would be to simply preclude the existence of loops in the set of arcs, but that is stronger than we need.

We say that the collection of nodes $(z_0, z_1, \ldots z_{k+1})$ is a loop if it contains $k + 1$ distinct nodes with $z_\ell z_{\ell+1} \in \mathcal{A}$ for $\ell = 0, \ldots k$, and $z_0 = z_{k+1}$. We assume:

Assumption 4. *If $(z_0, z_1, \ldots, z_{k+1})$ is a loop, then $p > G_{z_0 z_1} \circ G_{z_1 z_2} \circ \ldots \circ G_{z_k z_{k+1}}(p)$ for all p.*

Assumption 4 ensures there is no way to achieve positive payoffs by moving flow through a loop, with no net change in the quantities. If such a loop existed, continual movement along this loop would constitute a money pump, giving rise to infinite payoffs. Equivalently, if Assumption 4 fails, then it may be impossible to assign prices to nodes in such a way as to ensure the absence of positive-payoff arcs.

Conversely, if Assumption 4 holds, one can show that it is possible to assign prices to nodes so that no arc admits a positive payoff. The argument is by construction, following the lines of a construction that appears in the proof of Lemma 2 below. One first arbitrarily fixes the price at a single node, letting all other prices be $-\infty$. One then successively updates prices at nodes, setting the price at each node z just high enough to ensure that no arc emanating from z admits a positive payoff. Assumption 4 ensures that this process terminates in a specification of prices satisfying the no-positive-payoff requirement.

3.2. The Existence Result

Assumption 2 ensures that it is possible to transport the required mass from sources to destinations. Assumption 4 ensures that it is possible to attach prices to nodes so that there are no positive-payoff arcs. In equilibrium, we require some coordination between these two constructions, in that an equilibrium flow must transport the required mass from sources to targets using only zero-payoff arcs. It would accordingly be no surprise if additional conditions were required to ensure the existence of an equilibrium flow. However, these assumptions suffice:

Theorem 1. *Let the equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$ and quantity q satisfy Assumptions 1-4. Fix a node z with $q_z \neq 0$ and a price θ. Then there exists an equilibrium outcome with $p_z = \theta$.*

The node z and price θ are arbitrary (as long as $q_z \neq 0$). If we did not fix the price at some node, then we would have multiple equilibrium outcomes.[4] For example, Let $\mathcal{Z} = \{x, y\}$, $\mathcal{A} = \{xy\}$, $G_{xy}(p_y) = p_y$, and $q = (-1, 1)$. Then any outcome (q, μ, p) with $q = (-1, 1)$, $\mu_{xy} = 1$ and $p_x = p_y$ is an equilibrium outcome. This is the functional equivalent of the familiar observation that the conditions for competitive equilibrium in an exchange economy set relative but not absolute prices.

[4] At the end of Section 2.2, we noted that that if (q, μ, p) is an equilibrium outcome, then so is $(\lambda q, \lambda\mu, p)$ for any $\lambda > 0$. Here, we are noting that there will in general exist multiple equilibrium outcomes, exhibiting different prices, for a *fixed* quantity q.

3.3. Proof of Theorem 1

The proof of Theorem 1 associates the equilibrium flow problem with a bipartite matching problem, and then exploits existence results for the bipartite matching problem. This association may be useful for other purposes.

3.3.1. Equilibrium Outcomes and Bipartite Solutions

Given an equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$, define the set of source nodes as $\mathcal{X} = \{x \in \mathcal{Z} : q_x < 0\}$ and the set of destination nodes as $\mathcal{Y} = \left\{y \in \mathcal{Z} : q_y > 0\right\}$. Given nodes $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, we say that the sequence $(z_0, \ldots, z_{k+1})$ is a path from x to y if $z_0 = x$, $z_{k+1} = y$, and $z_\ell z_{\ell+1} \in \mathcal{A}$ for $\ell = 0, \ldots k$. Let $\tilde{\mathcal{A}}$ be the set of pairs $xy \in \mathcal{X} \times \mathcal{Y}$ for which such a path exists.

Given an equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$, we define an associated bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$, where the connection function $\tilde{G}$ is defined by letting, for $xy \in \tilde{\mathcal{A}}$,

$$\tilde{G}_{xy}\left(p_y\right) = \sup_{\substack{z_0, z_1, z_2, \ldots, z_{k+1} \\ \text{s.t. } z_0 = x,\ z_{k+1} = y \\ z_\ell z_{\ell+1} \in \mathcal{A}}} G_{xz_1} \circ G_{z_1 z_2} \circ \ldots \circ G_{z_k y}\left(p_y\right).$$

By definition of $\tilde{\mathcal{A}}$, the set over which we are taking the supremum is nonempty, and therefore we have necessarily $\tilde{G}_{xy}\left(p_y\right) > -\infty$. In principle, we cannot preclude the possibility that $\tilde{G}_{xy}\left(p_y\right) = +\infty$, which could happen with a sequence of paths visiting an increasing and unbounded number of nodes, in which case the path must contain a loop. However, it is straightforward to verify that Assumption 4 ensures that $\tilde{G}_{xy}(p_y)$ is bounded and the maximum is attained.

The bipartite equilibrium flow problem thus contains the source nodes and destination nodes of the original problem. There is an arc from a source node to a target node in the bipartite problem if there is a path between these two nodes in the original problem. The connection function in the bipartite problem is defined by letting $\tilde{G}_{xy}(p_y)$ be the composition of the connection functions along one of the paths from x to y in the original problem, namely the path in the original problem that maximizes $\tilde{G}_{xy}(p_y)$. Intuitively, we ask which path from x to y in the original problem maximizes the value of moving flow from x to y, and use this path to induce the function $\tilde{G}_{xy}$ in the bipartite problem.

Given a function q assigning quantities to the nodes in $\mathcal{Z}$, we simplify notation by also using q to denote the restriction of q to the nodes $\mathcal{X} \cup \mathcal{Y}$. Notice that in restricting q to $\mathcal{X} \cup \mathcal{Y}$, we exclude a node $z \in \mathcal{Z}$ if and only if $q_z = 0$.

Lemma 2. *Let Assumptions 1–4 hold. Then there exists an equilibrium outcome (q, μ, p) for the equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$ if and only if there exists an equilibrium outcome $(q, \tilde{\mu}, \tilde{p})$ for the associated bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$ such that p and $\tilde{p}$ agree on $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. [Only if:] Let (q, μ, p) be an equilibrium outcome of the equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$. We derive an equilibrium outcome for the associated bipartite equilibrium flow problem with the required properties.

Let Π_{xy} be the set of paths in $\mathcal{A}$ beginning at x and ending at y. Let $\tilde{\mu}_{xy} = \sum_{\rho \in \Pi_{xy}} \mu^\rho$, where μ^ρ is the flow associated with path ρ.[5] It is straightforward from the mass balance condition for the equilibrium outcome (q, μ, p) that for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ we have

$$\begin{aligned} -\sum_{y \in \mathcal{Y}} \tilde{\mu}_{xy} &= q_x \\ \sum_{x \in \mathcal{X}} \tilde{\mu}_{xy} &= q_y. \end{aligned}$$

This ensures that q and $\tilde{\mu}$ satisfy the mass balance equation (1) in the bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$.

We next argue that for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $xy \in \tilde{\mathcal{A}}$, we have

$$p_x \geq \tilde{G}_{xy}\left(p_y\right).$$

To see this, we note that for any given path $x = z_0, z_1, z_2, ..., z_{k+1} = y$ from x to y, we have (from the no-positive-payoff equilibrium condition (2)) $p_{z_\ell} \geq G_{z_\ell z_{\ell+1}}\left(p_{z_{\ell+1}}\right)$ and thus

$$p_x \geq G_{xz_1} \circ G_{z_1 z_2} \circ \ldots \circ G_{z_k y}\left(p_y\right).$$

Taking the supremum over paths from x to y, we have $p_x \geq \tilde{G}_{xy}\left(p_y\right)$. This ensures that $(q, \tilde{\mu}, \tilde{p})$ satisfies the equilibrium condition (2).

[5] This sum could conceivably be infinite, as there may be an infinite number of paths from x to y. However, all but finitely many of the paths from x to y must involve a loop, and Assumption 4 implies that an equilibrium outcome cannot induce a positive flow along a loop.

Finally, suppose we have $\tilde{\mu}_{xy} > 0$. Then there must exist a path $(z_0, z_1, z_2, \ldots, z_{k+1})$ from x $(= z_0)$ to y $(= z_{k+1})$ along which the flow is strictly positive. Thus, along this path $p_{z_\ell} = G_{z_\ell z_{\ell+1}}\left(p_{z_{\ell+1}}\right)$, and hence

$$p_x = G_{xz_1} \circ G_{z_1 z_2} \circ \ldots \circ G_{z_k y}\left(p_y\right).$$

Because $p_x \geq \tilde{G}_{xy}\left(p_y\right) \geq G_{xz_1} \circ G_{z_1 z_2} \circ \ldots \circ G_{z_k y}\left(p_y\right)$, it follows that equality holds everywhere and thus

$$p_x = \tilde{G}_{xy}\left(p_y\right).$$

This ensures that $(q, \tilde{\mu}, \tilde{p})$ satisfies the equilibrium condition (3). Therefore, letting $\tilde{p}$ be the restriction or p to $\mathcal{X} \cup \mathcal{Y}$, we have the equilibrium outcome $(q, \tilde{\mu}, \tilde{p})$ for the bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$, i.e., satisfying:

$$\begin{array}{rcl} -\sum_{y \in \mathcal{Y}} \tilde{\mu}_{xy} & = & q_x \\ \sum_{x \in \mathcal{X}} \tilde{\mu}_{xy} & = & q_y \\ \tilde{p}_x & \geq & \tilde{G}_{xy}\left(\tilde{p}_y\right) \\ \tilde{\mu}_{xy} > 0 & \Longrightarrow & \tilde{p}_x = \tilde{G}_{xy}\left(\tilde{p}_y\right). \end{array} \tag{5}$$

[If:] Suppose $(q, \tilde{\mu}, \tilde{p})$ is an equilibrium outcome of the bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$, and hence satisfies (5). We find an associated equilibrium (q, μ, p) of the equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$, where q retains its specification on $\mathcal{X} \cup \mathcal{Y}$ and is otherwise zero.

For each pair $xy \in \tilde{\mathcal{A}}$ with $\tilde{\mu}_{xy} > 0$, consider the (by construction nonempty) set of paths from x to y satisfying $\tilde{p}_x = \tilde{G}_{xy}(\tilde{p}_y)$. Choose any one of these paths, and let the flow μ route a flow of size $\tilde{\mu}_{xy}$ along this path. This ensures that μ satisfies the mass balance equilibrium condition (1). For an arc $zz' \in \mathcal{A}$, $\mu_{zz'}$ is defined as the sum of the flows along the selected paths that contain zz', for every origin-destination pair $xy \in \tilde{\mathcal{A}}$.

We now define an iterative procedure that generates the prices p. For the initial step, we let

$$p_z^0 = \begin{cases} \tilde{p}_z & \text{if } z \in \mathcal{Y} \\ -\infty & \text{otherwise} \end{cases},$$

with the induction step given by

$$p_z^{t+1} = \max\left\{p_z^t, \max_{y: zy \in \mathcal{A}}\left\{G_{zy}\left(p_y^t\right)\right\}\right\}.$$

We then complete the argument with the following lemma:

Lemma 3. *Let Assumptions 1–4 hold. For t sufficiently large (but finite), (q, μ, p^t) is an equilibrium outcome of the equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$.*

Proof. We establish this result in the following series of steps:

1. After a finite number of steps, $p_z^t > -\infty$ for all z.

   To see this, let $x \rightsquigarrow y$ denote a path from node x to node y, let $l(x \rightsquigarrow y)$ denote the number of arcs along this path, let $G_{z\rightsquigarrow y}(p_y)$ be the composition of the functions G along the arcs in this path, and let $d(x, y)$ be the number of arcs in the *shortest* path from x to y. Then we have,

   $$p_z^t = \max_{y\in\mathcal{Y}} \max_{z\rightsquigarrow y: l(z\rightsquigarrow y)\leq t} G_{z\rightsquigarrow y}(p_y),$$

   letting the inner maximum be $-\infty$ if $\{z \rightsquigarrow y : l(z \rightsquigarrow y) \leq t\}$ is empty. Hence, if there is a directed path of length at most k from z to $\mathcal{Y}$, then the price of z will be updated to a finite price (and never thereafter reduced) after at most k steps. Assumption 2 (for nodes z with $q_z > 0$) and Assumption 3 (for nodes z with $q_z = 0$) ensure that for every node $z \notin \mathcal{Y}$, there is a path from node z to $\mathcal{Y}$. As a result, all prices will be updated to finite values, ensuring $p_z^t > -\infty$ for all z, after $\max_z \min_{y\in\mathcal{Y}} d(z, y)$ steps.

2. The price p_y of a node $y \in \mathcal{Y}$ is never updated by the algorithm.

   Assume that one updates the price of node $y \in \mathcal{Y}$ at some step. Then there is a path $y \rightsquigarrow y'$ from y to some $y' \in \mathcal{Y}$ with $p_y < G_{y\rightsquigarrow y'}(p_{y'})$. Let $x \in \mathcal{X}$ be an element of $\mathcal{X}$ such that $\tilde{\mu}_{xy} > 0$. Then one has $p_x = \tilde{G}_{xy}(p_y)$, and hence $p_x < \tilde{G}_{xy} \circ G_{y\rightsquigarrow y'}(p_{y'}) = G_{x\rightsquigarrow y'}(p_{y'}) \leq \tilde{G}_{xy'}(p_{y'})$, a contradiction.

3. There exists a T such that $p^t = p^T$ for all $t \geq T$.

   If the price at node z is updated at step t, then it must be that

   $$p_z^t = \max_{y\in\mathcal{Y}} \max_{z\rightsquigarrow y: l(z\rightsquigarrow y)\leq t} G_{z\rightsquigarrow y}(p_y) > p_z^{t-1}$$

   for some node $y \in \mathcal{Y}$. Assumption 4 ensures that $\max_{z\rightsquigarrow y: l(z\rightsquigarrow y)\leq t} G_{z\rightsquigarrow y}(p_y)$ is achieved by path $z \rightsquigarrow y$ that contains no loop. There are a finite number of nodes $y \in \mathcal{Y}$ and for each pair of nodes $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$ a finite number of paths $z \rightsquigarrow y$ that contain no loops. This ensures that for each node $z \in \mathcal{Z}$, the price p_z is updated a finite number of times. Hence, there exists a step T after which no further updating occurs.

4. For a node $y \in \mathcal{Y}$, $p_y^T = \tilde{p}_y$. For $x \in \mathcal{X}$, $p_x^T = \tilde{p}_x$.

   The first statement follows from steps 2 and 3. For the second statement, if there exists a x such that $p_x^T \neq \tilde{p}_x$, then:

   - If $p_x^T > \tilde{p}_x$, then there exists a node $y \in \mathcal{Y}$ and a path $x \rightsquigarrow y$ such that $G_{x \rightsquigarrow y}\left(\tilde{p}_y\right) = p_x^T > \tilde{p}_x = \max_{y' \in \mathcal{Y}} \tilde{G}_{xy'}\left(\tilde{p}_{y'}\right) \geq G_{x \rightsquigarrow y}\left(\tilde{p}_y\right)$, a contradiction.
   - If $p_x^T < \tilde{p}_x$, then there is a node $y \in \mathcal{Y}$ and a path $x \rightsquigarrow y$ such that $\tilde{p}_x = G_{x \rightsquigarrow y}\left(p_y^T\right)$. Moving along path $x \rightsquigarrow y$, take the last z such that $p_z^T \neq \tilde{G}_{zy}\left(p_y^T\right)$, and let z' be its successor. One has by definition $p_z^T < \tilde{G}_{zy}\left(p_y^T\right) = G_{zz'}\left(\tilde{G}_{z'y}\left(p_y^T\right)\right) = G_{zz'}\left(p_{z'}^T\right)$, which contradicts the fact that the algorithm has reached stationarity.

5. We let the stationary prices p^T be our price vector p. The construction of the prices ensures that for every arc $zz' \in \mathcal{A}$, we have $p_z \geq G_{zz'}(p_{z'})$, and hence the no-positive-payoff equilibrium condition (2) holds. Moreover, if $\mu_{zz'} > 0$, then the arc zz' is contained in a path from source node x to destination node y for some $xy \in \tilde{\mathcal{A}}$ with $\tilde{\mu}_{xy} > 0$. We then have $p_x = \tilde{G}_{xy}(p_y)$ and hence $p_x = G_{x \rightsquigarrow y}(p_y)$, which suffices to ensure $p_z = G_{zz'}(p_{z'})$ and hence the equilibrium condition (3) is met. □

This completes the proof of Lemma 2. □

3.3.2. *Bipartite Existence Result*

The bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$ and the quantity q satisfy Hall's (1935) condition if, for every subset $\mathcal{B}$ of $\mathcal{X}$, we have

$$-\sum_{x \in B} q_x \leq \sum_{\{y \in \mathcal{Y} \,:\, \exists x \in B \text{ with } xy \in \tilde{\mathcal{A}}\}} q_y. \tag{6}$$

Intuitively, the set B is a set of nodes from which from which the total quantity $-\sum_{x \in B} q_x$ (≥ 0) must flow. The term on the right side sums over the set of nodes to which this quantity might possibly flow, and the condition requires that the latter set has at least the capacity to receive the desired flow.

Lemma 4. *Let Assumptions 1, 3 and 4 hold. Fix a node z with $q_z \neq 0$ and a price $\theta \in \mathbb{R}$. If the bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$ and quantity q satisfy Hall's condition, then there exists an equilibrium outcome $(q, \tilde{u}, \tilde{p})$ for $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$ with $\tilde{p}_z = \theta$.*

Proof. The proof first extends the set of arcs $\tilde{\mathcal{A}}$ to the set $\mathcal{X} \times \mathcal{Y}$ by introducing an arc for every $xy \in (\mathcal{X} \times \mathcal{Y}) \setminus \tilde{\mathcal{A}}$, with the connection function $\tilde{G}$ extended to such arcs (without introducing new notation) by letting

$$\tilde{G}_{xy}(p_y) = p_y - n$$

for some integer n. Our approach will be to establish the existence of an equilibrium outcome in the equilibrium flow problem created by added these additional arcs, and then to ensure that for sufficiently large n, flow along any such arc not in $\tilde{\mathcal{A}}$ is so unprofitable that the equilibrium outcome will not exhibit such flow. This will ensure that the outcome is also an equilibrium outcome of the bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$.

We refer to the bipartite equilibrium flow problem with the additional arcs as the expanded (bipartite, hereafter omitted from the name) equilibrium flow problem, denoted by $(\mathcal{X} \cup \mathcal{Y}, \mathcal{A}^E, \tilde{G})$. Notice that if an equilibrium flow in this expanded problem directs flow along an arc xy that is not contained in $\tilde{\mathcal{A}}$, then it must be that $p_x = p_y - n$.

Let $\left(q, \mu^E_{xy}, p\right)$ be an equilibrium outcome of the expanded equilibrium flow problem, i.e., let $\left(q, \mu^E_{xy}, p\right)$ satisfy

(i) $-\sum_y \mu^E_{xy} = q_x$ for all $x \in \mathcal{X}$ and $\sum_x \mu^E_{xy} = q_y$ for all $y \in \mathcal{Y}$
(ii) $p_x \geq \tilde{G}_{xy}\left(p_y\right)$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$
(iii) $\mu^E_{xy} > 0$ implies $p_x = \tilde{G}_{xy}\left(p_y\right)$, with $p_x = p_y - n$ if $xy \notin \tilde{\mathcal{A}}$.
(iv) $p_z = \theta$.

The existence of this outcome follows from Proposition 6 of Nöldeke and Samuelson. The key to applying this proposition is to note that the function $G_{xy}(p_y)$ is increasing, continuous, and as p_y varies, sweeps out the entire real line. Alternatively, one could first work with a regularized version of the equilibrium flow problem, following the lines of Galichon et al. (2019), for which a constructive proof of existence can be given, and then exploit results from Galichon et al. (2015) to provide a limiting argument giving existence in the underlying equilibrium flow problem.

We begin with an intermediate result. Consider the arcs defined by

$$\mathcal{A}^* = \left\{xy \in \tilde{\mathcal{A}}\right\} \cup \left\{yx : x \in \mathcal{X}, y \in \mathcal{Y}, \mu^E_{xy} > 0\right\}$$

The set $\mathcal{A}^*$ thus adds to the set of arcs $\mathcal{A}$ from the bipartite equilibrium flow problem an arc yx from node y to node x whenever the arc xy is *not* contained in $\tilde{\mathcal{A}}$ but carries positive flow in the equilibrium outcome of the extended equilibrium flow problem. These added arcs may not appear in the extended equilibrium flow problem, nor will we use them to construct yet another equilibrium flow problem, but the set $\mathcal{A}^*$ is useful in formulating the following intermediate result:

Lemma 5. *Let Assumptions 1, 3 and 4 hold and let the bipartite equilibrium flow problem* $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$ *and quantity* q *satisfy Hall's condition. Assume* $x^*y^* \notin \tilde{\mathcal{A}}$ *and* $\mu^E_{x^*y^*} > 0$*. Then there exists no partition of the set of nodes* $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ *into sets* A *and* B *such that* $y^* \in A$, $x^* \in B$ *and there are no arcs of* $\mathcal{A}^*$ *from* B *to* A.

The question is whether we can partition the set of nodes $\mathcal{Z}$ so that the set B containing node x^* is retaining (given the arcs $\mathcal{A}^*$) *and* excludes the node y^*. The set B must contain more than simply the node x^*, since there is some arc x^*y in $\mathcal{A}^*$. We can accordingly think of successively adding nodes to the set B in an effort to make it retaining. The proof uses the fact that $\mu^E_{x^*y^*} > 0$ is an equilibrium outcome of the extended equilibrium flow problem to show that this effort leads to a contradiction.

Proof. We suppose that a partition with the desired properties exists, and seek a contradiction. Let $A^{\mathcal{X}} = A \cap \mathcal{X}$ and $A^{\mathcal{Y}} = A \cap \mathcal{Y}$, and define $B^{\mathcal{X}}$ and $B^{\mathcal{Y}}$ in a similar fashion.

Hall's condition implies that the bipartite equilibrium flow problem has a flow satisfying mass balance. Because there are by assumption no arcs in $\mathcal{A}^*$ from $B^{\mathcal{X}}$ to $A^{\mathcal{Y}}$, this implies that $-q\left(A^{\mathcal{X}}\right) \geq q\left(A^{\mathcal{Y}}\right)$.

Similarly, by assumption there exists a flow μ^E satisfying the mass balance equation in the extended equilibrium flow problem. Because there are by assumption no arcs in $\mathcal{A}^*$ from $B^{\mathcal{Y}}$ to $A^{\mathcal{X}}$, this implies that $-q\left(B^{\mathcal{X}}\right) \geq q\left(B^{\mathcal{Y}}\right)$.

Summing the previous two inequalities yields $-q(\mathcal{X}) \geq q(\mathcal{Y})$, but as this holds as an equality, we get an equality in the two previous inequalities. Thus $-q(A^{\mathcal{X}}) = q\left(A^{\mathcal{Y}}\right)$ and $-q(B^{\mathcal{X}}) = q\left(B^{\mathcal{Y}}\right)$.

Now $-q(A^{\mathcal{X}}) = \sum_{x \in A^{\mathcal{X}}} \sum_{y \in \mathcal{Y}} \mu^E_{xy}$. In addition, for $x \in A^{\mathcal{X}}$ and $y \in B^{\mathcal{Y}}$, we have $\mu^E_{xy} = 0$, since otherwise there would be an arc of $\mathcal{A}^*$ from B to A. Hence

$$-q\left(A^{\mathcal{X}}\right) = \sum_{x \in A^{\mathcal{X}}} \sum_{y \in A^{\mathcal{Y}}} \mu^E_{xy}.$$

Because μ^E satisfies mass balance, we also have

$$q\left(A^{\mathcal{Y}}\right) = \sum_{x \in \mathcal{X}} \sum_{y \in A^{\mathcal{Y}}} \mu^E_{xy}.$$

Subtracting gives

$$0 = -q\left(A^{\mathcal{X}}\right) - q\left(A^{\mathcal{Y}}\right) = \sum_{x \in B^{\mathcal{X}}} \sum_{y \in A^{\mathcal{Y}}} \mu^E_{xy},$$

and hence for all $x \in B^{\mathcal{X}}$ and $y \in A^{\mathcal{Y}}$, $\mu^E_{xy} = 0$, which contradicts $x^* \in B^{\mathcal{X}}$, $y^* \in \mathcal{A}^Y$ and $\mu^E_{x^*y^*} > 0$. This contradiction proves the claim. □

The proof of Lemma 4 is then completed by the following lemma.

Lemma 6. *Let Assumptions 1, 3 and 4 hold and let the bipartite equilibrium flow problem* $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$ *and quantity* q *satisfy Hall's condition. For* n *large enough,* $\mu^E_{x^*y^*} > 0$ *implies* $x^*y^* \in \tilde{\mathcal{A}}$.

Proof. The proof makes use of Lemma 5 to establish a contradiction. Suppose we have $x^* \in \mathcal{X}$ and $y^* \in \mathcal{Y}$ with $\mu^E_{x^*y^*} > 0$ and $x^*y^* \notin \tilde{\mathcal{A}}$. We now seek a partition of the set of nodes. We start by partitioning the set of nodes $\mathcal{Z}$ into the sets $A = \{y^*\}$ and $B = \mathcal{Z} \backslash \{y^*\}$. Then by Lemma 5 there must be an arc in $\mathcal{A}^*$ from B to A. Call z_1 the origin point of that arc, and form a new partition with $A = \{y^*, z_1\}$ and $B = \mathcal{Z} \backslash \{y^*, z_1\}$. Once again, as long as the new candidate for the set A does not contain x^*, by Lemma 5 there must be an arc in $\mathcal{A}^*$ from B to A. We continue in this fashion, successively adding nodes to our candidate set A, until encountering an arc in $\mathcal{A}^*$ from x^* to some z_k in the preceding iteration to A, at which point the node x^* is added to the updated set A. At this point, the elements of A along with their arcs, taken in the reverse of the order in which they were added to A, constitute a path from x^* to y^*, given by

$$x^* = z_{k+1}, z_k, z_{k-1}, \ldots, z_2, z_1, z_0 = y^*.$$

By the construction of the path, any pair of successive nodes $z_{\ell+1}z_\ell$ from this path for which $\ell + 1$ is odd is an arc from a node in $\mathcal{X}$ to a node in $\mathcal{Y}$. This arc is included in the set of arcs $\mathcal{A}^*$ in the bipartite equilibrium flow problem, and so satisfies the no-positive-surplus equilibrium condition

$$p_{z_{\ell+1}} \geq \tilde{G}_{\ell+1,\ell}(p_{z_\ell}). \tag{7}$$

This includes the arc x^*z_k, as this must be an arc from a node in $\mathcal{X}$ (namely x^*) to a node in $\mathcal{Y}$, namely z_k, where k must be an even integer.

Again by the construction of the path, any pair of successive nodes $z_{\ell+1}z_\ell$ from this path for which $\ell + 1$ is even is an arc $z_{\ell+1}z_\ell \in \mathcal{A}^*\backslash\tilde{\mathcal{A}}$. Then by the construction of $\mathcal{A}^*$, the arc $z_\ell z_{\ell+1}$ (note the reversal of subscripts) carries a positive flow in the equilibrium outcome of the expanded equilibrium flow problem, and hence satisfies

$$p_{z_\ell} = p_{z_{\ell+1}} - n. \tag{8}$$

By assumption, we have $\mu^E_{x^*y^*} > 0$ with $x^*y^* \notin \tilde{\mathcal{A}}$, and hence we have

$$p_{x^*} = \tilde{G}_{x^*y^*}\left(p_{y^*}\right) = p_{y^*} - n. \tag{9}$$

Equation (9) shows that as n grows, $p_{y^*} - p_{x^*}$ diverges. We show that this gives a contradiction, establishing the result.

Write $p_{y^*} - p_{x^*}$ as

$$\begin{array}{rcllr}
p_{y^*} - p_{x^*} & = & p_{y^*} - p_{z_1} & (p_{z_1} \geq \tilde{G}_{z_1y^*}(p_{y^*})) & (1)\\
& + & p_{z_1} - p_{z_2} & (p_{z_2} = p_{z_1} + n) & (2)\\
& + & p_{z_2} - p_{z_3} & (p_{z_3} \geq \tilde{G}_{z_3z_2}(p_{z_2})) & (3)\\
& + & p_{z_3} - p_{z_4} & (p_{z_4} = p_{z_3} + n) & (4)\\
& + & p_{z_4} - p_{z_5} & (p_{z_5} \geq \tilde{G}_{z_5z_4}(p_{z_3})) & (5)\\
& + & p_{z_5} - p_{z_6} & (p_{z_6} = p_{z_5} + n) & (6)\\
& \vdots & & & \\
& + & p_{z_k} - p_{x^*} & (p_{x^*} \geq \tilde{G}_{x^*z_k}(p_{z_k})). &
\end{array}$$

The terms with odd numbers correspond to arcs from $\mathcal{X}$ to $\mathcal{Y}$ that are contained in $\tilde{\mathcal{A}}$. (The final $\cdot$ is an odd-numbered term.) The terms with even numbers correspond to arcs from Y to X, contained in $\mathcal{A}^* \setminus \tilde{\mathcal{A}}$. In each case, we have included in parentheses the relationship between the prices at the two nodes of the corresponding arc, taken from (7) or (8).

We now examine a sequence along which n grows arbitrarily large, with $\mu^E_{x^*,y^*} > 0$ along this sequence. Suppose first that as n grows, p_{y^*} approaches a limit. This imposes a lower bound on p_{z_1} (from condition (1) in our expansion of $p_{y^*} - p_{x^*}$), which in turn imposes a lower bound on p_{z_2} (from condition (2) in the expansion), and so on, leading to the conclusion that p_{x^*} does not diverge to $-\infty$, a contradiction.

A similar argument can be constructed, beginning from the assumption that p_{x^*} approaches a limit as n grows large, with a similar construction ensuring that it cannot be the case that $p_{y^*} \to +\infty$, giving a contradiction.

It remains to preclude the possibilities that p_{x^*} and p_{y^*} both approach $+\infty$ or both approach $-\infty$. We present the argument for the case in which both approach $+\infty$, with the other case being analogous. If p_{x^*} and p_{y^*} both approach $+\infty$, then we can divide $\mathcal{X}$ into disjoint sets $\underline{\mathcal{X}}$ and $\overline{\mathcal{X}}$ and can divide $\mathcal{Y}$ into disjoint sets $\underline{\mathcal{Y}}$ and $\overline{\mathcal{Y}}$ such that the prices in $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ diverge and the prices in $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ do not diverge. The sets $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ are nonempty, because they contain x^* and y^*. The set $\underline{\mathcal{X}} \cup \underline{\mathcal{Y}}$ is nonempty because it contains the node z whose price we have fixed. There must be no arcs in $\mathcal{A}$ from $\underline{\mathcal{X}}$ to $\overline{\mathcal{Y}}$ (since otherwise such an arc would generate positive payoff, a contradiction). There must exist no flow from nodes in $\overline{\mathcal{X}}$ to nodes in $\underline{\mathcal{Y}}$ (since these arcs earn negative payoff). Any flow from $\underline{\mathcal{X}}$ to $\underline{\mathcal{Y}}$ must occur along arcs in $\tilde{\mathcal{A}}$, since otherwise the arc must earn negative payoff. Hall's condition must then hold for the system whose nodes are $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$, and then must separately hold the system whose nodes are $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$.

We now repeat the previous existence argument separately for each of these systems, with one price $\underline{p}$ fixed at a node in the system whose nodes are $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ and price $\overline{p}$ fixed for the system whose nodes are $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$. This gives us an equilibrium for each system separately. We cannot be sure that we have an equilibrium of the combined systems, since there may be arcs in $\tilde{\mathcal{A}}$ from $\overline{\mathcal{X}}$ to $\underline{\mathcal{Y}}$. However, if we set $\underline{p}$ sufficiently low and $\overline{p}$ sufficiently high, these arcs will carry no flow, and we will indeed have an equilibrium in each of the disjoint sets. This again either gives a contradiction, or we have prices in the system whose nodes are $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ that diverge. In the latter case, we repeat this procedure, until obtaining a contradiction. □

This completes the proof of Lemma 4. □

3.3.3. Closing the Ring

The proof of Theorem 1 is then completed by the following:

Lemma 7. *Let Assumptions 1, 3 and 4 hold. The equilibrium flow problem* $(\mathcal{Z}, \mathcal{A}, G)$ *and quantity* q *satisfy Assumption 2, that* $q(B) \geq 0$ *for all retaining* B*, if and only if the associated bipartite equilibrium flow problem* $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$ *together with the quantity* q *satisfies Hall's marriage condition.*

Proof. [Only if:] Suppose the equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$ and quantity q are such that $q(B) \geq 0$ for all retaining sets. Suppose $\tilde{\mathcal{X}} \cup \tilde{\mathcal{Y}}$ is a subset of $\mathcal{X} \cup \mathcal{Y}$ for which, in the bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$, there are no arcs from nodes in $\tilde{\mathcal{X}}$ to nodes outside $\tilde{\mathcal{Y}}$. To establish Hall's condition, it suffices to show that $q(\tilde{\mathcal{X}} \cup \tilde{\mathcal{Y}}) \geq 0$ for any such set.

In the equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$, define a sequence of sets by letting $\mathcal{Z}_1 = \tilde{\mathcal{X}} \cup \tilde{\mathcal{Y}}$, letting $\mathcal{Z}_2$ be the union of $\mathcal{Z}_1$ and any node that is the endpoint of an arc whose origin lies in $\mathcal{Z}_1$; letting $\mathcal{Z}_3$ be the union of $\mathcal{Z}_2$ and any node that is the endpoint of an arc whose origin lies in $\mathcal{Z}_2$, and so on. This sequence terminates (since the set $\mathcal{Z}$ of nodes is finite) in a set $\mathcal{Z}^*$ that is retaining (otherwise the process would not terminate) and hence by assumption satisfies $q(\mathcal{Z}^*) \geq 0$. Moreover, $\mathcal{Z}^*$ contains no node z for which $q_z > 0$ that is not contained in $\tilde{\mathcal{Y}}$ (since otherwise there would be a link in the problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$ from some node in $\tilde{\mathcal{X}}$ to a node outside $\tilde{\mathcal{Y}}$, a contradiction). Then we have

$$0 \leq q(\mathcal{Z}^*) \leq q(\tilde{\mathcal{X}} \cup \tilde{\mathcal{Y}}),$$

giving Hall's condition.

[If:] For the other direction, let $\tilde{\mathcal{Z}}$ be a retaining set in equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$. Then

$$q(\tilde{\mathcal{Z}}) = q((\mathcal{X} \cap \tilde{\mathcal{Z}}) \cup (\mathcal{Y} \cap \tilde{\mathcal{Z}}))$$

Because $\tilde{\mathcal{Z}}$ is retaining in the equilibrium flow problem $(\mathcal{Z}, \mathcal{A}, G)$, the set $(\mathcal{X} \cap \tilde{\mathcal{Z}}) \cup (\mathcal{Y} \cap \tilde{\mathcal{Z}})$ is retaining in the bipartite equilibrium flow problem $(\mathcal{X} \cup \mathcal{Y}, \tilde{\mathcal{A}}, \tilde{G})$. Hence, Hall's condition gives $q((\mathcal{X} \cap \tilde{\mathcal{Z}}) \cup (\mathcal{Y} \cap \tilde{\mathcal{Z}})) \geq 0$, establishing the needed result that $q(\tilde{\mathcal{Z}}) \geq 0$. □

4. DISCUSSION

Section 1 argued that a number of familiar problems can be formulated as special cases of the equilibrium flow problem. This section offers an illustration.

Consider the following one-to-one matching market. Let $\mathcal{X}$ be a finite set of types of workers and $\mathcal{Y}$ a finite set of types of firms. There are n_x workers of each type $x \in \mathcal{X}$, and m_y firms of each type $y \in \mathcal{Y}$. We can think of n_x as identifying the number of workers of type x, or we can think of n_x as the mass of a continuum of workers of type x in an infinite population, as may be appropriate when modeling a competitive market.

A match involves a worker of type x, a firm of type y, and a wage w_{xy}. Such a match generates utility $\mathcal{U}_{xy}\left(w_{xy}\right)$ for the worker (increasing in w_{xy}) and $\mathcal{V}_{xy}\left(w_{xy}\right)$ for the firm (decreasing in w_{xy}). We allow workers and firms to be unmatched, with an unmatched worker receiving utility $\mathcal{U}_{x0}$ and an unmatched firm receiving utility $\mathcal{V}_{0y}$. A matching is a pair (μ, w), where μ is a vector identifying the mass μ_{xy} of matches between workers of type x and firms of type y, for each $xy \in \mathcal{X} \times \mathcal{Y}$, and w is a vector specifying the wage w_{xy} attached to each pair $xy \in \mathcal{X} \times \mathcal{Y}$.

We can reformulate this matching problem as an example of the equilibrium flow problem. The sets of types of workers and firms become the nodes in the network, augmented by a dummy node used to allow the possibility that a worker or firm remains unmatched. The quantities attached to the nodes represent the quantities of workers and firms of each type. In particular, let

$$
\begin{aligned}
\mathcal{Z} &= \mathcal{X} \cup \mathcal{Y} \cup \{0\} \\
\mathcal{A} &= ((\mathcal{X} \cup \{0\}) \times (\mathcal{Y} \cup \{0\})) \setminus \{(0,0)\} \\
q_z &= -n_z, \quad z \in \mathcal{X} \\
q_z &= m_z, \quad z \in \mathcal{Y} \\
q_0 &= \sum_{x \in \mathcal{X}} n_x - \sum_{y \in \mathcal{Y}} m_y.
\end{aligned}
$$

There is thus a node for each type of worker, a node for each type of firm, and a dummy node denoted by 0. There is an arc from each worker node to each firm node as well as to the dummy node. There is an arc from the dummy node to each firm node. Each node corresponding to a type of worker carries a negative quantity equal to the mass of workers of that type, while each node corresponding to a type of firm carries a positive quantity corresponding to the mass of that type. The dummy node carries the remaining mass, which ensures that the total $\sum_{z \in \mathcal{Z}} q_z$ equals zero. Because the quantity vector q identifies the negative of the quantity of each type of worker and the quantity of each type of firm, we can represent a match between a worker and a firm as a flow along the arc connecting the node containing that type of worker to the node representing the firm. In general, a flow along an arc signifies the mass of matches between the types residing at each node connected by the arc.

The price attached to a node identifies the utility of the agent at that node (if a worker) or the negative of the utility of the agent at that node (if a firm). Reminiscent of our treatment of quantities, the payoff vector p identifies the payoffs of workers and the *negative* of the payoffs of firms. An increase in the payoff p_y at firm node y thus corresponds to a smaller utility requirement from the firm at that node, making it more attractive for workers to traverse the arc terminating at the node. This allows us to write connection functions $G_{xy}(p_y)$ that are increasing functions. Formally, the connection function G describes the utility possibilities generated by a match, with

$$
\begin{aligned}
G_{xy}\left(p_y\right) &= \mathcal{U}_{xy} \circ \mathcal{V}_{xy}^{-1}\left(-p_y\right) \text{ for } x \in \mathcal{X}, y \in \mathcal{Y} \\
G_{x0}\left(p_0\right) &= \mathcal{U}_{x0} + p_0 \text{ for } x \in \mathcal{X} \\
G_{0y}\left(p_y\right) &= \mathcal{V}_{0y} + p_y \text{ for } y \in \mathcal{Y}.
\end{aligned}
$$

As required, $G_{xy}(p_y)$ is increasing in p_y.

We adopt the normalization $p_0 = 0$.[6] This ensures that if buyer x is unmatched, so that the equilibrium outcome involves flow along the arc $x0$, then the no-positive-payoff equilibrium condition ensures that $p_x = \mathcal{U}_{x0} + p_0 = \mathcal{U}_{x0}$, the payoff specified for an unmatched buyer. Similarly, if seller y is unmatched, then we have $p_0 = 0 = \mathcal{V}_{0y} + p_y$, and hence $-p_y = V_{0y}$.

[6] The existence result in Theorem 1 allows us to normalize the price at any node z for which $q_z \neq 0$. We can thus immediately apply Theorem 1 if the matching market is unbalanced, and hence the dummy node exhibits a nonzero quantity. However, if the market is balanced, then $q_0 = 0$. In this case, we can bring the equilibrium flow problem into the purview of Theorem 1 by introducing an additional buyer type $\hat{x}$ with $q_{\hat{x}} < 0$, with an arc from $\hat{x}$ only to the dummy node, so that this buyer must remain unmatched. To accommodate this additional type, we let the mass on the dummy node equal $-q_{\hat{x}}$ (> 0), and then can normalize $p_0 = 0$.

Remembering that p_y measures the negative of the seller's utility, we again have the seller receiving her unmatched payoff.

Following the standard definition, the matching (μ, w) is stable if

$$\sum_{y \in \mathcal{Y}} \mu_{xy} \leq n_x, \; \sum_{x \in \mathcal{X}} \mu_{xy} \leq m_y,$$

and

$$\begin{aligned}
\mu_{xy} > 0 \quad &\Longrightarrow \quad \left\{ \begin{array}{l} \mathcal{U}_{xy}(w_{xy}) = \max\left\{\max_{\tilde{y} \in \mathcal{Y}} \mathcal{U}_{x\tilde{y}}\left(w_{x\tilde{y}}\right), \mathcal{U}_{x0}\right\} \\ \mathcal{V}_{xy}(w_{xy}) = \max\left\{\max_{\tilde{x} \in \mathcal{X}} \mathcal{V}_{\tilde{x}y}\left(w_{\tilde{x}y}\right), \mathcal{V}_{0y}\right\} \end{array} \right\} \\
\sum_{y \in \mathcal{Y}} \mu_{xy} < n_x \quad &\Longrightarrow \quad \mathcal{U}_{x0} = \max\left\{\max_{\tilde{y} \in \mathcal{Y}} \mathcal{U}_{x\tilde{y}}\left(w_{x\tilde{y}}\right), \mathcal{U}_{x0}\right\} \\
\sum_{x \in \mathcal{X}} \mu_{xy} < m_y \quad &\Longrightarrow \quad \mathcal{V}_{0y} = \max\left\{\max_{\tilde{x} \in \mathcal{X}} \mathcal{V}_{\tilde{x}y}\left(w_{\tilde{x}y}\right), \mathcal{V}_{0y}\right\}
\end{aligned}$$

The first condition is the feasibility requirement that the quantity of each type of worker and firm that is matched is no more than the total quantity of this type in the market (with excess agents remaining unmatched). The second set of conditions is the stability requirement that no worker and firm can improve their utilities, either by matching with one another at some appropriate wage or by remaining unmatched.

Galichon et al. (2024, Lemma 6) prove the following:

Lemma 8. *The matching* (μ, w) *is stable in the matching market if and only if the associated outcome* (q, μ, p) *is an equilibrium outcome of the associated equilibrium flow problem.*

The proof is a relatively straightforward bookkeeping exercise, connecting the requirements for stability to their equivalents in the definition of an equilibrium flow.

To see how this result might be useful, we define the *equilibrium flow correspondence* $\mathcal{Q}$, which associates with each utility vector p the set of vectors q, i.e., the specifications of the sets of firms and workers, for which there exists an equilibrium flow outcome yielding the utilities specified by p. The point of Galichon et al. (2024) is to show that the equilibrium flow correspondence in general satisfies their condition of unified gross substitutes (Theorem 4), and hence has an inverse which is lattice valued (Corollary 1) and increasing (in the strong set order, Theorem 1). This implies, for example, that as the number of a given type of firms increases, the set of equilibrium payoffs of all firms decreases (in the strong set order) while the set of equilibrium payoffs of all workers increases. An increase in the number of workers has the reverse effects.[7]

More generally, let $\mathcal{Q}(p)$ be a correspondence that maps a vector p, specifying the price attached to each node of the equilibrium flow network, to the set of quantities q with the property that if $q \in \mathcal{Q}(p)$, then there exists an equilibrium outcome (q, μ, p). Again, $\mathcal{Q}$ has an inverse which is lattice valued and increasing. In a hedonic pricing context, this implies that adding a consumer to the market increases (again, at least weakly, in the strong set order) the set of equilibrium prices received by all sellers in the market. In a time-dependent routing context, it implies that adding a new supplier to the production network accelerates the delivery times attained by all final users. We thus view the equilibrium flow problem as a potentially useful tool for characterizing equilibrium outcomes in a variety of settings.

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[7] Demange & Gale (1985, Lemma 2 and Property 2) establish similar results for a model in which the quantities attached to the various nodes must take integer values, which we do not require. Dropping this requirement is useful in allowing one to accommodate a continuum of each type of agent, as is common in models of competitive matching (cf. Nöldeke & Samuelson 2024).

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